Groups, Buildings and Compactifications

Münster, August 9th - 11th 2017

This seminar is intended as a continuation of a joint seminar series of the math departments of Bielefeld, Karlsruhe, Münster and Regensburg.
This time, we study how buildings arise naturally as the boundary of semi-simple Lie groups. In general, we are interested in spaces with nice group actions (for certain classes of groups) and the construction of compactifications for those spaces.
Following the survey article [Ji], we consider the geodesic compactification of a symmetric space of non-compact type and identify its boundary with the Tits Building of a semi-simple Lie group.
Thereafter, we study Tits buildings associated to algebraic groups defined over $\mathbb{Q}$ which arise in a similar fashion as in the Lie group case.
Then, we turn to duality groups and universal spaces (for proper actions). We consider the Borel-Serre compactification of locally symmetric spaces and apply this to construct universal spaces for non-uniform lattices $\Gamma \subseteq G$ and obtain duality results for arithmetic subgroups of $G(\mathbb{Q})$.
At that point, we study how geometric properties of the group action on the boundary can be used to prove the Novikov Conjecture and verify these properties for arithmetic subgroups.
Finally, we study two examples which are generalisations of the arithmetic subgroup $\mathrm{GL}(2, \mathbb{Z})$, namely the Mapping Class Group $\mathrm{Mod}^+(\Sigma_g)$ and the group of outer automorphisms of a free group $\mathrm{Out}(F_n)$. Here, the natural geometric objects of interest are the Teichmüller space for $\mathrm{Mod}^+(\Sigma_g)$ and the so called Outer space for $\mathrm{Out}(F_n)$. Again, we study Borel-Serre like compactifications of these spaces and try to understand (the simplicial structure of) their boundaries and derive duality results.

Talk 1: Lie Groups and Symmetric Spaces
If you want to give this talk, it is helpful to be (at least a little bit) familiar with Lie groups and symmetric spaces. Give a short introduction to symmetric spaces (of non-compact type) and recall (shortly) Lie groups (in particular, semisimple Lie groups) and Lie algebras. More precisely, describe geodesics, the exponential map, the adjoint representation and flats in symmetric spaces. If time permits, talk about the Cartan decomposition and roots. Please communicate with the speaker of talk 3.
Literature: [Lin12] chapter 1 and 2, [Leu11], [GJT98] §2.1-§2.3, [Kna02], [WM15] §A6

Talk 2: Buildings
If you want to give this talk, it is helpful to be (at least a little bit) familiar with buildings. This talk introduces the basic facts about buildings and provides important examples. In particular, you should mention: Coxeter complexes, buildings as simplicial complexes, homotopy type of (spherical) buildings, spherical and affine examples (building corresponding to the Fano plane, trees), in particular stabilizer of simplices in the example [Bro89, Ex. V.1.A-D].
Literature: [Bro89] chapter II, IV (and references therein)

Talk 3: Geodesic Compactification of Symmetric Spaces
Explain the geodesic compactification of a symmetric space. Explain the relationship between points in $X(\infty)$ and proper parabolic subgroups. You can follow [Ji], chapter 2 up to Proposition 2.2 and fill in the details, which can be found in [BJ06]. You will need some Lie group backround such as the Cartan decomposition. Please communicate with the speaker of talk 1.
Literature: [BJ06] chapter I.2 up to I.2.17 (you can also look at [GJT98] chapter 3), [GJT98] §2.1-§2.3, [Ji]

Talk 4: Buildings corresponding to Lie Groups and Symmetric Spaces
Explain the construction of the building $\Delta(G)$ corresponding to a semisimple Lie group G. In order to do so, recall the concepts arising from Lie theory, such as the Bruhat decomposition. Show that this building $\Delta(G)$ is isomorphic to the simplicial complex constructed in talk 3. You can follow [Ji], pp. 11. If time permits, you can talk about Mostow rigidity, but you should focus on [Ji, Prop. 3.2].
Literature: [GJT98] chapter 3, [Ji]

Talk 5: Algebraic Groups and Tits Buildings over $\mathbb{Q}$
Introduce semisimple linear algebraic groups $\mathbf{G}$ in the sense it is used in [Ji], i.e. as $G \subseteq GL(n, \mathbb{C})$. In particular, introduce notions as defined over $\mathbb{Q}$, tori, $\mathbb{Q}$-split tori, $\mathbb{Q}$-parabolic subgroups and arithmetic subgroups. Give examples of these things. Explain the connection of algebraic and Lie groups. Equipped with the algebraic background, define the Tits building $\Delta_{\mathbb{Q}}(\mathbf{G})$ of $\mathbf{G}$ defined over $\mathbb{Q}$. You can follow the last section of chapter 3 of [Ji] and the very beginning of chapter 4.
Literature: [Bro89] appendix, [WM15] in particular for examples, [Ji]

Talk 6: Classifying Spaces and Duality Groups
Give a short introduction to the classifying spaces EG and $\underline{E}G$ for free and proper actions respectively for a group G including the proof of existence and uniqueness with emphasis on $\underline{E}G$, see for example [BCH94]. In the second part of the talk one should give an introduction to Duality Groups with the characterization as in [Bro94, VIII, 10.1] and the particular case of Poincaré-duality groups. Discuss the example of G being the fundamental group $\pi_1(M)$ of a closed, aspherical manifold M shortly.
Literature: [BCH94], [Bro94, Chapter VIII, Section 10]

Talk 7: Borel-Serre Compactification I
Introduce the horospherical decomposition of a symmetric space X and explain the geodesic action associated to a parabolic subgroup ([BJ06, III.5.3]).

Give the construction of the Borel-Serre partial compactification $\overline{X}^{BS}$ as in [BJ06, III.5.8].

Show the important properties of $\overline{X}^{BS}$ as listed in [Ji, Theorem 4.5]: $\overline{X}^{BS}$ is a real analytic manifold with corners which retracts into the interior X ([BJ06, III.5.11]), the boundary $\partial\overline{X}^{BS}$ can be decomposed into contractible components e_P which are parametrized by proper $\mathbb{Q}$-parabolic subgroups P (III.5.6) such that $e_{P_1} \subseteq \overline{e_{P_2}}$ if and only if $P_1 \subseteq P_2$ (III.5.12) and the action of a non-uniform lattice $\Gamma \subseteq G(\mathbb{Q})$ extends to a proper, cocompact, real analytic action on $\overline{X}^{BS}$ (III.5.13/14), thereby defining the Borel-Serre compactification $\Gamma\backslash\overline{X}^{BS}$ of $\Gamma\backslash X$. Also mention the [Ji, Corollary 4.6].

Literature: The general concept is explained in [Ji]. In [BJ06], the construction of $\overline{X}^{BS}$ is described in detail (see III.5 or III.9 for an alternative construction). See also [Sap97] for a nice geometric visualisation of the Borel-Serre compactification.

Talk 8: Borel-Serre Compactification II and Truncated Spaces

In this talk, we take a closer look at some applications of the Borel-Serre compactification. First, introduce the concept of truncated subspaces as another possible model for compactifications ([Ji], p.21-22). Then, show that a torsion-free arithmetic subgroup $\Gamma \subseteq G(\mathbb{Q})$ is a duality group, but not a Poincaré duality group ([Ji, Theorem 4.8]).

Prove that for a non-uniform lattice $\Gamma \subseteq G$ the partial Borel-Serre compactification $\overline{X}^{BS}$ is a Γ-cofinite $\underline{E}\Gamma$-space (Theorem 4.9). If time permits, you can also treat the structure of ends of $\Gamma\backslash X$ (Theorem 4.7).

Literature: The main source is chapter 4 in [Ji]. For Theorem 4.9 check [Ji07].

Talk 9: Novikov Conjecture

State the *Novikov Conjecture* of the homotopy invariance of higher signatures as in [Ros15, Conjecture 1.2]. *Formulate* the *Stable Borel Conjecture* and axioms that imply the Novikov Conjecture as in [Ji07, Theorem 1.1]. Then prove the Novikov Conjecture for arithmetic subgroups, see [Ji07, Theorem 3.1]. Follow [Ji07] with an eye on the results presented in talk 7 and 8 and cover the missing part.

Literature: [Ros15], [Ji07]

Talk 10: Mapping Class Group and Teichmüller Space

Give an introduction to the *Mapping Class Group* $Mod^+(\Sigma_g)$ of a closed oriented surface Σ_g of genus g and to the associated *Teichmüller space* T_g, see [Ji, Section 7], [FM12] or [Iva02]. Prove that T_g is a model for $\underline{E}Mod^+(\Sigma_g)$, see [Ji, Proposition 7.1] or [FM12, Theorem 12.2]. Please communicate intensely with the speaker of talk 11.

Literature: [Ji, Section 7,8], [FM12], [Iva02]

Talk 11: Curve Complex

Continue with Mapping Class Groups and Teichmüller Space following [Ji, Section 8]. Introduce the *Curve Complex* $\mathcal{C}(\Sigma_g)$ and discuss [Ji, Theorem 8.1, 8.2, 8.3], also see [Iva02]. Concerning the proofs of the theorems choose a reasonable amount of material you would like to present. Please communicate intensely with the speaker of talk 10.

Literature: [Ji, Section 8] and references therein, [Iva02]

Talk 12: $Out(F_n)$ and Outer Space

Introduce marked metric graphs and give the definition of Outer Space X_n. Explain how the topology of X_n looks like and construct the action of $\mathrm{Out}(F_n)$ on X_n.
Define the spine K of Outer Space. Show that K is a deformation retract of X_n and that K is contractible (sketch).
Literature: See [CV86] for precise definitions and statements, [Ji] and [Vog08] for motivation and [Vog02] for overview and some nice diagrams.

Talk 13: Truncated Outer Space
Introduce core graphs and give the definition of truncated Outer Space $X_n(\varepsilon)$ as in [Ji]. Moreover, you can ...

- ... discuss the Borel-Serre compactification of X_n (in [BF00]) and its relation to $X_n(\varepsilon)$.
- ... give the definition of the Core Graph Complex $\mathcal{CG}(F_n)$ and show that $\mathcal{CG}(F_n)$ is isomorphic to the Free Factor Complex.
- ... prove that $\mathrm{Out}(F_n)$ is a virtual duality group.

Note: After the introduction of core graphs and $X_n(\varepsilon)$ the speaker should pick one of the other topics and focus on that. The other two points should be mentioned but not be covered in detail.
Literature: See [BF00] for Core Graphs, the Borel-Serre Compactification of X_n and the result that $\mathrm{Out}(F_n)$ is a virtual duality group (Theorem 1.4). The construction of $X_n(\varepsilon)$ and $\mathcal{CG}(F_n)$ is explained in [Ji]. For the definition of the Free Factor Complex see [HV98]. Motivation and context is provided in [Vog02].

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If you are interested in participation or if you have any further questions, please contact us.

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