§ 1 Motivation

Claim: S^1 is the simplest interesting topological space!
What are the next-simple ones?

$(S^1)^n = T^n \rightsquigarrow SL_n\mathbb{Z}$, arithmetic groups
surface $\Sigma \rightsquigarrow \mathrm{Mod}(\Sigma)$
<u>graphs</u> $\rightsquigarrow$ <u>$\mathrm{Out}(F_n)$</u>

$$\mathrm{Out}(F_n) := \mathrm{Aut}(F_n)/\mathrm{Inn}(F_n)$$

$\mathrm{Mod}($  $) \cong \mathrm{Out}(F_2) \cong GL_2(\mathbb{Z})$

<u>Automorphisms of free groups</u> were studied...
... already by Nielsen (1910s)
... by Whitehead (1930s - 50s) using topological models
... using combinatorial group theory (60s, 70s)

[1976] for information. But we would like to note that the history of the theory of $A(F_n)$ illustrates the fact that, even in the theory of infinite groups, there exist special groups defined for purely algebraic reasons which attract research work over long periods of time in spite of gaps of several decades in the sequence of papers that deal with them. In Chapter II.10 on

– "The history of combinatorial group theory", Chandler – Magnus 1982

PROBLEM 5. *Determine the structure of* Aut F, *of its subgroups, especially its finite subgroups, and its quotient groups, as well as the structure of individual automorphisms.*

– "Problems in combinatorial group theory", Lyndon 1987

GGT: Study groups by realizing them as symmetries of geometric objects.

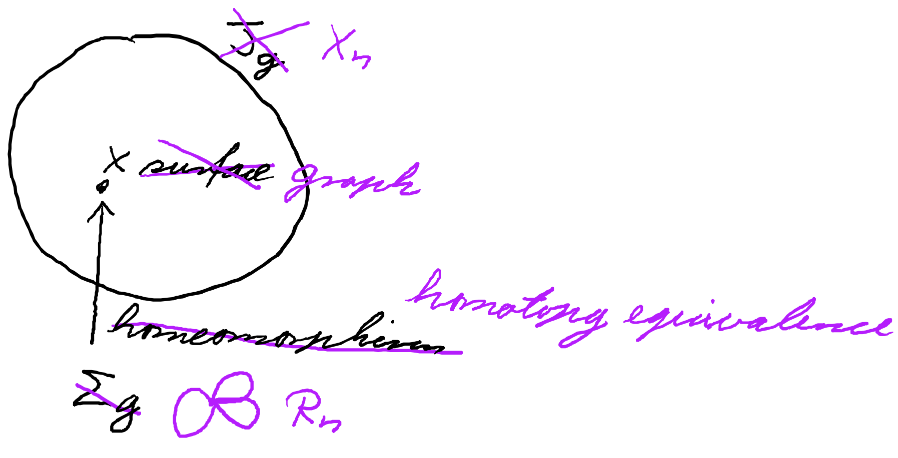
Culler - Vogtmann '86: Outer Space X_n (CV_n)

⇝ contractible space with "nice" (= proper, cocompact if we restrict to spine) action of $Out(F_n)$

Idea: $Mod(\Sigma_g)$ ⟷ Teichmüller Space T_g

$Out(F_n)$ ⟷ Outer Space X_n

⇝ mimic proofs for $Mod(\Sigma)$ using Teichmüller space



Examples of applications:

- Culler - Vogtmann (1986): $vcd(Out(F_n)) = 2n-3$
- Bestvina - Feighn (2000): $Out(F_n)$ is a virtual duality group.
- Bestvina - Guirardel - Horbez (2017): $Out(F_n)$ satisfies the

Novikov conjecture.

§2 Definition of Outer Space

In what follows, a *graph* ...

... can have loops

... is always connected

... is not allowed to have vertices of valence 0, 1 or 2.

$\mathbb{R}$-graph: Graph with metric by assigning a length to each edge.

Fix n, a rose R_n and an identification $F_n = \pi_1(R_n)$.

Def.: G an $\mathbb{R}$-graph. A hom. equ.

$$g : R_n \to G$$

is called a *marking*.

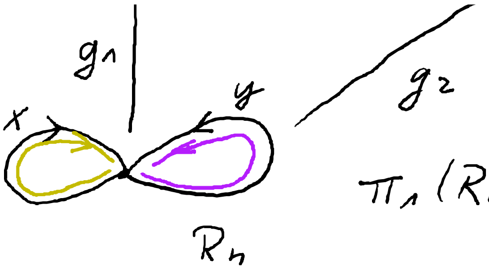
Two *markings* $g_1 : R_n \to G_1$ and $g_2 : R_n \to G_2$ *are equiv.* if $\exists$ isometry $i : G_1 \to G_2$ s.t.

$$\begin{tikzcd} G_1 \arrow[rr, "i"] & & G_2 \\ & R_n \arrow[ul, "g_1"] \arrow[ur, "g_2"'] \arrow[u, phantom, "\circlearrowleft"] & \end{tikzcd}$$

up to free homotopy.

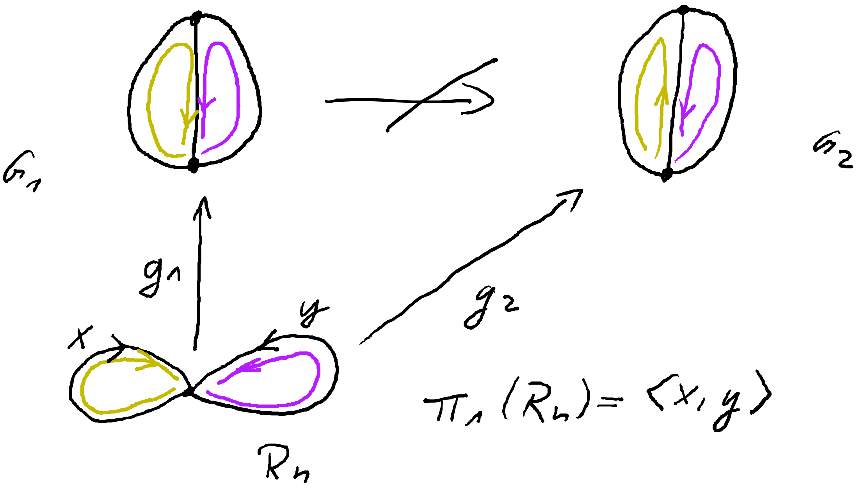
An equ. class (G, g) is a *marked graph*.

Ex.:



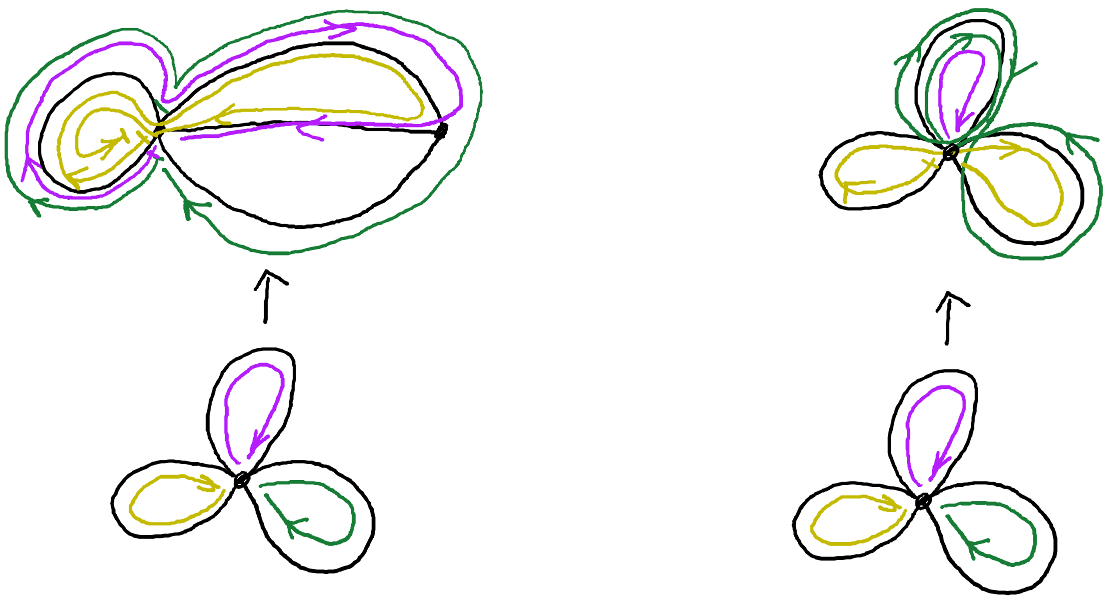
$\pi_1(R_n) = \langle x, y \rangle$

$\Rightarrow (G_1, g_1) = (G_2, g_2)$



$\pi_1(R_n) = \langle x, y \rangle$

$\Rightarrow (G_1, g_1) \neq (G_2, g_2)$



Def.: X_n = as a set $\left\{ \text{marked graphs of total edge length } 1 \right\}$

$Out(F_n) \curvearrowright X_n$: Take $[\alpha] \in Out(F_n)$, realise it by $A: R_n \to R_n$ and define

$[\alpha].(G,g) = (G, g \circ A)$

$\rightsquigarrow$ acts by changing the marking

Marking $g: R_n \to G$ determines length function on conj classes in F_n.

$l: \mathcal{C} \longrightarrow \mathbb{R}$

$[w] \longmapsto$ length of shortest loop representing $g_*(w)$

$(g_* : \pi_1(R_n) = F_n \longrightarrow \pi_1(G))$

Fact (Morgan-Shalen): l determines (G,g).

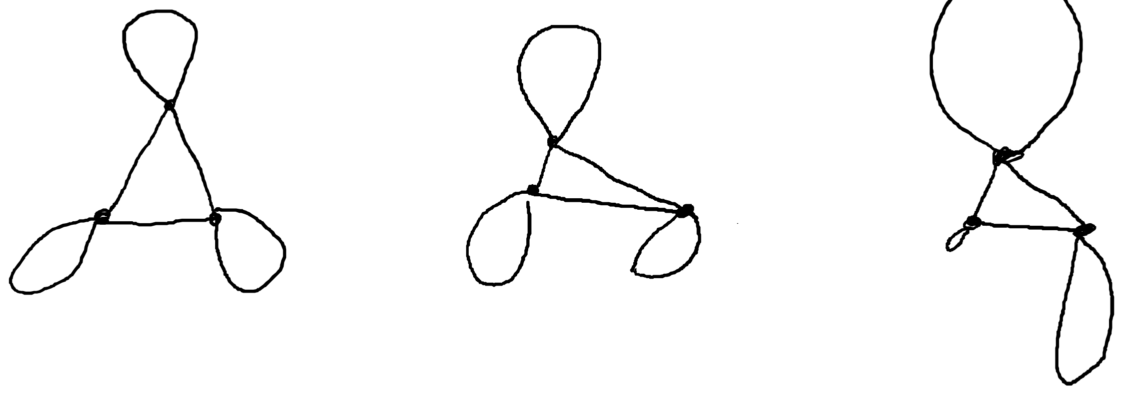
$\Rightarrow X_n \hookrightarrow \cancel{\mathbb{R}^{\mathcal{C}}}\ \mathbb{P}^{\mathcal{C}}$

$\rightsquigarrow$ Give X_n the subspace topology

$\rightsquigarrow$ action $Out(F_n) \curvearrowright X_n$ continuous

Fix (G,g), change only metric

$\rightsquigarrow$ family S of marked graphs



$x \in S$: determined by edge lengths

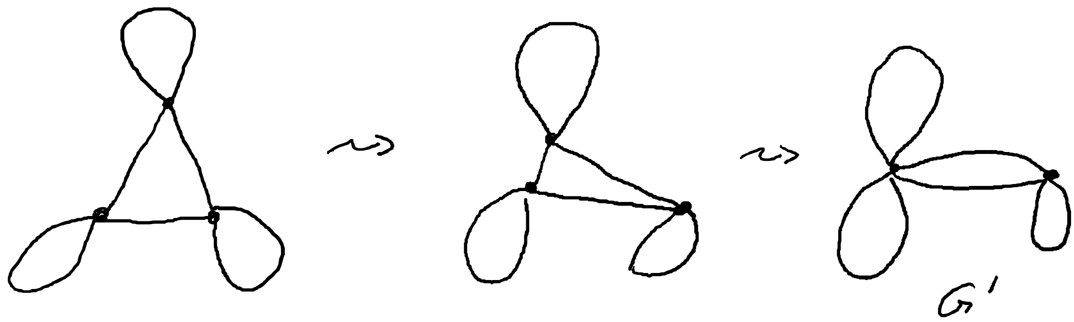
$\rightsquigarrow$ one coordinate for each edge $e \in E(G)$

$\rightsquigarrow$ open simplex of dimension
$\# E(G) - 1$

$\rightsquigarrow$ X_n is union of open simplices
of dimension $\leq 3n-4$
(Euler char)

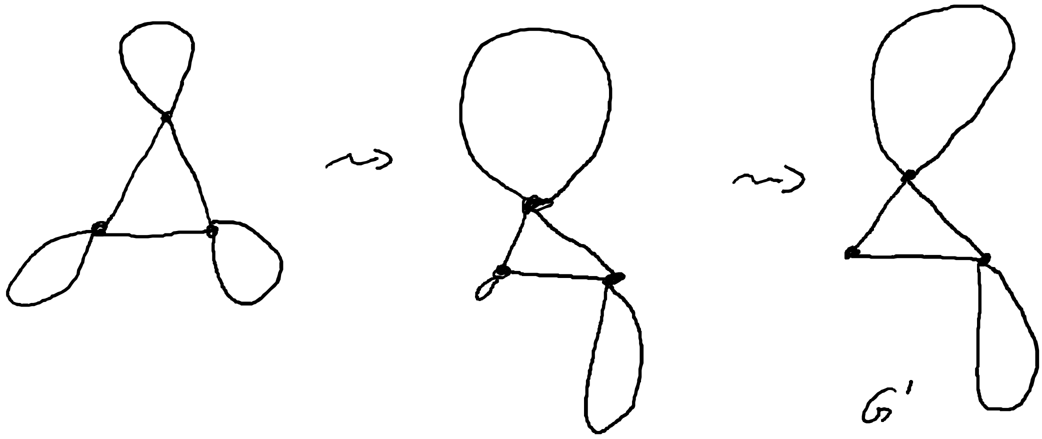
<u>What happens if we shrink an edge to length $= 0$?</u>

Possibility 1:



$\rightsquigarrow$ go to a codim. 1 face of the simplex

Possibility 2:



$\rightsquigarrow$ $\pi_1(G') \neq F_n$

$\Rightarrow$ $(G', g') \notin X_n$

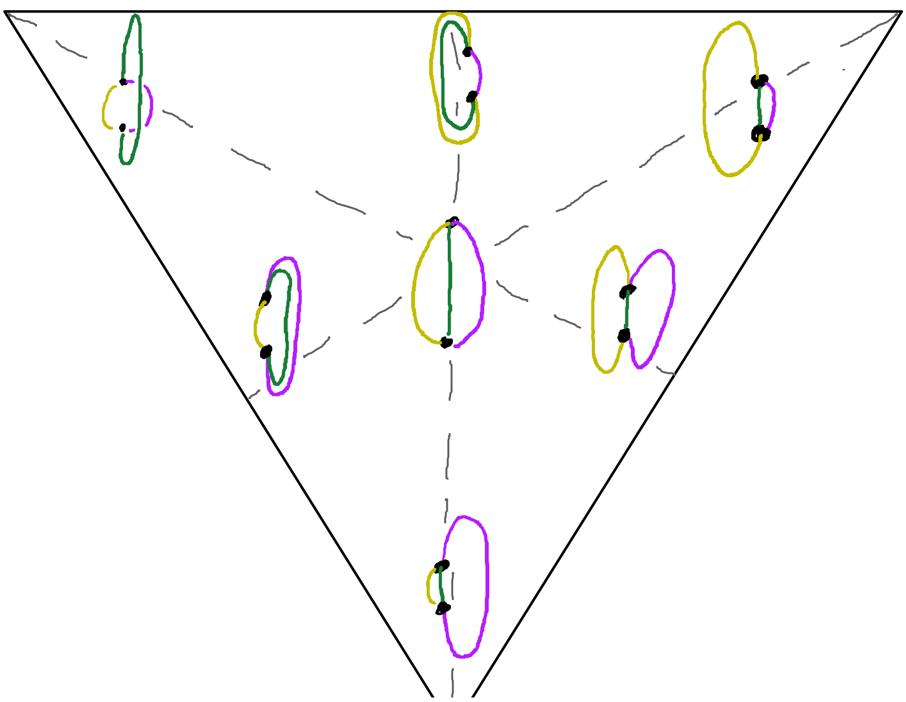
$\rightsquigarrow X_n$ is not a simpl. compl.

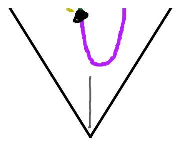
some faces are missing

§3 The spine K_n of X_n

Open simplices of X_n form a POSet,
define $\underline{K_n}$ as its geometric realisation.
$\dim(K_n) = 2n-3$

vertex in K_n $\triangleq$ open simplex in X_n
$\triangleq$ family of marked graphs
$\triangleq$ marked graph where all edges have the same length

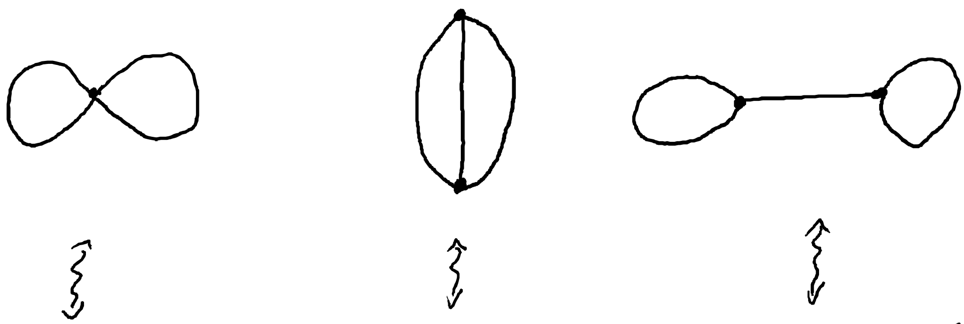




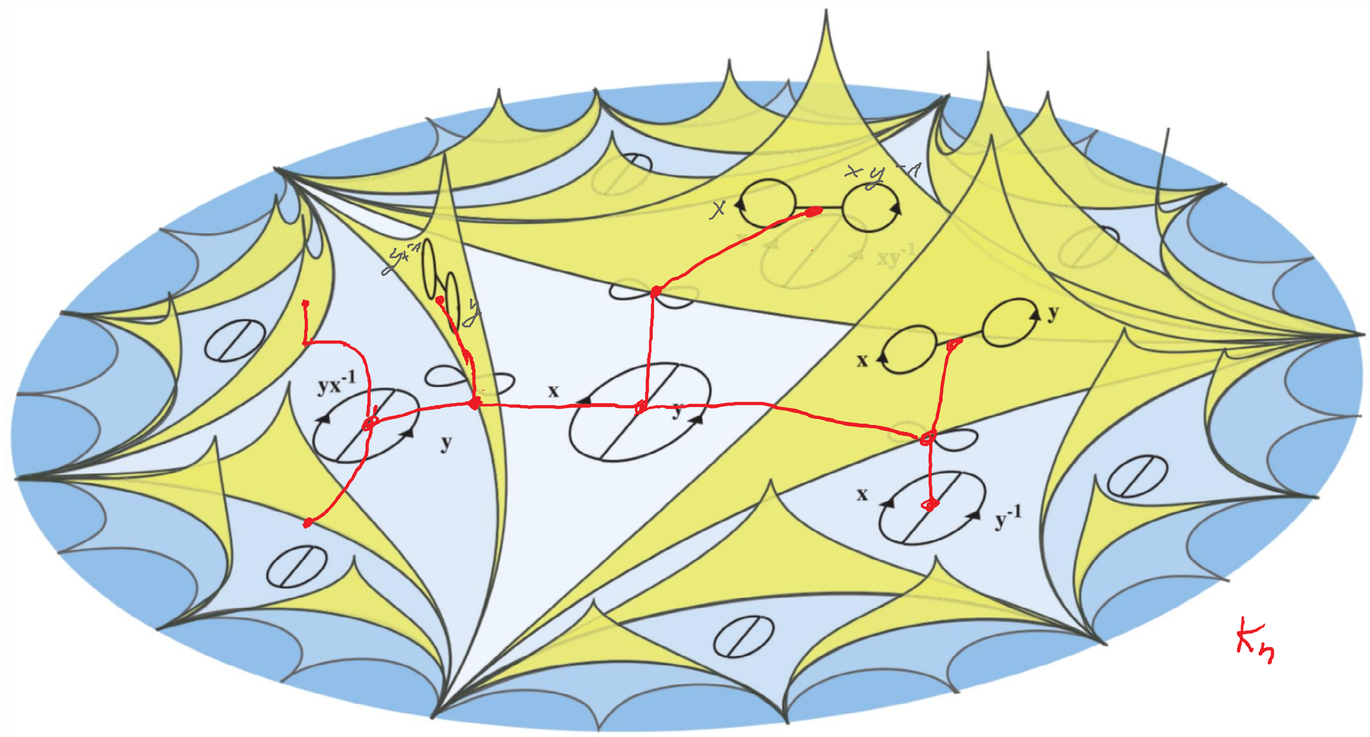
$Out(F_n) \curvearrowright K_n$ simplicial, cocompact

Example: X_2

Possible graphs with fundamental group $= F_2$:
(no vertices of valence ≤ 2 allowed)

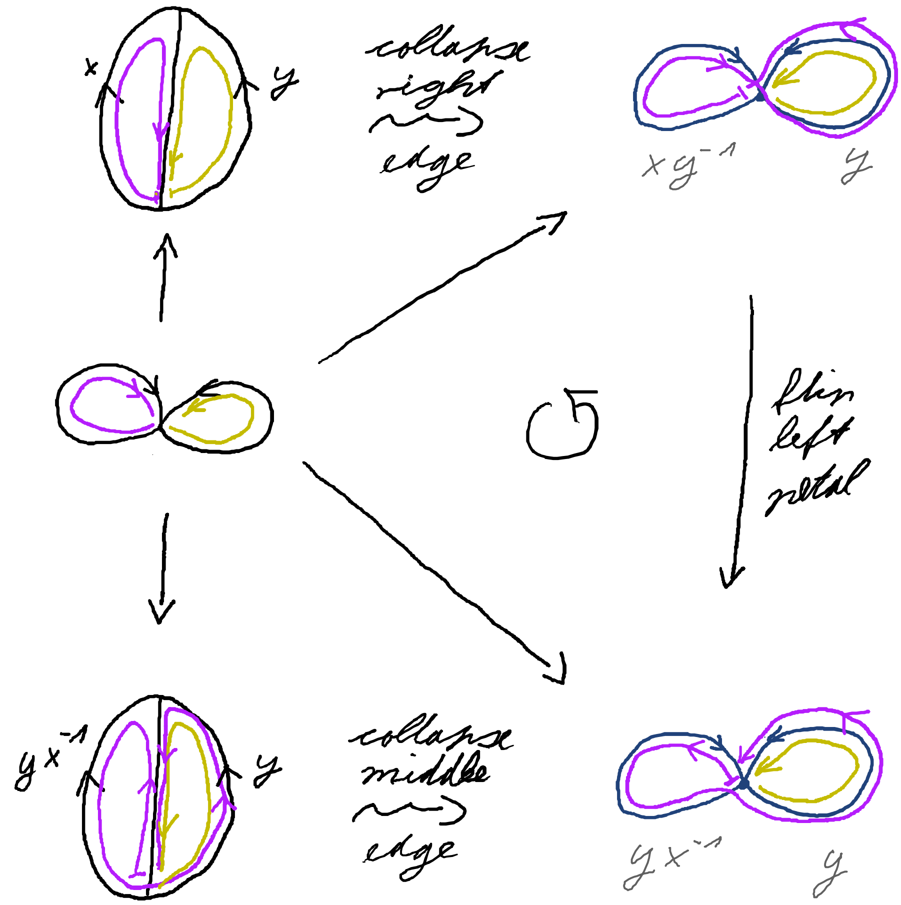


1-simplex 2-simplex 2-simplex



Picture stolen from "What is... Outer Space?" – Vogtmann

$\rightsquigarrow X_n$ is not a manifold



Thm: K_n is connected
Pf by Stallings folds
Central role: Roses

Lemma: K_n is retract of X_n.
Pf.: Push in from missing faces.

⇝ Prove contractibility of X_n by showing
Thm: K_n is contractible
Steps of proof:

1. Every vertex is adjacent to a rose
   $\Rightarrow K_n = \bigcup_{R \text{ rose}} st(R, g)$
2. Define a norm on the set of marked roses.

3. Glue the stars together according to the norm.

$\rightsquigarrow$