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On the K_0 of a p-adic group

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Abstract. This article deals with various topics related with Grothendieck groups, invariant distributions, parabolic and compact inductions... for a p-adic group G. The main result is a description of the K_0 of the Hecke algebra $\mathcal{H}$ of G in terms of discrete series of Levi subgroups, which has an interesting behavior with regard to parabolic restriction and induction. A similar description – but no more compatible with these parabolic functors – is obtained for $\overline{\mathcal{H}} = \mathcal{H}/[\mathcal{H}, \mathcal{H}]$ and the Hattori rank map gets an easy description in this dictionary.

We follow a beautiful idea of J. Bernstein consisting in comparing two natural filtrations on these objects, one of combinatorial nature and one of topological nature. The combinatorial filtrations are related to the structure of Levi subgroups in G and have counterparts concerning many classical objects of interest as the Grothendieck group of finite length G-modules $R(G)$, the set Ω^{sr} of regular semi-simple conjugacy classes, and the variety $\Theta(G)$ of infinitesimal characters. These filtrations will turn out to be "compatible", in a sense to be specified, with regard to all the classical operations or morphisms between these objects.

Contents

1 Introduction

As already said in the abstract, this article investigates some "level 0 linearizations" of the Hecke algebra of a p-adic group G with regard to its Levi structure. In this "introductory" part, we will begin with a collection of the many results on the subjects (published or unpublished). The main sources are articles from Bernstein ([7], [6], [5]), Kazhdan ([23]), Schneider-Stuhler ([25], see also Bezrukavnikov [8]), Vignéras [27] and Blanc-Brylinsky [9], etc...

The second part deals with "combinatorics" – as named by Bernstein – on parabolic restriction and induction. Assume fixed some set of standard Levi subgroups in G, then for each of the three objects $\mathcal{A} = K_0, \ \overline{H}$ or $\mathcal{R}$ (see abstract) and for all tower $M < L < G$ of standard Levi subgroups the functors of "standard" parabolic restriction and induction induce morphisms $\mathcal{A}(M) \rightleftarrows \mathcal{A}(L)$. Taking up all these morphisms and the relations they fulfill, we can equip these objects with an additional structure, called "Hopf system" (from ideas of Zelevinski), of modules over some quiver. Then, inspired by [6, 5.5] we define the so-called combinatorial filtrations; they are filtrations in the category of Hopf systems. Adding some assumptions on the Hopf system, so that we can use (slight generalization of) the results of [6, 5.5], we can describe the associated graduates of these filtrations as "trivial Hopf systems"– by definition induced from some simpler quiver.

The third part introduces the filtration "à la Harish-Chandra" of the set of regular semi-simple conjugacy classes and investigates its compatibility with the combinatorial filtrations previously defined. We are also lead to give a generalization of a result of Kazhdan which had been expressed for groups with compact center; as a matter of fact, we are faced with a constant problem: in order to use induction on the depth of Levi subgroups, we must get rid of this compactness assumption on the center.

In the fourth part, following ideas of Bernstein, is investigated the K_0 of the p-adic group G, *i.e.* the Grothendieck group of finitely generated projective G-modules. Among the profusion of Bernstein's ideas which will be used, the fundamental one may be that of introducing a topological filtration (the "dévissage" of [10] applied to the variety of infinitesimal characters) on this K_0 and studying it with combinatorial tools. This idea has been spreading through conferences (see [18]) and private communications for a few years. It has lead Bernstein to prove some important properties in K-theory as the injectivity of the so-called "rank map" or the fact that K_0 is generated over $\mathbb{Q}$ by induced modules from compact open subgroups, etc... Here we (try to) give the detailed proofs of these facts. Then we proceed to get an explicit description of $K_0 \otimes \mathbb{Q}$ as explained in the abstract; this is actually a description of the Hopf system structure, hence including the behavior with regard to parabolic induction and restriction.

The fifth part adapts the discussion of part four to the case of $\overline{\mathcal{H}}$. Certainly some results are contained in [18] where the topological filtration is also used, but it seems that we go further. In particular, starting from a categorical definition of $\overline{\mathcal{H}}$ we show the existence of a short exact sequence for $\overline{\mathcal{H}}$ with regard to quotient categories similar to the "localization exact sequence" for K_0 (but unfortunately with heavy assumptions...). Then we get a description of $\overline{\mathcal{H}}(G)$ similar to that of $K_0(G)$, but unfortunately not compatible with induction and restriction: as a Hopf system, we actually only get a description of the graduate module associated to the topological filtration.

The sixth part is an evidence of the author's ignorance. It was intended to provide a proof of the innocent-looking statement of 4.13; the strategy consisted in studying representations of G over "big fields" (actually fields of functions of complex algebraic varieties) by means of direct image in algebraic $\mathcal{K}$-theory. It was redacted before the conference [18], which contained a much shorter argument (although incompletely stated), was revealed to the author by G. Henniart. The interest of this part is to define "discrete" infinitesimal characters for arbitrary fields and to show that they are actually closed points of the variety of discrete $\mathbb{C}$-infinitesimal characters. It is completely independent of the rest of this work.

Acknowledgements I am indebted to J. Bernstein for explaining me some of his ideas. Special thanks are due to Peter Schneider: this work started from a discussion during a stay at Münster and his comments were a constant motivation. Finally I thank my advisor Marie-France Vignéras for her support and for supplying me with the proof of 1.6.

From now on, G is a reductive group over a non-archimedean local field F. We will have soon to assume its characteristic to be zero although certainly in many places it may be avoidable. G-modules are smooth and the coefficient field is $\mathbb{C}$ unless it is specified. We write $Mod(G)$ for the category they form and $Mod_f(G)$ for the category of finitely generated G-modules. We write $\mathcal{H}(G)$ for the convolution algebra of locally constant measures with compact support. We fix a minimal parabolic P_0 with Levi decomposition $M_0.U_0$ and we write $M < G$ for "M is a standard Levi subgroup of G". G^0 will be the subgroup generated by compact elements and $\Psi(G) = Hom_{\mathbb{Z}}(G/G^0, \mathbb{C}^*)$ will be the torus of unramified characters of G. For a compact open subgroup H, we write $\mathcal{H}(G, H)$ for the convolution sub-algebra of $\mathcal{H}(G)$ of bi-H-invariant functions and V^H for the subspace of H-invariants elements in the G-module V. Recall from [7, 3] that there exists a system of neighborhoods of 1_G formed by open compact subgroups H enjoying the property:

$(*)$: The full subcategory

$$Mod_H(G) = \{V \in Mod(G), \quad V^H \text{ generates } V\}$$

is abelian and the functor

$$\begin{aligned} Mod_H(G) &\to \mathcal{H}(G, H) - Mod \\ V &\mapsto V^H \end{aligned}$$

is an equivalence of categories.

1.1 *The objects:* Here are the main heroes

- $\mathcal{K}(G)$: The Grothendieck group of finitely generated projective G-modules. Since the category of G-modules was shown (by Bernstein, see [27, Prop. 37]) to be of finite cohomological dimension, we are allowed to identify $\mathcal{K}(G)$ with the Grothendieck group $\mathcal{K}(Mod_f(G))$ of finitely generated G-modules. If H satisfies condition $(*)$, then $\mathcal{K}(\mathcal{H}(G,H))$ is naturally a direct summand of $\mathcal{K}(G)$ and we have $\mathcal{K}(G) = \varinjlim \mathcal{K}(G,H)$.
- $\overline{\mathcal{H}}(G) := \mathcal{H}/[\mathcal{H},\mathcal{H}] = \mathcal{H}_G$ the G-coinvariants quotient of $\mathcal{H}$. We have $\overline{\mathcal{H}}(G) = \varinjlim \overline{\mathcal{H}}(G,H)$, the limit being taken over those H's satisfying $(*)$. Moreover for such a H, $\overline{\mathcal{H}}(G,H)$ is a direct summand of $\overline{\mathcal{H}}(G)$ (see [24]).
- $\mathcal{R}(G)$: The Grothendieck group of finite length G-modules.

It will be convenient to consider the complex groups $\mathcal{K}_{\mathbb{C}} := \mathcal{K} \otimes_{\mathbb{Z}} \mathbb{C}$ and $\mathcal{R}_{\mathbb{C}} := \mathcal{R} \otimes_{\mathbb{Z}} \mathbb{C}$. They are connected by different morphisms:

1.2 *The relations:*

- The rank map: $Rk : \mathcal{K}(G) \longrightarrow \overline{\mathcal{H}}(G)$ which sends a finitely generated projective P to the image in $\overline{\mathcal{H}}$ of the trace of the idempotent of $\mathcal{M}_n(\mathcal{H})$ defining it as a quotient of $\mathcal{H}^n$. (One can pass through some ring $\mathcal{H}(G,H)$, H satisfying $(*)$ to recover the standard situation of a unital algebra).
- The canonical pairing (up to the choice of a Haar measure):

$$
\begin{array}{rl}
<,>: & \overline{\mathcal{H}}(G) \times \mathcal{R}_{\mathbb{C}}(G) \to \mathbb{C} \\
 & \qquad (f,\pi) \mapsto \Theta_\pi(f)
\end{array}
$$

  where Θ_π is the character of the finite length G-module π.
- The Euler-Poincare map:

$$
\begin{array}{rl}
EP: & \mathcal{R}(G) \to \mathcal{K}(G) \\
 & \qquad \pi \mapsto [\pi \otimes \mathbb{C}[G/G^0]] = \sum_{i=1}^{k} (-1)^i [P_i]
\end{array}
$$

  where we let G act diagonally on the tensor product and

$$
0 \longrightarrow P_k \longrightarrow \cdots \longrightarrow P_0 \longrightarrow \pi \otimes \mathbb{C}[G/G^0] \longrightarrow 0
$$

  is any finitely generated projective resolution of $\pi \otimes \mathbb{C}[G/G^0]$.

1.3 *Alternative description of $\overline{\mathcal{H}}(G)$:* (See [27]) By general theory of rings with units and the property $\overline{\mathcal{H}}(G) = \varinjlim \overline{\mathcal{H}}(G,H)$, we have the following description of $\overline{\mathcal{H}}(G)$: it is the free $\mathbb{C}$-vector space with basis the pairs (P,u) where P is a finitely generated projective G-module and $u \in End_G(P)$, modulo the relations:

- $(P \oplus P', u \oplus u') = (P, u) + (P', u')$
- $(P, u) + (P, v) = (P, u + v)$ and $(P, \lambda.u) = \lambda(P, u)$
- $(P, fg) = (P', gf)$ if $f : P' \longrightarrow P$ and $g : P \longrightarrow P'$

In this context the map Rk is just $P \mapsto (P, 1_P)$ and we have

$$< (P, u), \pi >= Tr(u|Hom_G(P, \pi))$$

In particular when the center of G is compact, we have the following: for any irreducible G-modules π, π':

$$< Rk \circ EP(\pi), \pi' >= \sum_k (-1)^k \, dim(Ext^k(\pi, \pi')).$$

Note that the property of finite cohomological dimension implies that we can forget the word "projective" in the definition above (see 5.2). In particular we may consider the following $\overline{\mathcal{H}}$-theoretic analogue of the map EP above ($\lambda \in \mathbb{C}[G/G^0]$ acts as a G-equivariant endomorphism on $\pi \otimes \mathbb{C}[G/G^0]$ by right multiplication):

$$\begin{aligned} \overline{EP}: \quad \mathcal{R}(G) \otimes \mathbb{C}[G/G^0] &\to \overline{\mathcal{H}}(G) \\ \pi \otimes \lambda &\mapsto [\pi \otimes \mathbb{C}[G/G^0], \lambda] = \textstyle\sum_{i=1}^k (-1)^i [P_i, \lambda_i] \end{aligned}$$

for any endomorphism of finitely generated projective resolutions

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & P_k & \longrightarrow & \cdots\cdots & \longrightarrow & P_0 & \longrightarrow & \pi \otimes \mathbb{C}[G/G^0] & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \lambda_k} & & & & \downarrow{\scriptstyle \lambda_0} & & \downarrow{\scriptstyle \lambda} & & \\ 0 & \longrightarrow & P_k & \longrightarrow & \cdots\cdots & \longrightarrow & P_0 & \longrightarrow & \pi \otimes \mathbb{C}[G/G^0] & \longrightarrow & 0 \, . \end{array}$$

Now we give known properties of these objects and relations.

Theorem 1.4 *(Trace Paley-Wiener, see [6]) The pairing $<,>$ provides a morphism $\overline{\mathcal{H}}(G) \longrightarrow \mathcal{R}_{\mathbb{C}}(G)^*$ whose image is the space of linear forms ϕ on $\mathcal{R}_{\mathbb{C}}(G)$ satisfying:*

i) There exists a compact open subgroup K such that $(\phi, \pi) = 0$ if $\pi^K = 0$.
ii) For each Levi subgroup M and finite length M-module σ, the map

$$\begin{aligned} \Psi(M) &\to \mathbb{C} \\ \chi &\mapsto (\phi, i_M^G(\sigma.\chi)) \end{aligned}$$

is algebraic.

We now discuss the injectivity of the morphism of 1.4. It is obviously equivalent to the following "spectral density property"

(SDP) $\forall f \in \mathcal{H}(G), (\forall \pi \in \mathcal{R}(G), < f, \pi >= 0) \Rightarrow f \in [\mathcal{H}(G), \mathcal{H}(G)]$

This property has a geometric analogue: let Ω^{sr} be the set of conjugacy classes of regular semi-simple elements in G. For $f \in \mathcal{H}(G)$ and $\omega \in \Omega^{sr}$, we will write $\Phi(f, \omega)$ for the orbital integral of f on ω (The same notation will be used for $\overline{f} \in \overline{\mathcal{H}}(G)$ since it doesn't depend on the choice of a representative function f of $\overline{f}$ in $\mathcal{H}(G)$). The Haar measure on ω will be specified in the context but we will mainly be concerned with vanishing of such orbital integrals. We will call "geometrical density property" the following property:

$$\text{(GDP)} \quad \forall f \in \mathcal{H}(G), \ \left(\forall \omega \in \Omega^{sr}, \ \Phi(f, \omega) = 0\right) \Rightarrow f \in [\mathcal{H}(G), \mathcal{H}(G)]$$

Theorem 1.5 *i) Assume Char$F = 0$, then (GDP) holds for G.*
ii) Assume Char$F = 0$ or G is split, then (SDP) holds for G.

The standard proof of i) is [20, Thm 10] although it is given there only for Lie Algebras. As for ii) it is proved via a Global Trace Formula argument in [23, Appendix] for characteristic 0 and via Close Fields in [24] with this split assumption that is certainly not necessary.

Now we turn to the properties of *Rk*; the main result is the so-called abstract Selberg principle of Blanc and Brylinsky:

Theorem 1.6 *The image of Rk is contained in the set $\overline{\mathcal{H}}_c(G)$of $f \in \overline{\mathcal{H}}(G)$ such that $\Phi(f, \omega) = 0$ for any conjugacy class ω of a regular non-compact element. It is equal to this set whenever (GDP) holds for G.*

Proof: The inclusion *im Rk* $\subset \overline{\mathcal{H}}_c(G)$ is the most difficult assertion here and is proven in [9]. To show the reversal inclusion, we give an argument of Vignéras (see [28]): let $\mathcal{L}$ be a set of representatives of G-conjugacy classes of maximal open compact subgroups, then in the following commutative diagram,

$$\begin{tikzcd}
\mathcal{K}_{\mathbb{C}}(G) \arrow[r, "Rk"] & \overline{\mathcal{H}}(G) \\
\bigoplus_{L \in \mathcal{L}} \mathcal{K}_{\mathbb{C}}(L) \arrow[u, "\sum ind_L^G"] \arrow[r, "Rk"] & \bigoplus_{L \in \mathcal{L}} \overline{\mathcal{H}}(L) \arrow[u, "\sum \mathrm{ext}_L^G"']
\end{tikzcd}$$

where ext_L^G is the extension by zero, the bottom map is an isomorphism. Hence it is sufficient to show that given any $f \in \mathcal{H}_c(G)$, one can construct a sum $f_c = \sum_{L \in \mathcal{L}} f_L$ with $f_L \in \mathcal{C}^\infty(L)$ such that $\Phi(f, \omega) = \Phi(f_c, \omega)$ for any regular semi-simple conjugacy class and then apply (GDP). To do this, let Ω_c^{sr} be the set of compact regular semi-simple elements. We can cover the set *Supp* $f \cap \Omega_c^{sr}$ by a finite disjoint union of sets of the form gYg^{-1} with $g \in G$ and such that Y is open in a certain $L \in \mathcal{L}$. Then define

$$f_L(x) = \sum_{\substack{gYg^{-1} \\ x \in Y \subset L}} f(g^{-1}xg).$$

$\square$

As a consequence, we see that the image of $\mathcal{K}_{\mathbb{C}}(G)$ in $\overline{\mathcal{H}}(G)$ and that of the subspace of $\mathcal{K}_{\mathbb{C}}(G)$ generated by compact induced representations coincide. This fact will be useful later. It will also be shown later on that Rk is injective.

1.7 *Parabolic induction:* In this paragraph, the symbol $\mathcal{A}$ may stand for any of the symbols $\overline{\mathcal{H}}, \mathcal{K}, \mathcal{R}$. Given a standard Levi subgroup M, one can use the exact normalized functors of parabolic induction (resp. restriction) to get morphisms $\mathcal{A}(M) \xrightarrow{i_M^G} \mathcal{A}(G)$ (resp. $\mathcal{A}(G) \xrightarrow{r_G^M} \mathcal{A}(M)$). Indeed, it is known that these functors respect the following properties of representations:

- *To be Finitely generated:* Immediate for r_G^M and [7, 3.11] for i_M^G.
- *To be Projective:* Frobenius reciprocity for r_G^M and Bernstein's "second adjunction" theorem in [4, Main Theorem] for i_M^G.
- *To be of finite length:* Jacquet's first lemma for r_G^M and [2, 2.8] for i_M^G.

Similarly we will write $\overline{i_M^G}$ and $\overline{r_G^M}$ the morphisms obtained from induction and restriction along the opposite parabolic $M.\overline{P_0}$ of $M.P_0$. Then Frobenius reciprocity and Bernstein's second ajunction theorem yield the adjunction formulas

$$(1.8) \qquad \begin{aligned} &< r_G^M(P,u), \sigma >_M =< (P,u), i_M^G(\sigma) >_G \text{ and} \\ &< i_M^G(Q,u), \pi >_G =< (Q,u), \overline{r_G^M}(\pi) >_M \end{aligned}$$

From [6] we have the following equality

$$(1.9) \qquad r_G^N \circ i_M^G = \sum_{w \in W_G^{NM}} i_{N\cap M^w}^N \circ w \circ r_M^{M\cap N^{w^{-1}}}$$

where W_G^{NM} is the set of representatives of $W_N \backslash W_G / W_M$ of minimal length (see [2, 2.11]).

The definition of the morphisms i and r for $\mathcal{A} = \overline{\mathcal{H}}$ implicitly uses the alternative description of $\overline{\mathcal{H}}$. But we can formulate r_G^M in terms of functions using the adjunction property, 1.3 and Van Dijk's formula ([26]):

Proposition 1.10 *Let $M < G$ and assume (SDP) holds for M. If $\overline{f} \in \overline{\mathcal{H}}(G)$ has representative $f \in \mathcal{H}(G)$, then the smooth function on M defined by*

$$r_G^M(f)(m) = \delta_P^{\frac{1}{2}}(m) \int_{Rad_{(M.P_0)}} \int_K f(kmnk^{-1})dk.dn$$

where K is a maximal open compact subgroup, is a representative of $r_G^M \overline{f} \in \overline{\mathcal{H}}(M)$ in $\mathcal{H}(M)$.

2 Combinatorics and filtrations

2.1 *Hopf systems*[1]*:* As was already pointed out in the last paragraph, there are common features between the objects $\mathcal{R}$, $\mathcal{K}$ and $\overline{\mathcal{H}}$ with respect to the structure of Levi subgroups in the reductive group G. We can make this formal with the following definition (which may appear somewhat heavy):

Definition 2.2 *We will call Hopf-system on G any data* $(\mathcal{A}(.), r, i, w)$ *consisting of a family of abelian groups* $(\mathcal{A}(N))_{N<G}$ *together with morphisms*

- $i_N^M : \mathcal{A}(N) \longrightarrow \mathcal{A}(M)$ *and* $r_M^N : \mathcal{A}(M) \longrightarrow \mathcal{A}(N)$ *for any* $N < M$ *(induction and restriction).*
- $w : \mathcal{A}(N) \longrightarrow \mathcal{A}(N^w)$ *for any* $w \in W_G$ *such that* $N^w = w(N) < G$ *(conjugation).*

satisfying the properties:

i) $i_N^M \circ i_L^N = i_L^M$ *and* $r_N^L \circ r_M^N = r_M^L$ *for any triples* $L < N < M$.

ii) $r_M^L \circ i_N^M = \sum_{w \in W_M^{NL}} i_{L\cap N^w}^L \circ w \circ r_N^{N\cap L^{w^{-1}}}$ *if* $L, N < M < G$.

iii) $w' \circ w = (w'w) : \mathcal{A}(N) \longrightarrow \mathcal{A}(N^{w'w})$ *if* $w(N) < G$ *and* $w'w(N) < G$.

iv) Any $w \in W_N$ *acts trivially on* $\mathcal{A}(N)$*. Moreover if* $w(N) < G$ *and* w *has minimal length in* wW_N *then (see 2.12 i))* $w \circ i_M^N = i_{w(M)}^{w(N)} \circ w$ *and* $w \circ r_N^M = r_{w(N)}^{w(M)} \circ w$.

A morphism of Hopf systems is naturally defined as a family of morphisms $\mathcal{A}(N) \longrightarrow \mathcal{A}'(N)$ *which commute with induction, restriction and conjugation.*

Remarks:

- Our main examples will be of course $\mathcal{R}$, $\mathcal{K}$ and $\overline{\mathcal{H}}$; the only property that wasn't yet checked is iv). Note that $w \in W_G$ has minimal length in wW_N if and only if $w(N \cap P_0) \subset P_0$ and in our cases of interest, i_M^N and r_N^M are induced by usual parabolic induction and restriction functors, more precisely , if $V \in Mod(M)$

  $$w(i_M^N(V)) = w(Ind_{M.(P_0\cap N)}^N(\delta.V)) \simeq Ind_{w(M).(P_0\cap w(N))}^{w(N)}(\delta^w V^w) = i_{w(M)}^{w(N)}(V^w).$$

  The same remark works for restriction hence the property iv) is fulfilled by $\mathcal{R}$, $\mathcal{K}$ and $\overline{\mathcal{H}}$.
- Note that Rk is an example of morphism of Hopf systems.
- The definition above enables to speak of the category of Hopf systems on G. It is quite clear that it is an abelian category. See 2.13 for a more "modern" interpretation in terms of quivers.

[1] This terminology comes from the works of Bernstein and Zelevinski

2.3 *Filtrations:* Let $\mathcal{A}$ be a Hopf system on G, then the family of morphisms r and i yields the following filtrations on $\mathcal{A}$. For a Levi subgroup M put $d(M) = dim\,\Psi(M)$. Then we define

- $\mathcal{A}_i(G) = \sum_{d(M)>i} im\ i_M^G$ (decreasing)
- $\mathcal{A}^i(G) = \bigcap_{d(M)>i} ker\ r_G^M$ (increasing)

We also write $\mathcal{A}_{d(G)}(G) = \mathcal{A}_I(G)$ and $\mathcal{A}^R(G) = \mathcal{A}^{d(G)}(G)$. These filtrations have the following easy properties:

- They are respected by any morphism of Hopf system.
- $i_M^G(\mathcal{A}^i(M)) \subset \mathcal{A}^i(G)$ and all other combinations with lower or upper filtrations, induction or restriction hold true. The proof of the non-trivial cases uses property 1.9. Hence these are actually filtrations in the category of Hopf systems on G.

2.4 *Combinatorics:* Strengthening a little the discussion in [6, 5.5], we can give a more handy definition of the filtrations above. We first define for each $M < G$

$$T_M = i_M^G \circ r_G^M \text{ and } P_M = |\mathcal{N}_{W_G}(M)/W_M|$$

where W_G stands for the Weil group of G. Then we have to choose (and fix throughout the paper) for each d an ordering $\mathfrak{o}_d = (M_0, M_1, \cdots M_{n_d})$ of the set of Levi subgroups of given depth $d(M) = d$, and put $T_d = T_d(\mathfrak{o}_d) := \prod_{i=0}^{n_d}(T_{M_i} - P_{M_i})$ then define

$$A_d = A_d(\mathfrak{o}) := T_{d(M_0)-1} \circ \cdots \circ T_d \text{ and } A^d = A^d(\mathfrak{o}) = T_d \circ \cdots \circ T_{d(M_0)-1}$$

Note that all these morphisms respect both filtrations.

Proposition 2.5 *Here, $\mathcal{A}$ is a $\mathbb{Q}$-Hopf system on G (i.e. $\mathcal{A}(N)$ is a $\mathbb{Q}$-vector space for each $N < G$):*

i) *Assume $i_{w(M)}^G \circ w = i_M^G$ for any $M < G$, $w \in W_G$ such that $w(M) < G$. Then there exist an integer P_d and rationals $c_d(M) \in \frac{1}{P_d}\mathbb{Z}$ such that*

$$A_d = P_d(1 - \sum_{d(M)>d} c_d(M)T_M)$$

Moreover the endomorphism $\pi_d := \frac{1}{P_d}A_d$ is a projector and $\mathcal{A}_d(G) = ker\ \pi_d$ (note that im π_d a priori depends on the chosen ordering).

ii) *Assume $r_G^{w(M)} = w \circ r_G^M$ for any $M < G$, $w \in W_G$ such that $w(M) < G$. Then there exist an integer P^d and rationals $c^d(M) \in \frac{1}{P^d}\mathbb{Z}$ such that*

$$A^d = P^d(1 - \sum_{d(M)>d} c^d(M)T_M)$$

Moreover the endomorphism $\pi^d := \frac{1}{P^d}A^d$ is a projector and $\mathcal{A}^d(G) = im\ \pi^d$.

iii) Assume the two conditions above are satisfied, then $A_d = A^d$ *and* $\mathcal{A}(G) = \mathcal{A}^d \oplus \mathcal{A}_d$.

Proof: i): From the hypothesis, we have the following multiplication rule concerning the T_M's (see [6, 5.4]):

$$T_N T_M = \sum_{w \in W_G^{NM}} T_{M_w}$$

where $M_w = M \cap N^w$. This shows the existence of P_d and the $c_d(M)$'s, hence the implication $A_d X = 0 \Rightarrow X \in \mathcal{A}_d$. Moreover, the discussion in [6, 5.5] shows that A_d annihilates $\mathcal{A}_d$ (Note the notations are different from that of [6]). Hence

$$A_d \circ A_d = P_d.A_d - A_d \circ (\sum_{d(M)>d} c_d(M) T_M)) = P_d.A_d$$

and the projector property follows.

ii): The proof is exactly "dual" to that of i). In particular we get the multiplication rule

$$T_N T_M = \sum_{w \in W_G^{NM}} T_{N_w}$$

and the equality $im\ A^d = \mathcal{A}^d$.

From the latter and calculating $A_d \circ A^d$ by two different ways, we find $P^d.A_d = P_d A^d$. From the definition, the P's are easily seen to coincide and iii) follows. □

Proposition 2.6 *i)* $\mathcal{R}_{\mathbb{C}}(G)$ *fulfills condition i) in 2.5.*
ii) If (SDP) holds for any $M < G$*, then* $\overline{\mathcal{H}}(G)$ *fulfills condition ii) in 2.5.*
iii) If (GDP) holds for any $M < G$*, then* $\mathcal{K}_{\mathbb{C}}(G)$ *fulfills iii) in 2.5.*

Proof: i) is well known for complex representations, a proof is given in [6, 5.4]. ii) follows immediately by the adjunction property 1.8. iii) is much deeper and will be shown later. It should hold only under (SDP) assumption. □

Remarks:

- From now on, if a Hopf system is said to fulfill condition i), ii) or iii) of 2.5, it should be understood that it satisfies this condition for any $N < G$ (and not only G): in particular there are analoguous projectors π_N^d or π_d^N defined just as for G: note that in general they depend *a priori* on the choice of an ordering of the set of d-deep Levi subgroups in N. Anyway, if we are in the case of condition iii), then π_N^d and π_d^N coincide and are actually "canonically" defined by the properties $\mathcal{A}^d(N) = im\ \pi_N^d$ and $\mathcal{A}_d(N) = ker\ \pi_d^N$. In this case we hence have equalities

$$\pi_d^G \circ i_N^G(X) = i_N^G \circ \pi_d^N(X) \text{ and } r_G^N \circ \pi_G^d(Y) = \pi_N^d \circ r_G^N(Y)$$

so that we may view the system $(\pi^d_N)_{N<G}$ as an endomorphism of Hopf system.

- As for the filtration $\mathcal{R}^\bullet_{\mathbb{C}}(G)$, in a previous version of these notes, I wrote that

$$\mathcal{R}^d_{\mathbb{C}}(G) = \bigoplus_{d(\Theta_{\mathfrak{s}})\leqslant d} \mathcal{R}_{\mathbb{C}}(G, \Theta_{\mathfrak{s}})$$

(With the notations of 4.1). Thinking more carefully to this, it didn't appear so evident to me. In fact I think this can be shown for $GL(n)$, using techniques (Zelevinski's theory of segments, or types and representation theory of Hecke algebras) that won't fit in a more general case but I don't see any general argument.

2.7 *Trivial Hopf systems:* There is an easy way to construct Hopf systems on G. Let us first define the following object for any $M, L < N < G$:

$$W_N(M, L) := \{w \in W_N, \quad w(M) < L\}$$

Now, assume given for each $M < G$ an abelian group A_M, and a system of isomorphisms $A_M \xrightarrow{w} \mathcal{A}_{M^w}$ (if $w(M) < G$) compatible with multiplication in W_G and such that W_M acts trivially on A_M (see also 2.13). Then we may put for any $N < G$

$$\mathcal{A}(N) := \Big(\bigoplus_{M<N} A_M\Big)/\sim_N$$

where $\sim_N$ is the equivalence relation identifying (M, x) and (M^w, x^w) for $M < N$, $x \in A_M$ and $w \in W_N$ such that $w(M) < N$. Then we define induction and restriction by:

$$i^G_N(M, x) := (M, x) \text{ and } r^N_G(M, x) := \sum_{w\in W_G(M,N)} \frac{1}{|W_N(M^w, N)|}(M^w, x^w)$$

(the reader will check that they are well defined). As for conjugation morphisms, suppose $w(N) < G$ and write $\overline{w}$ for the minimal length element in wW_N, then from 2.12, $\overline{w}$ carries standard Levi subgroups of N to standard Levi subgroups of $w(N)$ and thus induces our desired morphism $w : \mathcal{A}(N) \longrightarrow \mathcal{A}(N^w)$. Now we claim:

Lemma 2.8 *The system thus obtained is a Hopf system and will be noted* $\mathfrak{H}(M \mapsto A_M)$.

Proof: The only nontrivial things to be checked are the transitivity of restriction and the composition $r \circ i$ (the compatibility of conjugation with muliplication in W_G is insured by 2.12 ii)). For the first one, we have

$$\begin{aligned} r_N^L \circ r_G^N(M,x) &= \sum_{w\in W_G(M,N)} \frac{1}{|W_N(M^w,N)|} \sum_{y\in W_N(M^w,L)} \frac{1}{|W_L(M^{yw},L)|}(M^{yw},x^{yw}) \\ &= \sum_{z\in W_G(M,L)} \frac{1}{|W_L(M^z,L)|} \sum_{y\in W_N(M^z,N)} \frac{1}{|W_N(M^{y^{-1}z},N)|}(M^z,x^z) \\ &= \sum_{z\in W_G(M,L)} \frac{1}{|W_L(M^z,L)|}(M^z,x^z) \end{aligned}$$

where the second line is obtained by resummation. As for the composition $r\circ i$, fix Levi subgroups $M<N<G$ and $L<G$ and $x\in A_M$, on one hand we have

$$r_G^L \circ i_N^G(M,x) = \sum_{w\in W_G(M,L)} \frac{1}{|W_L(M^w,L)|}(M^w,x^w),$$

on the other hand, putting $N_y = N\cap L^{y^{-1}}$, we have

$$\sum_{y\in W_G^{NL}} i^L_{L\cap N^y}\circ y\circ r_N^{N_y}(M,x) = \sum_{y\in W_G^{NL}} \sum_{z\in W_N(M,N_y)} \frac{1}{|W_{N_y}(M^z,N_y)|}(M^{yz},x^{yz})$$

Now since W_L-conjugates are identified in $\mathcal{A}(L)$, the right hand side above is also equal to

$$\sum_{y\in W_G^{NL}} \sum_{z\in W_N(M,N_y)} \sum_{w_l\in W_L(M^{yz},L)} \frac{1}{|W_L(M^{yz},L)|}\frac{1}{|W_{N_y}(M^z,N_y)|}(M^{w_lyz},x^{w_lyz})$$

Hence the claim follows from the following fact:

Fact 2.9 *Every $w\in W_G(M,L)$ can be decomposed into a product $w=w_lyz$ with $y\in W_G^{NL}$, $z\in W_N(M,N_y)$ and $w_l\in W_L(M^{yz},L)$. Moreover, any other such decomposition $w=w_l'y'z'$ verifies*

- $y'=y$
- *There exists some $t\in W_{N_y}(M^z,N_y)$ such that $z'=tz$ and $w_l'=w_lyt^{-1}y^{-1}$.*

Proof: (of the fact) Let l be the length function on the Weyl group W_G. Using [2, 6.3] and [11, ch.IV, ex 1.3], we know that for any pair of Levi subgroups $L,N<G$, each $x\in W_G$ decomposes as a product $x=x_lyx_n$ with $x_l\in W_L$, $y\in W_G^{NL}$, $x_n\in W_N$ and $l(x)=l(x_l)+l(y)+l(x_n)$. So start applying this to our w with respect to the pair L,M, hence $w=w_ly_1w_m$. From the properties of W_G^{ML} we get $y_1w_m(M)=y_1(M)<L$, hence $w_l^{-1}\in W_L(M^w,L)$. Now apply again the decomposition property to y_1 with respect to the pair L,M: we get $y_1=y_lyy_n$, but the length equality above together with the fact that

y_1 is of least length in $W_L y_1$ imply that $y_l = 1$. Hence putting $z = y_n w_m$, we get a decomposition $w = w_l yz$ satisfying $yz(M) = w_l^{-1} w(M) < L \cap N^y$, hence $z(M) < M \cap L^{y^{-1}}$ and $w_l \in W_L(M^{yz}, L)$ as required.

Now the remaining assertion follows from the fact that W_G^{LN} is a set of representatives of the double classes $W_L x W_N$ in W_G and from the equality $W_N \cap W_L^{y^{-1}} = W_{N_y}$ (see [12, 2.7,pp 64-65]). □

□

Remark: It is a kind of "trivial" Hopf system. It may be checked that it fulfills the property iii) of 2.5.

As was noticed before, the terms of both filtrations $\mathcal{A}^*$ and $\mathcal{A}_*$ are again Hopf systems so that we may attach to any Hopf system $\mathcal{A}$ the associated graduate $gr^* \mathcal{A} := \bigoplus_d \mathcal{A}^d / \mathcal{A}^{d-1}$ and $gr_* \mathcal{A} = \bigoplus_d \mathcal{A}_{d-1}/\mathcal{A}_d$, which are again Hopf systems. Then we have the following fact:

Proposition 2.10 *Let $\mathcal{A}$ be a $\mathbb{Q}$-Hopf system,*

i) If $\mathcal{A}$ fulfills 2.5 i), then $gr_*(\mathcal{A}) \simeq \mathfrak{H}\,(M \mapsto \mathcal{A}(M)/\mathcal{A}_I(M))$.
ii) If $\mathcal{A}$ fulfills 2.5 ii) then $gr^*(\mathcal{A}) \simeq \mathfrak{H}\left(M \mapsto \mathcal{A}^R(M)\right)$.
iii) If $\mathcal{A}$ satisfies both conditions, then $\mathcal{A} \simeq gr^*(\mathcal{A}) \simeq gr_*(\mathcal{A})$.

Proof: We begin proving point ii). We first notice that if $x \in \mathcal{A}^{d(M)}(M)$ and $w(M) < G$, then $y = i_M^G(x) - i_{w(M)}^G \circ w(x) \in \mathcal{A}^{d(M)-1}(G)$: as a matter of fact, if $N < G$ and $d(N) = d(M)$, then

$$r_G^N y = \sum_{y \in W_G(M,N)/W_M} x^y - \sum_{z \in W_G(M^w,N)/W_{M^w}} x^{zw} = 0.$$

Hence the following map:

$$\begin{aligned} \alpha_G : \quad \left(\bigoplus_{M<G} \mathcal{A}^{d(M)}(M)\right) &\to gr^* \mathcal{A}(G) \\ (M, x) &\mapsto i_M^G(x) \end{aligned}$$

induces a well-defined map $(\bigoplus \mathcal{A}^{d(M)}(M))_{/\sim} \longrightarrow gr^* \mathcal{K}_{\mathbb{C}}(G)$ still denoted by α_G.

Conversely, note that the morphism π_G^d defined in 2.5 (see also the remark below 2.6) induces the canonical projection $gr^* \mathcal{A}(G) \longrightarrow \sum_{i \leqslant d} gr^i \mathcal{A}(G)$ and define

$$\begin{aligned} \beta_G : \quad gr^* \mathcal{A}(G) &\to \left(\bigoplus_{M<G} \mathcal{A}^{d(M)}(M)\right)_{/\sim} \\ X &\mapsto \sum_{M<G} \frac{|W_M|}{|W_G(M,G)|} \left(r_G^M \circ \pi_G^{d(M)}(X)\right) \end{aligned}$$

Then an easy computation shows that $\beta_G \circ \alpha_G = \mathrm{Id}$, whereas the following facts:

- $\pi_G^d - \pi_G^{d-1}$ is the canonical projector $gr^* \mathcal{A}(G) \longrightarrow gr^d \mathcal{A}(G)$.
- $X = \sum_d (\pi_G^d - \pi_G^{d-1})(X)$
- $gr^d \mathcal{A}(G) = \bigoplus_{d(M)=d} ker\,(T_M - P_M)$ (M in a set of representatives of association classes): this is a corollary of the proof of 2.5 ii).

show that $\alpha_G \circ \beta_G = \mathrm{Id}$. Now we have to check compatibility with restriction and induction: for the latter this is quite easy since for $M < N < G$ and $x \in \mathcal{A}^R(M)$ we have

$$\alpha_G \circ i_N^G(M, x) = \alpha_G(M, x) = i_M^G(x) = i_N^G \circ \alpha_N(M, x)$$

As for restriction, first note that $r_G^M \circ \pi_G^{d(M)} = \pi_M^{d(M)} \circ r_G^M$ on $gr^* \mathcal{A}(G)$, hence

$$\beta_N \circ r_G^N(X) = \sum_{M<N} \frac{|W_M|}{|W_N(M, N)|} \pi_M^{d(M)} \circ r_G^M(X)$$

whereas

$$\begin{aligned} r_G^N \circ \beta_G(X) &= \sum_{M<G} \frac{|W_M|}{|W_G(M, G)|} \left(\sum_{w \in W_G(M,N)} \frac{1}{|W_N(M^w, N)|} w\big(\pi_M^{d(M)} \circ r_G^M(X)\big) \right) \\ &= \sum_{M<G} \frac{|W_M|}{|W_G(M, G)|} \left(\sum_{w \in W_G(M,N)} \frac{1}{|W_N(M^w, N)|} \pi_{M^w}^{d(M)} \circ r_G^{M^w}(X) \right) \\ &= \sum_{M<N} \frac{|W_M|}{|W_N(M, N)|} \left(\sum_{y \in W_G(M,G)} \frac{1}{|W_G(M, G)|} \pi_M^{d(M)} \circ r_G^M(X) \right) \end{aligned}$$

(We get the second line thanks to property ii) of 2.5 and the third one by resummation.) Hence we get the compatibility required. It remains to prove compatibility with conjugation, so choose $M < N$ and $w \in W_G$ such that $w(N) < N$, assume moreover that w has minimal length in wW_N, so that $w(M) < w(N)$ by 2.12 i), we have for all $x \in \mathcal{A}^{d(M)}(M)$:

$$w \circ \alpha_N(M, x) = w\big(i_M^N x\big) = i_{M^w}^{N^w} x^w = \alpha_{N^w}(M^w, x^w)$$

from axiom iv) in the definition of Hopf systems.

The proof of point i) is parallel to the foregoing one and will be omitted. As for point iii), note first that i), ii) and the isomorphism $\mathcal{A}^{d(M)}(M) \simeq \mathcal{A}(M)/\mathcal{A}_{d(M)}(M)$ provided by the direct sum decomposition of 2.5 iii) imply that $gr^* \mathcal{A} \simeq gr_* \mathcal{A}$, hence it remains only to show that $\mathcal{A} \simeq gr^*(\mathcal{A})$. For this we can use the remark below 2.6 which asserts that for each d, the collection $(\pi_N^d)_{N<G}$ provides an endomorphism noted π^d of the Hopf system $\mathcal{A}$, hence from the multiplication rule $\pi^{d'} \circ \pi^d = \pi^d \circ \pi^{d'} = \pi^d$ if $d \leqslant d'$, we get formally $\mathcal{A} = \bigoplus_d im\,(\pi^d - \pi^{d-1}) \simeq gr^* \mathcal{A}$. □

2.11 *About minimal length representatives:* As was already pointed out and used, the length function on the Weyl group W_G enables to have nice representatives in right, left or double classes modulo a subgroup W_N for $N < G$. As an example, recall the definition of $W_G(M,L)$ in 2.7 for $M, L < G$, and set

$$\overline{W}_G(M,L) := \{w \in W_G(M,L), \quad w \text{ has minimal length in } wW_M\},$$

we record here the following fact:

Fact 2.12 *i) If $w \in \overline{W}_G(N,G)$ and $M < N$, then $w(M) < w(N)$.*
ii) The multiplication in W_G induces a well-defined map $\overline{W}_G(N,L) \times \overline{W}_G(M,N) \longrightarrow \overline{W}_G(M,L)$.

Proof: Just observe that if $w \in \overline{W}_G(N,G)$, then $w \in W_G^{NN^w}$, hence in particular for any $M < N$, $w \in W_G^{MN^w}$ so that from [2, 2.11 ii)], $w(M) \cap w(N) = w(M) < w(N)$, hence i). As for ii), recall that w has minimal length in wW_N if and only if $w(N \cap P_0) \subset P_0$ ([2, 2.11]), hence taking $(y,z) \in \overline{W}_G(N,L) \times \overline{W}_G(M,N)$, we get $yz(M \cap P_0) \subset y(N \cap P_0) \subset P_0$ and by i), $M^z < N \Rightarrow M^{yz} < N^y < L$ hence we get ii). $\square$

2.13 *Quivers:* The remark below the definition of Hopf systems claimed that the latters form an abelian category. Actually we can formalize this a little more and define a quiver with relations $\mathcal{L}$ such that the category of Hopf systems identifies with the category of $\mathbb{Z}\mathcal{L}$-modules (this has only an aesthetical interest and this paragraph can be completely omitted). Indeed, define the set of vertices of $\mathcal{L}$ as the set of standard Levi subgroups, assign to each $w \in \overline{W}_G(M,N)$ an arrow $w_M^N : M \to N$ and an arrow $\overline{w}_N^M : N \to M$ and require the relations (using fact 2.12 ii))

i) $v_N^L \circ w_M^N = (vw)_M^L$ and $\overline{w}_N^M \circ \overline{v}_L^N = \overline{vw}_L^M$ for any $M, N, L < G$ and $(w,v) \in \overline{W}_G(M,N) \times \overline{W}_G(N,L)$.

ii) $\overline{1}_M^L \circ 1_N^M = \sum_{w \in W_M^{NL}} w^L_{N \cap L^{w^{-1}}} \circ \overline{1}_N^{N \cap L^{w^{-1}}}$ for any $L, N < M$.

iii) $\overline{w}_{M^w}^M = (w^{-1})_{M^w}^M$ for any $w \in \overline{W}_G(M,G)$.

In the dictionnary between Hopf systems and modules on $\mathcal{L}$, w_M^N corresponds to $i_{w(M)}^N \circ w$ and $\overline{w}_N^M$ corresponds to $w^{-1} \circ r_N^{w(M)}$.

Now define a quiver $\mathcal{Q}$ by keeping the same set of vertices as that of $\mathcal{L}$ and deleting each arrow $M \longrightarrow N$ such that M is not associated to N. Keep the same relations as above (but note the second one becomes empty): we get a quiver whose connected components are association classes of Levi subgroups and whose representations are exactly the families $M \mapsto A_M$ used in 2.7 to construct "trivial" Hopf systems. Now let $\mathcal{P}_{\mathcal{L}}$ and $\mathcal{P}_{\mathcal{Q}}$ be the quotient-path algebras associated to these quivers with relations; by definition we have a morphism $\mathcal{P}_{\mathcal{Q}} \longrightarrow \mathcal{P}_{\mathcal{L}}$. Then it can be checked that our recipe to get "trivial" Hopf system is just induction, that is:

$$\mathfrak{H}(M \mapsto A_M) \simeq \mathcal{P}_{\mathcal{L}} \otimes_{\mathcal{P}_{\mathcal{Q}}} (M \mapsto A_M).$$

3 Some harmonical analysis

3.1 *A filtration on the set of conjugacy classes:* Recall Ω^{sr} is the set of semi-simple regular conjugacy classes in G. For any integer d we define the increasing filtration

$$\Omega^{sr}_d = \{\omega \in \Omega^{sr},\ \forall M < G,\ \ d(M) > d \Rightarrow \omega \cap M = \emptyset\}$$

Note that $\Omega^{sr}_{d(G)}$ is the set of elliptic conjugacy classes and will also be denoted by Ω^{sr}_{ell}.

The following result shows that the filtrations $\overline{\mathcal{H}}^{\bullet}$ and $\mathcal{R}_{\mathbb{C}\bullet}$ are "compatible" in a certain sense with the pairing $<,>$ and relate them with the filtration on conjugacy classes above.

Theorem 3.2 *Writing $E^{\perp}$ for the orthogonal of E w.r.t the pairing $<,>$, we have*

i) *Adjunction property: we can choose orderings $\mathfrak{o}$ and $\mathfrak{o}'$ of Levi subgroups of G in 2.5 such that $A_d = A_d(\mathfrak{o})$ and $A^d = A^d(\mathfrak{o}')$ be adjoint w.r.t the pairing $<,>$:*

$$\forall (f, x) \in \overline{\mathcal{H}}(G) \times \mathcal{R}_{\mathbb{C}}(G),\ \ < A^d f, x >=< f, A_d x > .$$

ii) *$\overline{\mathcal{H}}^d(G)^{\perp} = \mathcal{R}_{\mathbb{C}d}(G)$ and $\mathcal{R}_{\mathbb{C}d}(G)^{\perp} = \overline{\mathcal{H}}^d(G)$ if (SDP) holds for any $M < G$.*

iii) *$\overline{\mathcal{H}}^d(G) = \{f \in \overline{\mathcal{H}}(G), \forall \omega \in \Omega^{sr} \setminus \Omega^{sr}_d,\ \Phi(f, \omega) = 0\}$ if (GDP) holds for any $M < G$.*

Proof: For i) choose first an arbitrary ordering $\mathfrak{o}_d = (M_0, M_1, \cdots M_{n_d})$ of d-deep Levi subgroups and put $\mathfrak{o}'_d = (w_0(M_0), w_0(M_1), \cdots, w_0(M_{n_d}))$ where w_0 stands for the longest element in W_G. Now put $\overline{T_M} = \overline{i^G_M} \circ \overline{r^M_G}$; we have

$$\overline{T_M} = i^G_{w_0(M)} \circ w_0 \circ {w_0}^{-1} \circ r^{w_0(M)}_G = T_{w_0(M)}$$

Hence from the adjunction formulas in 1.8 and the fact that i^G_M and $\overline{i^G_M}$ coincide on $\mathcal{R}(M)$, we see that $\overline{T_M}$ and T_M – hence $T_d(\mathfrak{o}_d)$ and $T_d(\mathfrak{o}'_d)$ – are adjoint w.r.t the pairing $<,>$ hence we get i). As a consequence, ii) follows from the projector properties of 2.5 i) and ii) which hold true since $\mathcal{R}_{\mathbb{C}}(G)$ and $\overline{\mathcal{H}}(G)$ fulfill the conditions respectively required there.

As for iii) we will need the following well known result: suppose $x \in G^{sr} \cap M$ for $M < G$. Then $x \in M^{sr}$, and writing $\omega_M(x)$ (resp. $\omega_G(x)$) for the conjugacy class of x in M (resp. G), we can normalize suitably the Haar measures on ω_G and ω_M to have:

$$\forall f \in \overline{\mathcal{H}}(G),\ \ \Phi(f, \omega_G(x)) = \Phi\big(r^M_G f, \omega_M(x)\big)$$

From this, the inclusion $\subset$ in iii) is clear. For the other inclusion $\supset$, suppose f is in the RHS of the equality of iii) and fix $M < G$ such that $d(M) = d$.

It is clear that for any $x \in M \cap G^{sr}$ we have from the above formula $\Phi(r_G^M f, \omega_M(x)) = 0$. In order to get $r_G^M f = 0$ we have to prove the same vanishing for any $x \in M^{sr}$ and then use (GDP). So choose such a x and also a Cartan subgroup Γ containing x. It is known (cf. [26]) that the function defined on $\Gamma \cap M^{sr}$ by

$$\gamma \mapsto |D_M(\gamma)|^{\frac{1}{2}} \int_{M/\Gamma} (r_G^M f)(m\gamma m^{-1}) dm^*$$

where dm^* is some invariant measure on M/Γ is locally constant on $\Gamma \cap M^{sr}$. But since it is zero on $\Gamma \cap G^{sr}$ and since the latter is dense in $\Gamma \cap M^{sr}$, this function is actually zero. Hence $\Phi(r_G^M f, \omega_M(x)) = 0$ for any $x \in M^{sr}$ and $r_G^M f = 0$. □

Note that in the case where G has compact center and $d = d(G) = 0$, this result specializes to a well known result of Kazhdan ([23]). Actually in this case, one has more precise results. We will be mainly interested in the following one which describes the composite $Rk \circ EP$ (recall that as in 2.3, $\mathcal{A}^R(G) = \mathcal{A}^{d(G)}(G)$ and $\mathcal{A}_I(G) = \mathcal{A}_{d(G)}(G)$):

Theorem 3.3 *(Kazhdan-Schneider-Stuhler) Assume G has compact center and $Char(F) = 0$. Then $Rk \circ EP$ induces an isomorphism:*

$$Rk \circ EP : \mathcal{R}_{\mathbb{C}}(G)/\mathcal{R}_{\mathbb{C}I}(G) \xrightarrow{\sim} \overline{\mathcal{H}}^R(G)$$

We will need in the next section the following generalization of this result; we also assume the characteristic of F to be zero but it may be of course replaced by the assumption that (GDP) holds for any $M < G$.

Theorem 3.4 *Assume $Char(F) = 0$, and define $\overline{\mathcal{H}}_c^R(G)$ to be the subspace of $f \in \overline{\mathcal{H}}(G)$ such that $\Phi(f, \omega(x)) = 0$ for x non compact or non elliptic, then:*

i) ker $(Rk \circ EP) = \mathcal{R}_{\mathbb{C}I}(G) + \ker Res_G^{G^0}$ and im $(Rk \circ EP) = \overline{\mathcal{H}}_c^R(G)$ where $Res_G^{G^0}$ is the morphism $\mathcal{R}(G) \longrightarrow \mathcal{R}(G^0)$ induced by the restriction functor $Mod(G) \longrightarrow Mod(G^0)$.

ii) ker $(\overline{EP}) = \mathcal{R}_I(G) \otimes \mathbb{C}[G/G^0] + \langle \psi\pi \otimes \psi\lambda - \pi \otimes \lambda \rangle_{\pi,\lambda,\psi}$ where the last summand is the space generated by all elements as in the brackets and we let $\Psi(G)$ act on $\mathbb{C}[G/G^0]$ by $(\psi\lambda)(g) := \psi(g)\lambda(g)$. Moreover im $\overline{EP} = \overline{\mathcal{H}}^R(G)$.

Proof: Note first that since $Rk \circ EP(\pi) = \overline{EP}(\pi \otimes 1)$, assertion *i*) is a formal consequence (a particular case) of assertion *ii*). The proofs of the image assertion and the inclusion $\supset$ in the kernel assertion make use of some results of Sects. 4 and 5 and so will be given in 5.13. We are thus here only interested in the inclusion $\subset$ of the kernel assertion of *ii*). However we will need the following simple remark: since the morphism

$$\begin{aligned} \pi \otimes \mathbb{C}[G/G^0] &\to \psi\pi \otimes \mathbb{C}[G/G^0] \\ v \otimes \mu &\mapsto v \otimes \psi\mu \end{aligned}$$

is a G-isomorphism which conjugates $\lambda \in End_G(\pi \otimes \mathbb{C}[G/G^0])$ and $\psi.\lambda \in End_G(\psi\pi \otimes \mathbb{C}[G/G^0])$, we have as a matter of fact $\langle \psi\pi \otimes \psi\lambda - \pi \otimes \lambda \rangle_{\pi,\lambda,\psi} \subset ker\ \overline{EP}$. Now, we introduce some new notations.

3.5 *Fixing a central character:* Let Z be the group of F-points of the connected component of the center of G. We fix a continuous character χ of Z and consider the objects:

- $Mod_\chi(G)$: the full subcategory of $Mod(G)$ consisting of those G-modules V on which Z acts through χ.
- $\mathcal{R}_\chi(G) = \{\pi \in \mathcal{R}(G),\ \ \chi_\pi = \chi\}$ where χ_π is the central quasi-character of π.
- $\mathcal{H}_\chi(G)$ the algebra of locally constant functions f with support compact modulo the center such that $f(zg) = \chi^{-1}(z) f(g)$ and equipped with the convolution product associated to the Haar measure d^*g on G/Z induced by that on G (call it dg) and the Haar measure on Z assigning volume 1 to the maximal open compact subgroup Z^0 of Z.
- $\mathcal{K}_\chi(G)$ the Grothendieck group of projective objects in $Mod_\chi(G)$.

There is a canonical epimorphism of algebras:

$$\begin{aligned} r_\chi : \quad \mathcal{H}(G) &\to \mathcal{H}_\chi(G) \\ f &\mapsto g \mapsto \int_Z \chi(z) f(zg) dg \end{aligned}$$

which induces a morphism $r_\chi : \overline{\mathcal{H}}(G) \longrightarrow \overline{\mathcal{H}}_\chi(G) := \mathcal{H}_\chi/[\mathcal{H}_\chi, \mathcal{H}_\chi]$. Moreover, the system of idempotents e_H in $\mathcal{H}(G)$ for H varying among compact open subgroups of G gives rise to a system of idempotents $r_\chi(e_H)$ in $\mathcal{H}_\chi(G)$ which enables to identify the category $Mod_\chi(G)$ with the category of $\mathcal{H}_\chi(G)$-modules whose elements are fixed by some idempotent $r_\chi(e_H)$. It was shown in [25] that this category is of finite cohomological dimension, hence we can define $EP_\chi : \mathcal{R}_\chi(G) \longrightarrow \mathcal{K}_\chi(G)$ and also a map $Rk_\chi : \mathcal{K}_\chi(G) \longrightarrow \overline{\mathcal{H}}_\chi(G)$.

Moreover, the alternative description of 1.3 is still valid for $Mod_\chi(G)$: this is for example implied by the fact that the results of [7] are true for a central extension of a p-adic group by a finite group (see [7, p. 16]). Hence we can give an alternative expression for r_χ: let $\mathbb{C}_\chi$ be the one-dimensional module of $\mathcal{H}(Z)$ corresponding to χ, then if P is a f.g. projective module of $Mod(G)$, the module $P_\chi = P \otimes_{\mathcal{H}(Z)} \mathbb{C}_\chi$ is a f.g. projective object of $Mod_\chi(G)$ on which any $u \in End_G(P)$ induces an endomorphism $u_\chi \in End_{Mod_\chi(G)}(P_\chi)$. Then r_χ is just induced by the map $(P, u) \mapsto (P_\chi, u_\chi)$.

We need two lemmas.

Lemma 3.6 *Let $\pi \in Irr(G)$ and let χ_π be the character of Z afforded by π, then for any continuous character χ of Z, we have*

i) If $\chi_{|Z^0} = \chi_{\pi\,|Z^0}$, then

$$r_\chi(\overline{EP}(\pi \otimes \lambda)) = \sum_{\psi_{|Z} = \chi\cdot\chi_\pi{}^{-1}} \psi(\lambda)\, Rk_\chi \circ EP_\chi(\psi\pi)$$

.

ii) If $\chi_{|Z^0} \neq \chi'_{|Z^0}$, then $r_\chi(\overline{EP}(\pi \otimes \lambda)) = 0$.

Proof: We begin with a review of *the resolution of Schneider and Stuhler*. Recall we have mentioned the fact that in [25] the category $Mod_{\chi_\pi}(G)$ was shown to be of finite cohomological dimension. Actually a more precise fact is proven there: there exists a projective resolution in $Mod_{\chi_\pi}(G)$:

$$0 \longrightarrow P_{s_G} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow \pi \longrightarrow 0$$

where s_G is the semi-simple rank of G and each P_i is a sum

$$P_i = \bigoplus_{k=1}^{d_i} ind_{\widetilde{K}_{i,k}}^G(\widetilde{\tau}_{i,k})$$

where the $\widetilde{K}_{i,k}$'s are open subgroups containing the center Z and compact modulo this center, and the $\widetilde{\tau}_{i,k}$'s are smooth finite dimensional representations of the $\widetilde{K}_{i,k}$'s such that $(\widetilde{\tau}_{i,k})_{|Z}$ is a multiple of χ_π.

Mackey formulas: We fix a pair $(\widetilde{K}, \widetilde{\tau})$ as above and call K the maximal open compact subgroup of $\widetilde{K}$, τ the restriction of $\widetilde{\tau}$ to K and τ_Z the restriction of $\widetilde{\tau}$ to KZ. Now put $n_{\widetilde{K}} = [G : G^0\widetilde{K}]$, then since $V \otimes \mathbb{C}[G/G^0Z] = ind_{G^0Z}^G \circ Res_G^{G^0Z} V$ we deduce from the Mackey formulas ([29, I.5.5]):

$$ind_{\widetilde{K}}^G(\widetilde{\tau}) \otimes \mathbb{C}[G/G^0] \simeq ind_K^G(\tau)^{n_{\widetilde{K}}}$$

In particular, $ind_{\widetilde{K}}^G(\widetilde{\tau}) \otimes \mathbb{C}[G/G^0]$ is a projective object in $Mod(G)$ hence the sequence obtained with the notations of the last paragraph above

$$\begin{aligned} 0 \longrightarrow P_{s_G} \otimes \mathbb{C}[G/G^0] \longrightarrow \cdots \\ \longrightarrow P_0 \otimes \mathbb{C}[G/G^0] \longrightarrow \pi \otimes \mathbb{C}[G/G^0] \longrightarrow 0 \end{aligned}$$

is a projective resolution of $\pi \otimes \mathbb{C}[G/G^0]$ in $Mod(G)$ which provides a formula for $\overline{EP}(\pi \otimes \lambda)$:

$$\overline{EP}(\pi \otimes \lambda) = \sum_k (-1)^k [P_k \otimes \mathbb{C}[G/G^0], \lambda]$$

Now we turn to the proof itself: let V be any finitely generated object of $Mod_{\chi_\pi}(G)$ then $V \otimes \mathbb{C}[G/G^0]$ is a finitely generated G-module and it is elementary to see that for any continuous character χ of Z, we have:

- $(V \otimes \mathbb{C}[G/G^0]) \otimes_{\mathcal{H}(Z)} \mathbb{C}_\chi \simeq V \otimes ind_{G^0Z}^G(\chi{\chi_\pi}^{-1})$ if $\chi_{|Z^0} = \chi_{\pi|Z^0}$.
- $(V \otimes \mathbb{C}[G/G^0]) \otimes_{\mathcal{H}(Z)} \mathbb{C}_\chi = 0$ if $\chi_{|Z^0} \neq \chi_{\pi|Z^0}$.

In the first point, $\chi{\chi_\pi}^{-1}$ is viewed as a character of G^0Z thanks to the projection $G^0Z \longrightarrow Z/Z^0$.

Now using the Schneider-Stuhler resolution of π and the resolution of $\pi \otimes \mathbb{C}[G/G^0]$ deduced from it as above, and given the equality $r_\chi[P, u] = [P_\chi, u_\chi]$, we see that:

$$r_\chi(\overline{EP}(\pi \otimes \lambda)) = \sum_k (-1)^k [P_k \otimes ind_{G_0Z}^G(\chi.{\chi_\pi}^{-1}), \lambda] \in \overline{\mathcal{H}}_\chi(G)$$

where λ acts on the right side of the tensor products. Now from standard representation theory of commutative groups, we have a direct sum decomposition

$$P_k \otimes ind_{G_0Z}^G(\chi.{\chi_\pi}^{-1}) = \bigoplus_{\psi_{|G^0Z} = \chi.{\chi_\pi}^{-1}} \psi.P_k$$

where λ acts on the summand $\psi.P_k$ by the scalar $\psi(\lambda)$. Hence the Lemma 3.6. □

Let G^{ell} be the set of elliptic elements in G, then we can associate to $f \in \mathcal{H}_\chi(G)$ the locally constant function on G^{ell} defined by:

$$f^\vee(x) = \int_{G/Z} f(gx^{-1}g^{-1})d^*g \ .$$

Since $\overline{\mathcal{H}}_\chi(G)$ is also the G-coinvariant quotient of $\mathcal{H}_\chi(G)$ (G acting by conjugation), $f^\vee$ actually doesn't depend on the class of f in $\overline{\mathcal{H}}_\chi(G)$.

Let Θ_π be the character function associated to an admissible representation. Then we have:

Lemma 3.7 *Let $\pi \in \mathcal{R}_\chi(G)$, then $Rk_\chi \circ EP_\chi(\pi)^\vee = \Theta_{\pi|G^{ell}}$.*

Proof: This "lemma" is actually a very deep result which was conjectured by Kazhdan and first showed by Schneider-Stuhler in [25]. Another independent proof is given in [8]. Unfortunately both proofs are given in the particular context of a reductive group with compact center. What we need here is a slight generalization to the context of a fixed central character, or equivalently to the case of a central extension of a reductive group by a finite subgroup. It is not clear whether the arguments in [8] extend to this case. But the proof of [25] makes use of a particular resolution of π which is defined in general. Actually, all the objects of [25, III-4.9-III-4.16] can be directly adapted to the central character situation, so that all arguments shall apply. Anyway, we won't check here the details since it would be a considerable digression to enter the world of [25]. Note however that the only "extra" result needed in [25] is prop. III.4-15 which is still true in the fixed character context, thanks to the existence of Rk_χ, already mentioned above. □

Now we can give the proof of the inclusion $\subset$: suppose $X = \sum_i x_i \otimes \lambda_i \in$ *ker* $(\overline{EP})$. First assume that the x_i's afford the same continuous character of Z^0. Hence, adding something in $\langle \psi\pi \otimes \psi\lambda - \pi \otimes \lambda\rangle_{\pi,\lambda,\psi}$, we may assume that $x_i \in \mathcal{R}_{\chi_0}(G)$ for some character χ_0 of Z. Now using Lemma 3.6 for some fixed character χ of Z and Lemma 3.7, we see that the (trace)-character of the element $\sum_i \sum_{\psi_{|Z}=\chi\chi_0{}^{-1}} \psi(\lambda_i)(\psi.x_i)$ of $R(G)$ is zero on the elliptic set. Hence by Theorem 3.2 ii) and iii), this actually means that

$$\sum_i \sum_{\psi_{|Z}=\chi\chi_0{}^{-1}} \psi(\lambda_i)(\psi.x_i) \in \mathcal{R}_{\mathbb{C}I}(G)\ .$$

Since $y \in \mathcal{R}_{\mathbb{C}I}(G) \Rightarrow \psi.y \in \mathcal{R}_{\mathbb{C}I}(G)$ for any $\psi \in \Psi(G)$, we get varying the χ:

$$\overline{EP}(X) = 0 \Rightarrow \forall \psi_1 \in \Psi(G),\ \ \sum_i \sum_{\psi_{|Z}=1} (\psi_1\psi)(\lambda_i)(\psi.x_i) \in \mathcal{R}_{\mathbb{C}I}(G)\ .$$

Now fix ψ_1, then we can write

$$\begin{aligned}
[G : G^0 Z]X &= \sum_{\psi_{|Z}=1} \sum_i \psi.x_i \otimes \psi\lambda_i \text{ modulo } \langle \psi\pi \otimes \psi\lambda - \pi \otimes \lambda\rangle_{\pi,\lambda,\psi} \\
&= \left(\sum_{\psi_{|Z}=1} \sum_i (\psi_1\psi)(\lambda_i)(\psi.x_i) \right) \otimes 1 \\
&\quad + \sum_i \sum_{\psi_{|Z}=1} \psi.x_i \otimes (\psi\lambda_i - (\psi_1\psi)(\lambda_i).1) \\
&\in \mathcal{R}_I(G) \otimes \mathbb{C}[G/G^0] + \mathcal{R}(G) \otimes \mathit{ker}\ \psi_1\ .
\end{aligned}$$

Note that from assertion *ii*) of Lemma 3.6 this is still true when we no longer assume X to afford a unique continuous character of Z^0. Hence we get for any $\psi_1 \in \Psi(G)$:

$$\mathit{ker}\ \overline{EP} = \langle \psi\pi \otimes \psi\lambda - \pi \otimes \lambda\rangle_{\pi,\lambda,\psi} + \mathcal{R}_I(G) \otimes \mathbb{C}[G/G^0] + \mathcal{R}(G) \otimes \mathit{ker}\ \psi_1\ .$$

Thus the kernel assertion of 3.4 follows from an application of Nakayama's lemma. □

4 Topological filtration on $\mathcal{K}(G)$

4.1 *The central algebra:* We freely use some standard facts of the theory of complex representations of p-adic groups. The main references are [7] and [6]. We write $\Theta(G)$ for the variety of infinitesimal characters of G and $\mathfrak{S}(G)$ for the set of inertia classes of cuspidal pairs. So we have

$$\Theta(G) = \bigcup_{\mathfrak{s} \in \mathfrak{S}(G)} \Theta_{\mathfrak{s}} \text{ and } \mathfrak{Z}(G) = \prod_{\mathfrak{s} \in \mathfrak{S}(G)} \mathfrak{Z}_{\mathfrak{s}}$$

where $\mathfrak{Z}_\mathfrak{s} = \mathcal{O}(\Theta_\mathfrak{s})$. This leads to a decomposition of the category $Mod(G) = \prod_{\mathfrak{s}\in\mathfrak{S}} Mod(\mathfrak{s})$ and of course to the corresponding decompositions $\mathcal{K}(G) = \bigoplus_\mathfrak{s} \mathcal{K}(G, \Theta_\mathfrak{s})$ and the same for $\mathcal{R}(G)$ and $\overline{\mathcal{H}}(G)$. Moreover, we have a decomposition:

$$\mathcal{R}(G, \Theta_\mathfrak{s}) = \bigoplus_{\theta\in\Theta_\mathfrak{s}} \mathcal{R}(G, \Theta_\mathfrak{s}, \theta)$$

with evident notations. Bernstein gives also in [3, Thm 31] a nice interpretation of $Mod(\mathfrak{s})$: let $(M, \rho) \in \mathfrak{s}$ and put

$$P_\mathfrak{s} = i_M^G(\rho \otimes \mathbb{C}[M/M^0]) \text{ and } \mathcal{H}^\mathfrak{s}(G) := End_G(P_\mathfrak{s})$$

Then the isomorphism class of $P_\mathfrak{s}$ doesn't depend on the choice of $(M, \rho) \in \mathfrak{s}$ ([3, Thm 37]) and $\mathcal{H}^\mathfrak{s}$ is a $\mathfrak{Z}_\mathfrak{s}$-algebra, finitely generated as a $\mathfrak{Z}_\mathfrak{s}$-module such that the functor

$$\begin{aligned} Mod(\mathfrak{s}) &\to \mathcal{H}^\mathfrak{s} - Mod \\ V &\mapsto Hom_G(P_\mathfrak{s}, V) \end{aligned}$$

is an equivalence of categories.

4.2 *An other filtration on $\mathcal{K}(G)$:* Let V be a finitely generated G-module and consider it as a $\mathfrak{Z}(G)$-module. Define $Supp\,(V)$ to be the support of the associated quasi-coherent sheaf. Note it is actually the support of a coherent sheaf, namely that associated to V^H where H is a sufficiently small open compact subgroup, hence it is closed and has finitely many irreducible components. Let $Mod_i(G)$ be the abelian category generated by those V satisfying: $dim\,Supp\,(V) \leqslant i$. We then put:

$$F^i\mathcal{K}(G) := im\left(\mathcal{K}(Mod_i(G)) \longrightarrow \mathcal{K}(G)\right)$$

This defines an increasing filtration of $\mathcal{K}(G)$, whose associated graded structure we note $G^i\mathcal{K}(G) := F^i\mathcal{K}(G)/F^{i-1}\mathcal{K}(G)$.

We get more into details: let Y be a $d(Y)$-dimensional irreducible subvariety of $\Theta(G)$ and let $Mod(G, Y)$ be the full abelian subcategory of f.g. G-modules V such that $\mathfrak{Z}(G)$ acts on V via its quotient $\mathbb{C}[Y]$ (note it is a stronger requirement than $Supp\ V \subset Y$). $Mod(G, Y)$ is Morita-equivalent to the category of modules over a $\mathbb{C}[Y]$-algebra of finite type, namely $\mathcal{H}^\mathfrak{s}(G) \otimes_{\mathfrak{Z}(G)} \mathbb{C}[Y]$ where $\mathfrak{s}$ is the unique inertia class such that $Y \subset \Theta_\mathfrak{s}$. Let $\mathcal{K}(G, Y)$ be the Grothendieck group of $Mod(G, Y)$: it is equipped with a filtration in the same way as $\mathcal{K}(G)$ and the natural inclusion induces a filtered morphism $\mathcal{K}(G, Y) \xrightarrow{i_*^Y} \mathcal{K}(G)$. Moreover we classically have:

Lemma 4.3 $G^d\mathcal{K}(G) = \sum_{d(Y)=d} i_*^Y(G^d\mathcal{K}(G, Y))$.

Proof: It is a standard fact that $F^d\mathcal{K}(G)$ is generated by the $[V]$'s for V of irreducible support Y of dimension d (see [10, lemme 17] for example or [14, 5.9.2] in an equivariant setting). So the only thing to show is that such a $[V]$ is in the image of i_*^Y. This is also standard; let $\mathcal{I}$ be the defining ideal of Y, then, V being finitely generated, there is an integer n such that $\mathcal{I}^nV = 0$. So looking at the filtration $0 = \mathcal{I}^nV \subset \cdots \subset \mathcal{I}V \subset V$, we see that in $\mathcal{K}(G)$ we have $[V] = \sum_i [\mathcal{I}^iV/\mathcal{I}^{i+1}V]$. Now $\mathcal{I}^iV/\mathcal{I}^{i+1}V$, being annihilated by $\mathcal{I}$ is an object of $Mod(G, Y)$, hence $[\mathcal{I}^iV/\mathcal{I}^{i+1}V]$ is in the image of i_Y^*. □

4.4 *Specialization to the generic point:* In this paragraph we specify the field of coefficients, e.g. $Mod(G, Y)$ becomes $Mod(G, \mathbb{C}, Y)$. We have the following "generic fiber" morphism:

$$\begin{aligned} Mod(G, \mathbb{C}, Y) &\to Mod(G, \mathbb{C}(Y), \theta_Y) \\ V &\mapsto V \otimes_{\mathfrak{Z}(G)} \mathbb{C}(Y) \end{aligned}$$

where the symbols have the following meanings:

- $\mathbb{C}(Y)$ is the quotient field of $\mathbb{C}[Y]$. The tensor product is taken w.r.t the canonical morphism $i_Y^* :\ \mathfrak{Z}(G) \longrightarrow \mathbb{C}[Y] \longrightarrow \mathbb{C}(Y)$. Hence the functor is actually exact.
- $Mod(G, \mathbb{C}(Y), \theta_Y)$ is the category of those finitely generated $G\mathbb{C}(Y)$-modules such that $\mathfrak{Z}(G)$ acts via the morphism i_Y^* above. θ_Y is thus a $\mathbb{C}(Y)$-infinitesimal character defined over $\mathbb{C}(Y)$ in the sense of 6.1 ii).

In this context, we have the following fundamental result:

Proposition 4.5 *The functor above induces an isomorphism:*

$$G^{d(Y)}\mathcal{K}(G, Y) \xrightarrow{\sim} \mathcal{R}(G, \mathbb{C}(Y), \theta_Y)\ .$$

Proof: This is a consequence of the standard "localization exact sequence" of [1, IX.(6.1)]: the only finiteness result required is the noetheriannity of $\mathcal{H}(G)$ which has been known since [7] and we can recover the situation of an algebra with unit, finitely generated as a $\mathbb{C}[Y]$-module by working within $Mod(\mathfrak{s})$ where $\mathfrak{s}$ is the unique inertia class such that $Y \subset \Theta_{\mathfrak{s}}$ and by using the equivalence of 4.1. □

Remark: Look at the case $Y = \Theta_{\mathfrak{s}}$ for some inertia class $\mathfrak{s} = [M, \rho]_G$. The previous discussion enables to describe the "top" quotient of the topological graduation:

$$\mathcal{K}(G, \Theta_{\mathfrak{s}}) \longrightarrow G^{d(M)}\mathcal{K}(G, \Theta_{\mathfrak{s}}) \simeq \mathcal{R}(\mathcal{H}_{\mathfrak{s}} \otimes_{\mathfrak{Z}_{\mathfrak{s}}} \mathbb{C}(\Theta_{\mathfrak{s}}))\ .$$

As a matter of fact, [7, 3.14] implies that $\mathcal{H}_{\mathfrak{s}} \otimes_{\mathfrak{Z}_{\mathfrak{s}}} \mathbb{C}(\Theta_{\mathfrak{s}})$ is a finite dimensional central simple algebra over $\mathbb{C}(\Theta_{\mathfrak{s}})$, hence its Grothendieck group is just $\mathbb{Z}$. The map $\mathcal{K}(G, \Theta_{\mathfrak{s}}) \longrightarrow \mathbb{Z}$ thus obtained is the same as that in [17] which was the starting point of these investigations.

4.6 *Moving parabolically:* We can define similar filtrations on $\mathcal{K}(M)$ and it can be checked that i_M^G and r_G^M are filtered morphisms. This is a direct consequence of [6, prop 2.4] as well as the following lemma:

Lemma 4.7 *The functor $T_M = i_M^G \circ r_G^M$ commutes with the action of $\mathfrak{Z}(G)$ on $Mod(G)$.*

Corollary 4.8 *Let Y be an irreducible subvariety of $\Theta(G)$. The system of functors $(T_M)_{M<G}$ induce on $\mathcal{K}(G,Y)$ as well as on $\mathcal{R}(G,\mathbb{C}(Y),\theta_Y)$ a system of (combinatoric) filtration-preserving endomorphisms $(T_M)_{M<G}$ which commute with i_*^Y and the isomorphism of 4.5.*

Proof: This is straightforward from the definitions and the previous lemma. □

Now we state the starting point of the theory of $\mathcal{K}(G)$:

Theorem 4.9 *The filtrations $\mathcal{K}_{\mathbb{C}}^{\bullet}(G)$ and $F^{\bullet}\mathcal{K}_{\mathbb{C}}(G)$ coincide.*

Proof: We begin with the inclusion $F^{\bullet}\mathcal{K}(G) \subset \mathcal{K}^{\bullet}(G)$, it follows from the next lemma.

We write Z for the set of F-points of a maximal F-split torus lying in the (algebraic) center of G. There is a canonical morphism

$$\mathcal{H}(Z) \longrightarrow \mathfrak{Z}$$

which provides, thanks to the inclusion $\mathbb{C}[Z/Z^0] \subset \mathcal{H}(Z)$, a morphism $\Theta(G) \longrightarrow \Psi(Z)$ and we note $Supp_Z V$ the image of $Supp\ V$ for a finitely generated G-module V (it need not be closed in general).

Lemma 4.10 *Let $V \in Mod(\mathfrak{s})$ be a finitely generated G-module. If $\dim Supp_Z V < d(G)$ then $[V] = 0$ in $\mathcal{K}(G)$.*

Proof: The hypothesis in the lemma imply that we can find a one-parameter subgroup $\mathbb{C}^* \xrightarrow{\alpha} \Psi(Z)$ such that $dim(im\ \alpha \cap Supp_Z V) = 0$. Now α comes from a morphism of groups: $Z/Z^0 \xrightarrow{\alpha^*} \mathbb{Z}$. Put $Z' := ker\ \alpha^*$. Recall that V is finitely generated over $G^0.Z$ since G/G^0Z is finite. Moreover the choice of α implies that only a finite number of characters of Z/Z' appear in the action of Z/Z' on V via $\mathbb{C}[Z/Z'] \subset \mathcal{H}(Z)$, and this implies that V is finitely generated over $G^0.Z'$.

We have $G/G^0.Z' \simeq \mathbb{Z} \oplus$ *finite group* and we define G' to be the kernel of the projection on the factor $\mathbb{Z}$. V is finitely generated over G' so that $V \otimes \mathbb{C}[G/G'] = ind_{G'}^G(V)$ is G-finitely generated. Now write $\mathbb{C}[G/G'] = \mathbb{C}[X, X^{-1}]$, we have the following exact sequence

$$0 \longrightarrow V \otimes \mathbb{C}[G/G'] \xrightarrow{\times(X-1)} V \otimes \mathbb{C}[G/G'] \xrightarrow{X \mapsto 1} V \longrightarrow 0$$

which shows $[V] = [V \otimes \mathbb{C}[G/G']] - [V \otimes \mathbb{C}[G/G']] = 0$ in $\mathcal{K}(G)$. □

Remark: The latter result could be seen in the following intuitive way: there is an action of $\Psi(G)$ on each object of interest for us $(\mathcal{R}(G), \overline{\mathcal{H}}(G), \Theta(G), \cdots)$ and in particular on $\mathcal{K}(G)$, but the latter is a discrete set whereas $\Psi(G)$ is continuous hence this action has to be trivial. In other words: for any finitely generated G-module V, we have $\forall \psi \in \Psi(G), \quad [\psi V] = [V]$ in $\mathcal{K}(G)$. Now since $Supp\,(\psi.V) = \psi\, Supp\ V$, it can be deduced that for V not to be 0 in $\mathcal{K}(G)$, $Supp\ V$ must be stable under $\Psi(G)$ hence of dimension at least $d(G)$.

Corollary 4.11 $F^{\bullet}\mathcal{K}(G) \subset \mathcal{K}^{\bullet}(G)$.

Proof: Let V be a f.g. G-module such that $dim\, Supp\ V = d$ and M a standard Levi subgroup such that $d(M) > d$, then $dim\, Supp\ r_G^M(V) < d(M)$, so that $[r_G^M(V)] = 0$ in $\mathcal{K}(M)$ by the lemma above. □

Before dealing with the reversal inclusion, we have to investigate a little more the geometry of $\Theta(G)$.

4.12 *A filtration on* $\Theta(G)$ *:* Here we want to precise 4.3, using the notion of discrete infinitesimal characters. The definition of "discrete" we will use slightly differs from that in [6]: a discrete infinitesimal character is here an element of:

$$\Theta^{disc}(G) := \{\theta \in \Theta(G),\ \ \mathcal{R}_{\mathbb{C}}(G,\theta) \neq \mathcal{R}_{\mathbb{C}I}(G,\theta)\}$$

and we will call "quasi-discrete" any element of

$$\Theta^{qdisc}(G) := \{\theta \in \Theta(G),\ \ \mathcal{R}(G,\theta) \neq \mathcal{R}_I(G,\theta)\}\ .$$

Note that we have:

$$(\theta \text{ non-discrete})\ \text{ if and only if }\ \left(A_{d(G)} = 0 \text{ on } \mathcal{R}(G,\theta)\right)$$

and for $GL(N)$ the two notions are actually equivalent, but they may differ in general. Now recall that the set $\Theta_{\mathfrak{s}}^{qdisc} = \Theta^{qdisc}(G) \cap \Theta_{\mathfrak{s}}$ was shown in [6] to be a finite union of $\Psi(G)$-orbits, hence is a closed subvariety of dimension $d(G)$. The same result holds of course for $\Theta_{\mathfrak{s}}^{disc}$.

Now we define for any d:

$$Y_d := \bigcup_{d(M)=d} i_M^G(\Theta^{disc}(M))$$

This is a closed subvariety of $\Theta(G)$ of pure dimension d (note that $Y_d \cap \Theta_{\mathfrak{s}}$ is either empty if $d > d(\Theta_{\mathfrak{s}})$ or has finitely many irreducible components). We have the following characterization of these subvarieties:

$$\Theta(G) \setminus (Y_{d(G)} \cup \cdots \cup Y_d) = \{\theta \in \Theta(G),\quad \mathcal{R}_{\mathbb{C}}(G,\theta) = \mathcal{R}_{\mathbb{C}d}(G,\theta)\}$$

This follows from the definition. Now, the fundamental tool is the following result:

Proposition 4.13 *Let Y be an irreducible algebraic subvariety of $\Theta(G)$ such that $A_d = 0$ on $\mathcal{R}_{\mathbb{C}}(G, \mathbb{C}, \theta)$ for θ in an open subset of Y. Then $A_d = 0$ on $\mathcal{R}_{\mathbb{C}}(G, \mathbb{C}(Y), \theta_Y)$.*

Proof: There are two ways to prove this statement. Here we explain what is suggested in [18]. In part 6, we give a proof using direct image in algebraic K-theory, which is more natural but also longer than the following argument.

First we can choose a subfield $\mathbb{K}$ of $\mathbb{C}$, finitely generated over $\mathbb{Q}$ satisfying the two conditions:

- The morphism $Y \hookrightarrow \Theta_{\mathfrak{s}}$ is defined over $\mathbb{K}$ (*i.e.* comes from a morphism $\underline{Y} \longrightarrow \underline{\Theta}_{\mathfrak{s}}$ of reduced geometrically irreducible schemes defined over $\mathbb{K}$).
- The $\mathfrak{Z}_{\mathfrak{s}}$-algebra $\mathcal{H}_{\mathfrak{s}}$ comes from a $\mathfrak{Z}_{\mathfrak{s}}^{\mathbb{K}}$-algebra (where $\underline{\Theta}_{\mathfrak{s}} = Spec\,(\mathfrak{Z}_{\mathfrak{s}}^{\mathbb{K}})$), *i.e.* $\mathcal{H}_{\mathfrak{s}} = \mathcal{H}_{\mathfrak{s}}^{\mathbb{K}} \otimes_{\mathfrak{Z}_{\mathfrak{s}}^{\mathbb{K}}} \mathfrak{Z}_{\mathfrak{s}}$.

Since the field of functions $\mathbb{K}(Y)$ at the generic point of $\underline{Y}$ is also finitely generated over $\mathbb{Q}$, we can choose a $\mathbb{K}$-embedding $\theta : \mathbb{K}(Y) \hookrightarrow \mathbb{C}$. It provides thus a complex closed point of $\underline{Y}$ (still noted θ). It is known that the set of complex closed points of $\underline{Y}$ which can be obtained in this fashion is Zarisky-dense in $Y = \underline{Y}(\mathbb{C})$ (indeed, call this set $c(\underline{Y}) \subset \underline{Y}(\mathbb{C})$; by Noether's normalization lemma one can find a finite epimorphism $\underline{Y} \xrightarrow{\phi} \mathbb{A}_d$ (the affine space of dimension $d = d(Y)$), then one easily sees that $\phi^{-1}(c(\mathbb{A}_d)) \subset c(\underline{Y})$ and $c(\mathbb{A}_d)$ is dense in $\mathbb{C}^d$ since it contains the set of those points whose coordinates are transcendant over $\mathbb{K}$. Hence the closure of $c(\overline{Y})$ is at least d-dimensional and thus equals $\underline{Y}(\mathbb{C})$ since $\underline{Y}$ is irreducible). In particular, by the hypothesis we can choose θ such that A_d vanishes on $\mathcal{R}_{\mathbb{C}}(G, \mathbb{C}, \theta)$.

Now from the fact that $\mathbb{C}$ is algebraically closed of infinite transcendence degree over $\mathbb{Q}$, we can extend θ to a $\mathbb{K}$-embedding of fields: $\overline{\theta} : \mathbb{C}(Y) \longrightarrow \mathbb{C}$. As a matter of fact, choose a transcendence basis $(x_i)_{i\in I}$ of $\mathbb{C}$ over $\theta(\mathbb{K}(Y))$ (resp. $(y_j)_{j\in J}$ of $\mathbb{C}(Y)$ over $\mathbb{K}(Y)$). Since I and J have the same cardinality we can embed $\mathbb{K}(Y)((y_j)_{j\in J})$ into $\mathbb{C}$ and extend to such a $\overline{\theta}$ as above since $\mathbb{C}(Y)$ is algebraic over $\mathbb{K}(Y)((y_j)_{j\in J})$. Hence from the isomorphism

$$\left(\mathcal{H}_{\mathfrak{s}}^{\mathbb{K}} \otimes_{\mathfrak{Z}_{\mathfrak{s}}^{\mathbb{K}}} \mathbb{C}(Y)\right) \otimes_{\mathbb{C}(Y),\overline{\theta}} \mathbb{C} \simeq \mathcal{H}_{\mathfrak{s}} \otimes_{\mathfrak{Z}_{\mathfrak{s}},\theta} \mathbb{C}$$

we can form the following diagram

$$\begin{tikzcd}
\mathcal{R}_{\mathbb{C}}(G, \mathbb{C}(Y), \theta_Y) \arrow[r, dotted] \arrow[d, equal] & \mathcal{R}_{\mathbb{C}}(G, \mathbb{C}, \theta) \arrow[d, equal] \\
\mathcal{R}_{\mathbb{C}}\left(\mathcal{H}_{\mathfrak{s}}^{\mathbb{K}} \otimes_{\mathfrak{Z}_{\mathfrak{s}}^{\mathbb{K}}} \mathbb{C}(Y)\right) \arrow[r, "\otimes_{\mathbb{C}(Y)}"] & \mathcal{R}_{\mathbb{C}}(\mathcal{H}_{\mathfrak{s}} \otimes_{\mathfrak{Z}_{\mathfrak{s}},\theta} \mathbb{C})
\end{tikzcd}$$

where the bottom map is given by base extension via $\overline{\theta}$. We thus get a morphism (the dotted one):

$$\overline{\theta}_* : \ \mathcal{R}_{\mathbb{C}}(G, \mathbb{C}(Y), \theta_Y) \longrightarrow \mathcal{R}_{\mathbb{C}}(G, \mathbb{C}, \theta)$$

which is injective by general theory of finite-dimensional algebras (see Appendix B) and obviously commutes with the T_M's. □

Corollary 4.14 *For any d, we have*

$$G^d\mathcal{K}_{\mathbb{C}}(G) = \bigoplus_{Y\subset Y_d, d(Y)=d} i_*^Y(G^d\mathcal{K}_{\mathbb{C}}(G,Y)) \ .$$

Proof: The meaning of this assertion is essentially that in order for an irreducible subvariety Y to provide a non-zero contribution in the formula of 4.3, it is necessary that Y be contained in Y_d. Taking this for granted, the sum of 4.3 becomes a direct sum because for each $\mathfrak{s} \in \mathfrak{S}(G)$ the variety $Y_d \cap \Theta_{\mathfrak{s}}$ has finitely many irreducible components which are in fact its connected components.

Now fix d and let Y be a d-dimensional closed irreducible subvariety of $\Theta(G)$ which is not contained in Y_d. The characterization of Y_d together with the previous proposition show that A_d is zero on $\mathcal{R}_{\mathbb{C}}(G, \mathbb{C}(Y), \theta_Y)$. Hence, from the expansion formula of A_d (see 2.5) and the Corollary 4.8, the operator

$$P_d(1 - \sum_{d(M)>d} c_d(M)T_M)$$

vanishes on $i_*^Y G^d\mathcal{K}(G,Y)$. As a consequence we get

$$P_d.i_*^Y G^d\mathcal{K}(G,Y) \subset \sum_{d(M)>d} i_M^G G^d\mathcal{K}(M) = 0$$

since $G^d\mathcal{K}(M) = 0$ for $d(M) > d$ by Corollary 4.11. Hence, killing the torsion by tensorizing with $\mathbb{C}$ (or $\mathbb{Q}$), we see that $G^d\mathcal{K}_{\mathbb{C}}(G,Y)$ provides a non-zero contribution in 4.3 only if Y is a d-dimensional subvariety of Y_d. □

4.15 *End of the proof of 4.9:* Going on with the above discussion, we see that for any irreducible Y:

$$A_{d(Y)-1} \text{ is zero on } \mathcal{R}_{\mathbb{C}}(G, \mathbb{C}(Y), \theta_Y)$$

So let V be a finitely generated G-module in $Mod(G, Y)$ whose contribution is non zero in 4.3, *i.e.* $i_*^Y[V] \neq 0$ in $G^{d(Y)}\mathcal{K}(G)$. We know from Lemma 4.10 that $i_*^Y[V] \in \mathcal{K}^{d(Y)}(G)$. Suppose that there exists some $d < d(Y)$ such that $i_*^Y[V] \in \mathcal{K}^d(G)$ then from the expansion formula

$$A_{d(Y)-1} = P_{d(Y)-1}(1 - \sum_{d(M)>d(Y)-1} c_{d(Y)-1}(M)T_M)$$

and the commutation $A_{d(Y)-1}i_*^Y = i_*^Y A_{d(Y)-1}$, we see that $A_{d(Y)-1}i_*^Y[V] = P_{d(Y)-1}i_*^Y[V]$ (since $d(Y) > d$), hence $P_{d(Y)-1}i_*^Y[V]$ is zero in $G^{d(Y)}\mathcal{K}(G)$. At this point we loose again the torsion tensorizing with $\mathbb{C}$ or $\mathbb{Q}$ to deduce $i_*^Y[V] = 0$ in $G^{d(Y)}\mathcal{K}_{\mathbb{C}}(G)$. This completes the proof of 4.9. □

4.16 *Level $d(G)$:* Once we have the equality of the combinatorial and topological filtrations on $\mathcal{K}_{\mathbb{C}}(G)$, we can use induction on the depth of Levi subgroups to investigate the properties of $\mathcal{K}_{\mathbb{C}}(G)$. But one has first to explicit the level $d(G)$. Write $\mathcal{K}_{\mathbb{C}}(G)_Y = i_*^Y \mathcal{K}_{\mathbb{C}}(G, Y)$, then from 4.14 and the fact that $F^{d(G)}\mathcal{K}_{\mathbb{C}}(G) = G^{d(G)}\mathcal{K}_{\mathbb{C}}(G)$, we get:

$$F^{d(G)}\mathcal{K}_{\mathbb{C}}(G) \simeq \bigoplus_{Y \in \Theta^{disc}(G)/\Psi(G)} \mathcal{K}_{\mathbb{C}}(G)_Y \tag{4.17}$$

From now on we fix a $\Psi(G)$-orbit Y in $\Theta^{disc}(G)$ and an element $\theta \in Y$. We write $\mathcal{N}_Y$ for the normalizer of θ in $\Psi(G)$ (since it doesn't depend on the choice of $\theta \in Y$). Note that the action of $\Psi(G)$ on $\mathcal{R}(G)$ induces an action of $\mathcal{N}_Y$ on $\mathcal{R}_{\mathbb{C}}(G, \theta)$. Then we claim

Proposition 4.18 *There exists an isomorphism $\phi : \mathcal{R}_{\mathbb{C}}(G, \mathbb{C}(Y), \theta_Y) \xrightarrow{\sim} \mathcal{R}_{\mathbb{C}}(G, \theta)^{\mathcal{N}_Y}$ such that the following diagram is commutative:*

$$\begin{tikzcd}
\mathcal{R}_{\mathbb{C}}(G, \theta)^{\mathcal{N}_Y} \arrow[r, "EP"] & \mathcal{K}_{\mathbb{C}}(G)_Y \\
\mathcal{R}_{\mathbb{C}}(G, \mathbb{C}(Y), \theta_Y) \arrow[u, "\phi"] & \mathcal{K}_{\mathbb{C}}(G, Y) \arrow[l, two heads] \arrow[u, "i_*^Y"']
\end{tikzcd}$$

Proof: First we observe the following general fact: assume $Y' \xrightarrow{\phi} Y$ is an étale morphism such that the extension $\phi^* : \mathbb{C}(Y) \hookrightarrow \mathbb{C}(Y')$ is Galois of automorphism group $\Gamma(Y'/Y)$ and splits (see Appendix B) the finite $\mathbb{C}(Y)$-dimensional algebra $\mathcal{H}_{\mathfrak{s}} \otimes_{\mathfrak{Z}_{\mathfrak{s}}, \theta_Y} \mathbb{C}(Y)$. Then tensorizing via ϕ^* we get an isomorphism

$$\mathcal{R}_{\mathbb{C}}(G, \mathbb{C}(Y), \theta_Y) \xrightarrow{\sim} \mathcal{R}_{\mathbb{C}}(G, \mathbb{C}(Y'), \theta_{Y'})^{\Gamma(Y'/Y)}$$

(where $\theta_{Y'} = \phi^*\theta_Y$). This is easily deduced from the point iv) of Appendix B.

Now let $Y' = \Psi(G)$. We define the "universal" unramified character ψ_{un} of G with values in $\mathbb{C}(G/G^0)$ by $\psi_{un}(g) = \overline{g} := 1_{gG^0}$. If $\pi \in Irr_{\mathbb{C}}(G)$, let $\pi \otimes \psi_{un}$ be the simple $\mathbb{C}(G/G^0)$-representation of G obtained by letting G act on both terms of the tensor product. Then if π has infinitesimal character θ, it is clear that $\mathfrak{Z}(G)$ acts on $\pi \otimes \psi_{un}$ via a morphism $\mathfrak{Z}(G) \xrightarrow{\omega} \mathbb{C}(G/G^0)$ characterized by the properties

- $im\ \omega \subset \mathbb{C}[G/G^0]$
- For any $\psi \in \Psi(G) = Hom_{\mathbb{C}-alg}(\mathbb{C}[G/G^0], \mathbb{C})$, we have the equality $\psi \circ \omega = \psi.\theta$ where the dot stands for the action of $\Psi(G)$ on $\Theta(G)$.

Put in other words, let ϕ_θ be defined by

$$\begin{aligned} \phi_\theta : \quad Y' = \Psi(G) &\to Y \\ \psi &\mapsto \psi.\theta \end{aligned}$$

then the map $\pi \mapsto \pi \otimes \psi_{un}$ induces a morphism

$$\mathcal{R}(G, \mathbb{C}, \theta) \longrightarrow \mathcal{R}(G, \mathbb{C}(Y'), \theta_{Y'}) \tag{4.19}$$

where $\theta_{Y'} = \phi_\theta^* \theta_Y$. This map is obviously an isomorphism, being the composition of the standard base extension from $\mathbb{C}$ to $\mathbb{C}(G/G^0)$ and the torsion by the universal unramified character ψ_{un}.

The map ϕ_θ is a Galois étale covering with Galois group $\mathcal{N}_Y = \mathcal{N}_{\Psi(G)}(\theta)$. The action of $\psi \in \mathcal{N}_Y$ on $\mathbb{C}(G/G^0)$ is given by $\overline{g}^\psi := \psi(\overline{g})\overline{g}$. Hence it is clear that $(\psi\pi) \otimes \psi_{un} = (\pi \otimes \psi_{un})^\psi$, or in other words, the isomorphism 4.19 is $\mathcal{N}_Y$-equivariant for the Galois action on $\mathcal{R}(G, \mathbb{C}(Y'), \theta_{Y'})$ and the usual action on $\mathcal{R}(G, \mathbb{C}, \theta)$. The existence of the isomorphism of the proposition thus follows.

We define $\mathcal{K}(G, Y')$ by $\mathcal{K}(G, Y') = \mathcal{K}(\mathcal{H}^{\mathfrak{s}} \otimes_{\mathfrak{Z}(G), \phi_\theta^*} \mathbb{C}[Y'])$ where $\mathfrak{s}$ is the unique inertia class such that $Y \subset \Theta_{\mathfrak{s}}$, and the morphism R_θ by

$$R_\theta : \mathcal{K}(G, Y') \longrightarrow \mathcal{R}(G, \mathbb{C}(Y'), \theta_{Y'}) \xrightarrow{T_\theta} \mathcal{R}(G, \theta)$$

where T_θ is the inverse isomorphism of 4.19. Now consider the following diagram where $i_*^{Y'}$ is induced by $Y' \xrightarrow{\phi_\theta} Y \hookrightarrow \Theta(G)$

$$\begin{tikzcd} \mathcal{K}(G, Y) \arrow[r, "\phi_\theta^*"] \arrow[d, "|\mathcal{N}_Y|.i_*^Y"'] & \mathcal{K}(G, Y') \arrow[d, "R_\theta"] \arrow[dl, "i_*^{Y'}"] \\ \mathcal{K}(G) & \mathcal{R}(G, \theta) \arrow[l, "EP"] \end{tikzcd} .$$

It is clear that the lower triangle is commutative (from the definitions). The upper one is also commutative since $i_*^{Y'} = i_*^Y \circ \phi_{\theta *}$ and $\phi_{\theta *} \circ \phi_\theta{}^*$ is just the multiplication by $|\mathcal{N}_Y|$ because we have $\mathbb{C}[G/G^0] \simeq \mathbb{C}[Y][\mathcal{N}_Y]$ as a $\mathbb{C}[Y]$-module. Now the commutative square thus obtained induces the following one where $i_Y^{Y'} = \frac{1}{|\mathcal{N}_Y|}(- \otimes_{\mathbb{C}[Y]} \mathbb{C}(G/G^0))$:

$$\begin{tikzcd} \mathcal{K}_{\mathbb{C}}(G, Y) \arrow[r, "i_Y^{Y'}"] \arrow[d, "i_Y^*"'] & \mathcal{R}_{\mathbb{C}}(G, \mathbb{C}(Y'), \theta_{Y'}) \arrow[d, "T_\theta"] \\ \mathcal{K}_{\mathbb{C}}(G) & \mathcal{R}_{\mathbb{C}}(G, \theta) \arrow[l, "EP"] \end{tikzcd}$$

Now the commutative diagram announced in the proposition follows from the facts that $im\ i_Y^{Y'} = \mathcal{R}_{\mathbb{C}}(G, \mathbb{C}(Y), \theta_Y)$ and that T_θ induces an isomorphism $\mathcal{R}_{\mathbb{C}}(G, \mathbb{C}(Y), \theta_Y) \simeq \mathcal{R}_{\mathbb{C}}(G, \mathbb{C}, \theta)^{\mathcal{N}_Y}$, as checked above. □

From now until the end of this section we will assume that Char $F = 0$ in order to use the piece of 3.4 we have already shown.

Corollary 4.20 *Rk is injective.*

Proof: We first prove that $Rk_{|F^{d(G)}\mathcal{K}_{\mathbb{C}}(G)}$ is injective: so let Y and θ as in Proposition 4.18 and $X \in \mathcal{K}_{\mathbb{C}}(G)_Y$. By this proposition, we can write $X = EP(x)$ for some $x \in \mathcal{R}_{\mathbb{C}}(G,\theta)^{\mathcal{N}_Y}$. Now $Rk(X) = 0$ implies that $x \in ker\,(Rk \circ EP)$ and by Theorem 3.4 we get $x \in \mathcal{R}_{\mathbb{C}I}(G)$, *i.e* $A_{d(G)}(x) = 0$, because $\mathcal{R}_{\mathbb{C}}(G,\theta)^{\mathcal{N}_Y} \cap ker\ Res_G^{G^0} = 0$. Hence we have $0 = A_{d(G)}(X) = P_{d(G)}X$ (since r_G^M is zero on $F^{d(G)}\mathcal{K}_{\mathbb{C}}(G)$ for any proper $M < G$) and thus $X = 0$.

We treat the general case: let $X \in G^d\mathcal{K}_{\mathbb{C}}(G)$ and $M < G$ with depth $d(M) = d$ such that $r_G^M(X) \neq 0$ (Theorem 4.9 gives the existence of such a M) then $r_G^M \circ Rk(X) = Rk(r_G^M(X)) \neq 0$ because of the discussion above applied to M, so $Rk(X) \neq 0$. □

Corollary 4.21 *Let $M < G$ and $w \in W_G$ such that $w(M) < G$ then $w \circ r_G^M = r_G^{w(M)}$ in $\mathcal{K}_{\mathbb{C}}(G)$.*

Proof: This comes from the same property on $\overline{\mathcal{H}}(G)$. This should remain true on $\mathcal{K}(G)$. □

Corollary 4.22 *$\mathcal{K}_{\mathbb{C}}(G)$ is generated by compactly induced representations of open compact subgroups.*

Proof: Use the discussion of the proof of Theorem 1.6. Note that one can deduce that $\mathcal{K}(G) \otimes_{\mathbb{Z}} \mathbb{Q}$ is generated by such induced representations but it is not precise enough to get the result for $\mathcal{K}(G)$. □

Corollary 4.23 *Let $M < G$ and $w \in W_G$ such that $w(M) < G$ then $i_{w(M)}^G \circ w = i_M^G$ on $\mathcal{K}_{\mathbb{C}}(M)$.*

Proof: (i) We call "M-pair" any couple (J,τ) with J an open compact subgroup of M and τ an irreducible smooth representation of J and write $V_\tau = Hom_J(\tau, V)$ for any $V \in Mod(M)$. Then from Corollary 4.22 and Theorem 4.20 we see it is sufficient to show:

For any finite length G-module π and any M-pair (J,τ) we have

$$dim_{\mathbb{C}}\left(r_G^M\pi\right)_\tau = dim_{\mathbb{C}}\left(w^{-1}\circ r_G^{w(M)}\pi\right)_\tau \ .$$

(ii) Now we claim: *It is enough to treat the case M maximal in G*. As a matter of fact one can decompose w as a product of elementary maps in the sense of [2, 2.17]. In this way we are reduced to the case M maximal.

(iii) We assume M maximal. We write $\overline{r_G^M}$ for induction with respect to the opposite parabolic P_0^-. By (i) we are reduced to show that for any finite length G-module and any M-pair (J,τ) we have:

$$dim\left(r_G^M\pi\right)_\tau = dim\left(\overline{r_G^M}\pi\right)_\tau \ .$$

We write ρ^+ the Hermitian contragredient of a representation ρ. It is a consequence of [13, 4.2.5] that if π is admissible then $\overline{r_G^M}(\pi^+) = (r_G^M\pi)^+$.

(It was generalized to any representation by Bernstein). Hence we have $dim\left(\overline{r_G^M \pi}\right)_\tau = dim\left(r_G^M \pi^+\right)_{\tau^+}$, but since τ is unitarizable hence isomorphic to τ^+, we are reduced to prove

$$dim\left(r_G^M \pi\right)_\tau = dim\left(r_G^M \pi^+\right)_\tau \ .$$

Now, we fix a standard Levi subgroup N of G and an irreducible tempered representation ρ of N. We write $B = \mathbb{C}[N/N^0]$ and consider the smooth B, M-representation $V = r_G^M \circ i_N^G(\rho \otimes B)$. It is B-admissible. Moreover one can see that V_τ is projective finitely generated as a B-module. (For example choose H an open subgroup of the conductor of τ and H_G a G-cover of H. Then from the Stabilization theorem of Bernstein ([4, 5.3]), we see that V^H, hence V_τ, is a direct factor of $i_N^G(\rho \otimes B)^{H_G}$ which by standard Mackey formulas (see [29, I.5.6]) is free over B.)

Hence V_τ has a B-rank r and for any unramified character χ of N, identified with a character of B, we have

$$dim_{\mathbb{C}}\, r_G^M(i_N^G(\rho.\chi))_\tau = r \ .$$

Now ρ is unitary, being tempered, thus is isomorphic to its Hermitian contragredient, so that: $r_G^M(i_N^G(\rho.\chi))^+ = r_G^M(i_N^G(\rho.\chi^+))$. Hence we are done with the case $\pi = i_N^G(\rho.\chi)$. By Langlands' theory, $R(G)$ is generated by such representations and this completes the proof. □

4.24 *Description of $\mathcal{K}_{\mathbb{C}}(G)$:* Now we give what seems to be the most explicit way of describing $\mathcal{K}_{\mathbb{C}}(G)$; viewed as a Hopf system, it turns out to be isomorphic to some "trivial" Hopf system as in Lemma 2.8. Let E be any $\mathbb{C}$-vector space with an algebraic action of a complex torus T, we note E_T the space of coinvariants *i.e.* $E_T := E/< te - e >_{e,x}$. In particular for a standard Levi subgroup M of G, the action of $\Psi(M)$ on $\mathcal{R}_{\mathbb{C}}(M)$ is algebraic and induces an action on $\overline{\mathcal{R}_{\mathbb{C}}}(M) = \mathcal{R}_{\mathbb{C}}(M)/\mathcal{R}_{\mathbb{C}I}(M)$. Now if $w \in W_G$ is such that $w(M) < G$, then conjugation by w induces a well-defined isomorphism $\mathcal{R}_{\mathbb{C}}(M) \longrightarrow \mathcal{R}_{\mathbb{C}}(M^w)$ which sends $\mathcal{R}_{\mathbb{C}I}(M)$ onto $\mathcal{R}_{\mathbb{C}I}(M^w)$. Since $w(\psi\pi) = w(\psi).w(\pi)$, the isomorphism w induces an isomorphism $\mathcal{R}_{\mathbb{C}}(M)_{\Psi(M)} \longrightarrow \mathcal{R}_{\mathbb{C}}(M^w)_{\Psi(M^w)}$ as well as an isomorphism $\overline{\mathcal{R}_{\mathbb{C}}}(M)_{\Psi(M)} \longrightarrow \overline{\mathcal{R}_{\mathbb{C}}}(M^w)_{\Psi(M^w)}$.

Theorem 4.25 *There exists an isomorphism of Hopf systems*

$$\Phi : \quad \mathcal{K}_{\mathbb{C}} \xrightarrow{\sim} \mathfrak{H}\left(M \mapsto \overline{\mathcal{R}_{\mathbb{C}}}(M)_{\Psi(M)}\right).$$

In particular we get the isomorphism

$$\Phi_G : \quad \mathcal{K}_{\mathbb{C}}(G) \xrightarrow{\sim} \left(\bigoplus_{M<G} \overline{\mathcal{R}_{\mathbb{C}}}(M)_{\Psi(M)}\right)_{/\sim_G}$$

where $(M, x) \sim_G (M', x')$ *if and only if* $\exists w \in W_G,\ M = w(M')$ *and* $x = w(x')$.

Proof: Since the abstract work has been done in part 2, we only have to use 2.10 iii) together with the next lemma. □

Lemma 4.26 *EP induces an isomorphism* $\overline{\mathcal{R}_{\mathbb{C}}(G)}_{\Psi(G)} \xrightarrow{\sim} F^{d(G)}\mathcal{K}_{\mathbb{C}}(G)$.

Proof: Fix $\theta \in \Theta^{disc}(G)$ and recall the morphism

$$\mathcal{R}(G,\theta)^{\mathcal{N}_Y} \xrightarrow{EP} \mathcal{K}_{\mathbb{C}}(G)_Y$$

from Proposition 4.18 and with the same notations. From this proposition it is surjective. On another hand it is clear that induced elements are in its kernel and actually 4.20 shows that *ker EP* $= \mathcal{R}_{\mathbb{C}I}(G,\theta)^{\mathcal{N}_Y}$. Now define $\mathcal{R}_{\mathbb{C}}(G,Y)$ as the subspace of $\mathcal{R}_{\mathbb{C}}(G)$ generated by those irreducible $\pi \in Irr(G)$ such that $\theta(\pi) \in Y$. We clearly have an isomorphism $\overline{\mathcal{R}_{\mathbb{C}}}(G,\theta)^{\mathcal{N}_Y} \simeq \overline{\mathcal{R}_{\mathbb{C}}}(G,Y)_{\Psi(G)}$ so that *EP* turns out to induce an isomorphism $\overline{\mathcal{R}_{\mathbb{C}}(G,Y)}_{\Psi(G)} \xrightarrow{\sim} F^{d(G)}\mathcal{K}_{\mathbb{C}}(G,Y)$ which is more precise than what we had claimed. □

4.27 *Remark on the "integrality" of the properties of* $\mathcal{K}$*:* The properties of $\mathcal{K}(G)$ have been set up here for the complexified $\mathcal{K}_{\mathbb{C}}(G)$. Of course they all remain valid if we consider only $\mathcal{K}(G) \otimes \mathbb{Q}$, since we only had to kill some torsion problems. It may be interesting to have a "bound" on these torsion problems, *i.e.* find some N_G such that these properties carry on for $\mathcal{K}(G) \otimes \mathbb{Z}[\frac{1}{N_G}]$. For that, we list up the points which required to kill the torsion:

i) Each time we had to use projector property of 2.5 and thus to invert the P_d's. For this, it is sufficient to require $|W_G|$ to be invertible. More precisely Proposition 2.5 may be rephrased for the ground ring $\mathbb{Z}[\frac{1}{|W_G|}]$ in the same words as for $\mathbb{C}$. In particular the various $\mathcal{R}^d(G) \otimes \mathbb{Z}[\frac{1}{|W_G|}]$ are direct factors in $\mathcal{R}(G) \otimes \mathbb{Z}[\frac{1}{|W_G|}]$ so that quasi-discrete characters are killed just as in the "complexified" case and don't contribute to $\mathcal{K}(G) \otimes \mathbb{Z}[\frac{1}{|W_G|}]$.
ii) In the proof of 3.4 (see below Lemma 3.7). It only requires that $|G/G^0Z|$ be invertible.
iii) In the first displayed isomorphism of the proof of 4.18. The problem is the Schur indices m_i of Appendix B point iv), but they must divide the order of the Galois group, hence in our case it is sufficient that $|G/G^0Z|$ be invertible.

Now put $N_G = |W_G|[G : G^0Z]$: we see that 4.9 is valid with $\mathbb{C}$ replaced by $\mathbb{Z}[\frac{1}{|W_G|}]$, while 4.18 remains true on $\mathbb{Z}[\frac{1}{N_G}]$. Hence *Rk* is an embedding of $\mathcal{K}(G) \otimes \mathbb{Z}[\frac{1}{N_G}]$ into $\overline{\mathcal{H}}(G)$, and 4.21 carries on. Remark that *on the contrary* our arguments do not insure the validity of 4.22 on $\mathcal{K}(G) \otimes \mathbb{Z}[\frac{1}{N_G}]$. We can only say that the quotient of $\mathcal{K}(G) \otimes \mathbb{Z}[\frac{1}{N_G}]$ by the submodule generated by compactly induced projectives is a torsion abelian group and this is enough to get 4.23. Now replacing N_G by the lowest common multiple of the N_M's for $M < G$, we see that the structure result is valid on $\mathbb{Z}[\frac{1}{N_G}]$.

5 Topological filtration on $\overline{\mathcal{H}}(G)$

This filtration has already been described in [18] (it is named there "dévissage"). Our treatment here will follow that of the previous section.

5.1 *Categorical cocenter:* (see [18]) Let $\mathcal{A}$ be an abelian, locally notherian (in the sense of [19, p.356]) category such that the subcategory $\mathcal{A}^f$ of notherian objects be a small category. The cocenter $\overline{\mathcal{H}}(\mathcal{A})$ is defined analogously to 1.3 as the quotient of the free abelian group on the symbols (V, u) where V is a noetherian object of $\mathcal{A}$ and $u \in End_{\mathcal{A}}(V)$ modulo the relations

- $(V, u) = (V_1, u_1) + (V_2, u_2)$ for any short exact sequence
$$0 \longrightarrow V_1 \xrightarrow{f} V \xrightarrow{g} V_2 \longrightarrow 0$$
such that $fu_1 = uf$ and $gu = u_2 g$.
- $(V, u) + (V, v) = (V, u + v)$
- $(V, fg) = (V', gf)$ if $f : V' \longrightarrow V$ and $g : V \longrightarrow V'$

It is a covariant object: if $\mathcal{A} \longrightarrow \mathcal{A}'$ is an exact noetherian functor (*i.e.* mapping notherian objects on noetherian objects), then it induces a morphism $\overline{\mathcal{H}}(\mathcal{A}) \longrightarrow \overline{\mathcal{H}}(\mathcal{A}')$.

If $\mathfrak{Z}(\mathcal{A})$ is the categorical center of $\mathcal{A}$, we let it act on $\overline{\mathcal{H}}(\mathcal{A})$ by $z(V, u) := (V, zu)$ and get a structure of $\mathfrak{Z}(\mathcal{A})$-module. In particular, if $\mathcal{A}$ is R-linear for some commutative ring R, $\overline{\mathcal{H}}(\mathcal{A})$ is a R-module. If $\mathcal{A}'$ is a full abelian subcategory of $\mathcal{A}$ then the morphism $\overline{\mathcal{H}}(\mathcal{A}') \longrightarrow \overline{\mathcal{H}}(\mathcal{A})$ associated to the inclusion functor is $\mathfrak{Z}(\mathcal{A})$-equivariant.

If $\mathcal{A}$ is the category of modules of some ring A, we shall abbreviate $\mathcal{H}(A) := \mathcal{H}(A - Mod)$.

5.2 *Examples:*

- For $\mathcal{A} = Mod(G)$ this definition coincides with that of 1.3 since the category $Mod(G)$ is of finite cohomological dimension.
- Assume A is a split[2] finite dimensional $\mathbb{K}$-algebra. Then it is easy to see that $\overline{\mathcal{H}}(A) = \mathcal{R}(A) \otimes_{\mathbb{Z}} \mathbb{K}$. Note it is clear when A is semi-simple since one is reduced to the case $\mathcal{M}_n(\mathbb{K})$. In the general case, one has to express any (V, u) as a sum of (π_i, λ_i) with π_i irreducible and $\lambda_i \in \mathbb{K}$. To do this we may proceed by induction on the length of V, noting that the socle of V must be stable by u so that $(V, u) = (\mathrm{Socle}(V), u) + (V/\mathrm{Socle}(V), u)$, and using the desired property for the semisimple modules.

[2] That is, $End_A(V) = \mathbb{K}$ for any simple A-module V

- In the case where A is no more assumed to be split, we may choose a finite Galois extension $\mathbb{K}'$ splitting A and let $Gal(\mathbb{K}'/\mathbb{K})$ act diagonally on $\overline{\mathcal{H}}(A \otimes \mathbb{K}') = \mathcal{R}_{\mathbb{C}}(A \otimes \mathbb{K}') \otimes \mathbb{K}'$. Then we have

$$\overline{\mathcal{H}}(A) \simeq (\mathcal{R}_{\mathbb{C}}(A \otimes \mathbb{K}') \otimes \mathbb{K}')^{Gal(\mathbb{K}'/\mathbb{K})}$$

  as we can easily see by reducing first to the semi-simple case. This is also isomorphic as a $\mathbb{K}$-vector space to $\bigoplus_\pi \mathcal{Z}(End_A(\pi))$ where the π's are the isomorphism classes of simple A-modules.

Complete proofs of the latter facts are given in A.5 for finite categories.

5.3 *The filtration on* $\overline{\mathcal{H}}(G)$*:* As in 4.2 we define $F^i\overline{\mathcal{H}}(G) := im\,(\overline{\mathcal{H}}(Mod_i(G)) \longrightarrow \overline{\mathcal{H}}(G))$ and note $G^*\overline{\mathcal{H}}(G)$ the graded $\mathfrak{Z}(G)$-module associated to this $\mathfrak{Z}(G)$-filtration. We will use the notations $\overline{\mathcal{H}}(G,Y)$, $\overline{\mathcal{H}}(G,\mathbb{C}(Y),\theta_Y)$, $\overline{\mathcal{H}}(G,\mathbb{C},\theta)$ for the cocenters of the respective (and already defined) categories $Mod(G,Y)$, etc... The first trick is the following analogue of 4.3:

Lemma 5.4 $G^d\overline{\mathcal{H}}(G) = \sum_{d(Y)=d} i_*^Y(G^d\overline{\mathcal{H}}(G,Y))$.

Proof: The arguments are the same as for 4.3. □

As in 4.4, we have a "generic fiber" morphism $\overline{\mathcal{H}}(G,Y) \longrightarrow \overline{\mathcal{H}}(G,\mathbb{C}(Y),\theta_Y)$. Note that the left hand side is a $\mathbb{C}[Y]$-module whereas the right hand side is $\mathbb{C}(Y)$-vector space. The analogue of 4.5 is:

Proposition 5.5 *The "generic fiber" morphism induces a monomorphism*

$$G^{d(Y)}\overline{\mathcal{H}}(G,Y) \hookrightarrow \overline{\mathcal{H}}(G,\mathbb{C}(Y),\theta_Y)$$

and an isomorphism:

$$G^{d(Y)}\overline{\mathcal{H}}(G,Y) \otimes_{\mathbb{C}[Y]} \mathbb{C}(Y) \xrightarrow{\sim} \overline{\mathcal{H}}(G,\mathbb{C}(Y),\theta_Y)$$

Proof: We need an analogue to [1, IX.(6.1)] with respect to $\overline{\mathcal{H}}$. Here is the result we will prove ($\mathcal{A}$ is again a locally notherian abelian category)

Proposition 5.6 *Let $\mathcal{S}$ be a localizing subcategory (in the sense of [19]) of $\mathcal{A}$ such that the quotient $\mathcal{A}/\mathcal{S}$ is locally finite (i.e is inductively generated by finite length objects, see [19, p.356]), and assume $\mathcal{A}$ and $\mathcal{A}/\mathcal{S}$ are $\mathfrak{Z}$-finite and $\mathbb{Q}$-linear (see appendix), then the sequence*

$$\overline{\mathcal{H}}(\mathcal{S}) \longrightarrow \overline{\mathcal{H}}(\mathcal{A}) \longrightarrow \overline{\mathcal{H}}(\mathcal{A}/\mathcal{S})$$

is exact.

Proof: See appendix. □

Now we may apply this to our situation: $\mathcal{A}$ is the category of all left modules over $\mathcal{H}_{\mathfrak{s}} \otimes_{\mathfrak{Z}_{\mathfrak{s}}} \mathbb{C}[Y]$, $\mathcal{S}$ is the category of all $\mathbb{C}[Y]$-torsion modules in $\mathcal{A}$. Then $\mathcal{S}$ is a localizing subcategory ([19, III.5] for example) so that $\mathcal{A}/\mathcal{S}$ is equivalent (see [1, IX.6]) to the category of all left modules over the finite dimensional $\mathbb{C}(Y)$-algebra $\mathcal{H}_{\mathfrak{s}} \otimes_{\mathfrak{Z}_{\mathfrak{s}}} \mathbb{C}(Y)$, hence is locally finite. Moreover the quotient functor is just tensorization by $\mathbb{C}(Y)$.

Thus we get the monomorphism and the injectivity of the isomorphim of 5.5. As for the surjectivity, it is quite clear since for any finitely generated object of $\mathcal{A}$ we have:

$$End_{\mathcal{A}/\mathcal{S}}(V \otimes \mathbb{C}(Y)) \simeq \mathbb{C}(Y) \otimes_{\mathbb{C}[Y]} End_{\mathcal{A}}(V).$$

$\square$

Now we shall apply the same discussion as for the topological filtration of $\mathcal{K}$, after noting that Corollary 4.8 applies word for word to the case of $\overline{\mathcal{H}}$:

Theorem 5.7 *The filtrations $\overline{\mathcal{H}}^{\bullet}(G)$ and $F^{\bullet}\overline{\mathcal{H}}(G)$ coincide.*

Proof: As a matter of fact, Lemma 4.10 is still valid as well as its proof, hence we get as in Corollay 4.11: $F^{\bullet}\overline{\mathcal{H}}(G) \subset \overline{\mathcal{H}}^{\bullet}(G)$.

Now we can use the fact that $\overline{\mathcal{H}}(G, \mathbb{C}, \theta) \simeq \mathcal{R}_{\mathbb{C}}(G, \mathbb{C}, \theta)$ insured by 5.2 to get the following equality:

$$\Theta(G) \setminus (Y_{d(G)} \cup \cdots \cup Y_d) = \{\theta \in \Theta(G), \quad A_d = 0 \text{ on } \overline{\mathcal{H}}(G, \mathbb{C}, \theta)\} \ .$$

As a consequence we can repeat the proofs of 4.13, 4.14, etc... with $\mathcal{R}_{\mathbb{C}}$ and $\mathcal{K}_{\mathbb{C}}$ replaced by $\overline{\mathcal{H}}$ (note that we don't have any more torsion problems) to get the theorem as well as the following decomposition

$$G^d\overline{\mathcal{H}}(G) = \bigoplus_{Y \subset Y_d, d(Y)=d} i_*^Y(G^d\overline{\mathcal{H}}(G, Y)) \ . \tag{5.8}$$

$\square$

5.9 *Level d(G):* As in the case of $\mathcal{K}_{\mathbb{C}}(G)$ we begin with the bottom of the filtration. Again we write $\overline{\mathcal{H}}(G)_Y := i_*^Y\overline{\mathcal{H}}(G, Y)$ and from 5.8 we get:

$$F^{d(G)}\overline{\mathcal{H}}(G) \simeq \bigoplus_{Y \in \Theta^{disc}(G)/\Psi(G)} \overline{\mathcal{H}}(G)_Y \ . \tag{5.10}$$

Hence we fix a $\Psi(G)$-orbit Y in $\Theta^{disc}(G)$ and an element $\theta \in Y$ and, as in the case of 4.18, we write $\mathcal{N}_Y$ for the normalizer of θ in $\Psi(G)$. The analogue of 4.18 is

Proposition 5.11 *There exists an isomorphism* ϕ : $G^d\overline{\mathcal{H}}(G,Y) \xrightarrow{\sim} (\mathcal{R}_{\mathbb{C}}(G,\theta)\otimes\mathbb{C}[G/G^0])^{\mathcal{N}_Y}$ *such that the following diagram is commutative:*

$$\begin{tikzcd}
(\mathcal{R}_{\mathbb{C}}(G,\theta)\otimes\mathbb{C}[G/G^0])^{\mathcal{N}_Y} \arrow[r, "\overline{EP}"] & \overline{\mathcal{H}}(G)_Y \\
G^d\overline{\mathcal{H}}(G,Y) \arrow[u, "\phi"] & \overline{\mathcal{H}}(G,Y) \arrow[l, two heads] \arrow[u, "i_*^Y"']
\end{tikzcd}$$

Proof: The proof will follow that of 4.18 but we have a little work to do. Assume $Y' \xrightarrow{\phi} Y$ is a Galois étale covering (and not only an étale morphism) with Galois group $\Gamma(Y'/Y)$ then we claim that the morphism induced by tensorization

$$\begin{array}{rrcl} \tau: & \overline{\mathcal{H}}(G,Y) & \to & \overline{\mathcal{H}}(G,Y') \\ & [V,u] & \mapsto & [V\otimes_{\mathbb{C}[Y]}\mathbb{C}[Y'],u\otimes 1] \end{array}$$

provides an isomorphism

$$G^{d(Y)}\overline{\mathcal{H}}(G,Y) \xrightarrow{\sim} G^{d(Y)}\overline{\mathcal{H}}(G,Y')^{\Gamma(Y'/Y)}.$$

(Recall that $\overline{\mathcal{H}}(G,Y')$ by definition is $\overline{\mathcal{H}}(\mathcal{H}^{\mathfrak{s}}\otimes_{\mathfrak{Z}(G),\phi^*}\mathbb{C}[Y']))$. First we have to precise the action of Γ on $\overline{\mathcal{H}}(G,Y')$: this Galois group acts by automorphisms of $\mathbb{C}[Y]$-algebra on $\mathbb{C}[Y']$, hence by automorphisms $1\otimes\gamma$ of algebra on $\mathcal{H}_{\mathfrak{s}}\otimes_{\mathfrak{Z}_{\mathfrak{s}}}\mathbb{C}[Y']\simeq\mathcal{H}_{\mathbb{C}[Y]}\otimes_{\mathbb{C}[Y]}\mathbb{C}[Y']$. Hence this action extends to an action by automorphisms on the category $Mod(G,Y')$ and then induces an action on $\overline{\mathcal{H}}(G,Y')$ by $\gamma[W,v]:=[W^\gamma,v]$. Recall $\overline{\mathcal{H}}(G,Y')$ carries a natural structure of $\mathbb{C}[Y']$-module; it turns out that the action of Γ is Γ-semilinear (*i.e.* $\gamma(zX)=z^\gamma\gamma(X), \quad \forall z\in\mathbb{C}[Y'], X\in\overline{\mathcal{H}}(G,Y')$).

Since $\mathbb{C}[Y']$ is a finitely generated $\mathbb{C}[Y]$-module, the forgetful functor $Mod(G,Y') \longrightarrow Mod(G,Y)$ is well defined (*i.e.* carries f.g. modules on f.g. modules) and induces the map

$$\begin{array}{rrcl} \kappa: & \overline{\mathcal{H}}(G,Y') & \to & \overline{\mathcal{H}}(G,Y) \\ & [W,v] & \mapsto & [W,v] \end{array} .$$

Now from the fact that $\mathbb{C}[Y']$ is free over $\mathbb{C}[Y]$ of rank $|\Gamma|$ we get

$$\kappa\circ\tau = |\Gamma|\,\mathrm{Id} \ .$$

And from the isomorphism $W\otimes_{\mathbb{C}[Y]}\mathbb{C}[Y']\simeq\bigoplus_{\gamma\in\Gamma}W^\gamma$ for any $W\in Mod(G,Y')$ (follows from Galois theory) we get

$$\tau\circ\kappa = Tr_\Gamma \ .$$

Where $Tr_\Gamma[W,v]:=\sum_{\gamma\in\Gamma}\gamma[W,v]$. The two last displayed formulas imply that τ induces an isomorphism $\overline{\mathcal{H}}(G,Y)\xrightarrow{\sim}\overline{\mathcal{H}}(G,Y')^\Gamma$ and the claim follows from the fact that τ and κ obviously preserve the respective topological filtrations.

Now we shall apply this to the case $Y' = \Psi(G) \xrightarrow{\phi_\theta} Y$ already defined in the proof of 4.18. First we have to compute $G^{d(Y)}\overline{\mathcal{H}}(G, Y')$; we will use the Proposition A.6 in the case where $\mathcal{A}_f = Mod(G, Y')$, $\mathcal{S}$ is the category of $\mathbb{C}[Y']$-torsion modules, so that $(\mathcal{A}/\mathcal{S})_f$ is equivalent to the category $Mod(G, \mathbb{C}(Y'), \theta_{Y'})$ (see the proof of 4.18 for definition of $\theta_{Y'}$). Proposition A.6 says that

$$G^{d(Y)}\overline{\mathcal{H}}(G, Y') \simeq \bigoplus_{\mathcal{V} \in Irr(G, \mathbb{C}(Y'), \theta_{Y'})} \overline{E}(\mathcal{V})$$

where

$$\overline{E}(\mathcal{V}) = \sum_{\substack{V \in Mod(G, Y') \\ V \otimes \mathbb{C}(Y') \simeq \mathcal{V}}} im\,(End_{G, \mathbb{C}[Y']}(V) \longrightarrow \overline{End_{G, \mathbb{C}(Y')}(\mathcal{V})})\ .$$

In our present case we know from 4.18 that $Mod(G, \mathbb{C}(Y'), \theta_{Y'})$ is equivalent to a *split* finite dimensional algebra over $\mathbb{C}(Y')$ hence $End_{G, \mathbb{C}(Y')}(\mathcal{V}) = \mathbb{C}(Y')$ and $\overline{E}(\mathcal{V})$ is integral over $\mathbb{C}[Y']$. Since the latter is integrally closed in $\mathbb{C}(Y')$, we get:

$$G^{d(Y)}\overline{\mathcal{H}}(G, Y') \simeq \mathcal{R}(G, \mathbb{C}(Y'), \theta_{Y'}) \otimes \mathbb{C}[Y']\ .$$

Now using the discussion around 4.19, we can state that the map

$$\begin{aligned} \mathcal{R}(G, \theta) \otimes \mathbb{C}[G/G^0] &\to G^{d(Y)}\overline{\mathcal{H}}(G, Y') \\ \pi \otimes \lambda &\mapsto [\pi \otimes \mathbb{C}[G/G^0], \lambda] \end{aligned} \tag{5.12}$$

where we let G act diagonally on $\pi \otimes \mathbb{C}[G/G^0]$ and λ act on the right term of the tensor product, is an isomorphism. Moreover, if we let $\mathcal{N}_Y$ act diagonally on $\mathcal{R}(G, \theta) \otimes \mathbb{C}[G/G^0]$ and by Galois action on $\mathcal{H}(G, Y')$, this isomorphism turns out as in 4.18 to be $\mathcal{N}_Y$-equivariant. Hence the isomorphism announced in the proposition. The rest of the proof is exactly as for 4.18. □

5.13 *Proof of Theorem 3.4:* First we deal with the kernel assertion. Recall it remains only to prove $\mathcal{R}_I(G) \otimes \mathbb{C}[G/G^0] \subset ker\ \overline{EP}$ but if $\pi = i_M^G(\sigma)$ in $\mathcal{R}_{\mathbb{C}}(G)$ then $\overline{EP}(\pi \otimes \lambda) = i_M^G([\sigma \otimes \mathbb{C}[G/G^0]_{|M}, \lambda])$ and by 5.7 must be zero.

Now we turn to the image assertion of 3.4. But from 5.7 again, we have $\overline{\mathcal{H}}^R(G) = F^{d(G)}\overline{\mathcal{H}}(G)$ hence it is clear that $im\ \overline{EP} \subset \overline{\mathcal{H}}^R(G)$ and the surjectivity follows from the previous proposition. □

Now we give a way to explicit $\overline{\mathcal{H}}(G)$ similar to that of Theorem 4.25:

Theorem 5.14 *There exists an isomorphism of Hopf systems:*

$$G^{\bullet}\overline{\mathcal{H}} \xrightarrow{\sim} \mathfrak{H}\left(M \mapsto (\overline{\mathcal{R}}(M) \otimes \mathbb{C}[M/M^0])_{\Psi(M)})\right).$$

In particular, let d be an integer $d(G) \leqslant d \leqslant d(M_0)$*, then:*

$$G^d\overline{\mathcal{H}}(G) \simeq \left(\bigoplus_{d(M)=d} \left(\overline{\mathcal{R}_{\mathbb{C}}}(M) \otimes \mathbb{C}[M/M^0]\right)_{\Psi(M)}\right)_{/\sim}$$

where $(M, x) \sim (M', x')$ *if and only if* $\exists w \in W_G,\ M = w(M')$*. Moreover the rank map* $G^d\, Rk : G^d\mathcal{K}_{\mathbb{C}}(G) \longrightarrow G^d\overline{\mathcal{H}}(G)$ *is just induced by the map* $\pi \mapsto \pi \otimes 1$.

Proof: Once again, we use the formalisation of part 2, more specifically 2.10 ii) together with the fact that $\overline{EP}$ induces an isomorphism $(\overline{\mathcal{R}}(G) \otimes \mathbb{C}[G/G^0])_{\Psi(G)} \xrightarrow{\sim} F^{d(G)}\overline{\mathcal{H}}(G)$ which is proven as Lemma 4.26. □

Remark: This property enables to get an expression similar to that of 4.25 for the whole $\overline{\mathcal{H}}(G)$ as a vector space. Anyway, we won't write it down since this expression won't be compatible with induction and restriction in the sense that the Hopf system $\overline{\mathcal{H}}(.)$ is *not* isomorphic to the "trivial" Hopf system obtained from the family $(\overline{\mathcal{R}}(M) \otimes \mathbb{C}[M/M^0])_{\Psi(M)}$, $M < G$ as in Lemma 2.8. The reason for this is the dependence of induction on the parabolic subgroup (property ii) of 2.5). Understanding the behaviour of induction in the dictionnary of the right hand side above may be an interesting question, somewhat mysterious –to the author– since it is connected to the upper filtration of $R(G)$, itself being connected to a good knowledge of the character-trace of a Jacquet module. It is likely that, whereas the lower filtration of $\mathcal{R}(G)$ has turned out to be linked with "Harish-Chandra's filtration w.r.t ellipticity" (see 3.2), the upper one be linked to "Deligne-Casselman-Clozel's filtration w.r.t compactness" (as in [15]).

6 Representations of G over an arbitrary extension of $\mathbb{C}$

6.1 The isomorphism 4.5, the proof of 4.9 and 4.25 were motivations to investigate some aspects of the representation theory of G on a field extension $\mathbb{K}$ of $\mathbb{C}$. We fix an algebraic closure $\overline{\mathbb{K}}$ of $\mathbb{K}$ and put $\Gamma_{\mathbb{K}} = Gal(\overline{\mathbb{K}}/\mathbb{K})$. The first easy remarks are:

i) Defining $\mathfrak{Z}(G, \mathbb{K})$ as the center of $Mod(G, \mathbb{K})$, it is clear from the property $\mathfrak{Z}(G, \mathbb{K}) = \varprojlim \mathcal{Z}(\mathcal{H}(G, H) \otimes \mathbb{K})$ (see [7, 1.5 ii)]) where the limit is taken over a system of H's satisfying the condition $(*)$ of the introduction and from the isomorphism $\mathcal{Z}(\mathcal{H}(G, H) \otimes \mathbb{K}) \simeq$

$\mathcal{Z}(\mathcal{H}(G, H)) \otimes \mathbb{K}$ (see [16, (2.38)] with the notheriannity of [7, 3.12]) that

$$\mathfrak{Z}(G, \mathbb{K}) \simeq \prod_{\mathfrak{s}} \mathfrak{Z}_{\mathfrak{s}} \otimes_{\mathbb{C}} \mathbb{K}.$$

In other words, Bernstein's decomposition of the category $Mod(G, \mathbb{K})$ into blocs remains parameterized by the same set of inertia classes, *i.e.* $(M, \rho) \mapsto (M, \rho \otimes_{\mathbb{C}} \mathbb{K})$ yields a bijection between inertia classes of cuspidal pairs and the equivalences of categories of 4.1 and $(*)$ carry on over $\mathbb{K}$. In particular every simple $\mathbb{K}$-module is $\mathbb{K}$-admissible.

ii) $\mathbb{K}$-*infinitesimal characters:* Consider $\Theta(G)$ as a reduced scheme over $\mathbb{C}$. Let $\Theta(G, \overline{\mathbb{K}})$ be the set of all $\overline{\mathbb{K}}$-points of $\Theta(G)$. As usual, such a point is said to be *defined over* $\mathbb{K}'$ if it is actually a $\mathbb{K}'$-point of $\Theta(G)$. Since $\Theta(G)$ is a union of algebraic schemes, each $\overline{\mathbb{K}}$-point is defined over a finite extension $\mathbb{K}'$ of $\mathbb{K}$. Define now the set of $\mathbb{K}$-infinitesimal characters $\Theta(G, \mathbb{K})$ to be the set of $\Gamma_{\mathbb{K}}$-orbits in $\Theta(G, \overline{\mathbb{K}})$ (in other words, the set of closed points in the topological space $\Theta(G) \times_{Spec_{\mathbb{C}}} Spec\,(\mathbb{K})$). We justify such a terminology: suppose first that $\mathbb{K} = \overline{\mathbb{K}}$ then by direct analogy with the complex case, we can attach to each simple $G, \mathbb{K}$-module an "infinitesimal character" $\theta(\pi) \in \Theta(G, \overline{\mathbb{K}})$ and we have the decomposition $\mathcal{R}(G, \mathbb{K}) = \bigoplus_{\theta \in \Theta(G, \overline{\mathbb{K}})} \mathcal{R}(G, \mathbb{K}, \theta)$. Now for any $\mathbb{K}$, let $\pi \in Irr_{\mathbb{K}}(G)$. Then by admissibility $\pi \otimes \overline{\mathbb{K}}$ is a finite length $G, \overline{\mathbb{K}}$-module, hence affords a finite subset $\theta(\pi) \subset \Theta(G, \overline{\mathbb{K}})$ of infinitesimal characters. Explicitly, the centralizer $End_G(\pi) = \mathbb{D}$ is a finite dimensional division algebra over $\mathbb{K}$ so that the action of the center $\mathfrak{Z}(G, \mathbb{K})$ on π gives a morphism $\kappa : \mathfrak{Z}(G, \mathbb{K}) \longrightarrow \mathcal{Z}(\mathbb{D})$ and $\theta(\pi)$ is the set of various embeddings $im\,(\kappa) \hookrightarrow \mathbb{K}$. This set is a $\Gamma_{\mathbb{K}}$-orbit in $\Theta(G, \overline{\mathbb{K}})$ so that we get a well defined "infinitesimal character map" $Irr_{\mathbb{K}}(G) \longrightarrow \Theta(G, \mathbb{K})$ and a decomposition

$$\mathcal{R}(G, \mathbb{K}) = \bigoplus_{\theta \in \Theta(G, \mathbb{K})} \mathcal{R}(G, \mathbb{K}, \theta)$$

which justifies our definition.

iii) *Algebra attached to an infinitesimal character:* Let θ be a $\mathbb{K}$-infinitesimal character, $\kappa \in \theta$ a point in $\Theta(G, \overline{\mathbb{K}})$ viewed as a map $\mathfrak{Z}(G, \mathbb{K}) \longrightarrow \overline{\mathbb{K}}$ and $\mathfrak{s}$ the unique inertia class such that $\kappa_{|\mathfrak{Z}_{\mathfrak{s}}} \neq 0$. Define $\mathcal{H}_\theta := \mathcal{H}^{\mathfrak{s}}(G, \mathbb{K})/(ker\ \kappa)$. This is a finite dimensional $\mathbb{K}$-algebra whose isomorphism class depends only on θ and by the equivalences of 4.1, $\mathcal{R}(G, \mathbb{K}, \theta)$ identifies with the Grothendieck group $\mathcal{R}(\mathcal{H}_\theta)$ of finite length $\mathcal{H}_\theta$-modules. This enables to define the *splitting field of an infinitesimal character* as such a one for this algebra. Note that even if θ is single point in $\Theta(G, \overline{\mathbb{K}})$ (and consequently θ is defined over $\mathbb{K}$), it need not be split over $\mathbb{K}$.

iv) *Base change*: let $\theta_{\mathbb{K}} \in \Theta(G, \mathbb{K})$ and $\mathbb{K}' \subset \overline{\mathbb{K}}$ a finite extension of $\mathbb{K}$. Then writing $\theta_{\mathbb{K}} = \cup_i \theta^i_{\mathbb{K}'}$ as a finite union of $\Gamma_{\mathbb{K}'}$-orbits, the base change from $\mathbb{K}$ to $\mathbb{K}'$ gives a morphism

$$\mathcal{R}(G, \mathbb{K}, \theta) \xrightarrow{\otimes \mathbb{K}'} \bigoplus_i \mathcal{R}(G, \mathbb{K}', \theta^i_{\mathbb{K}'})$$

which, thanks to the algebra interpretation in the preceding paragraph and Appendix B, turns out to be a monomorphism. For example, when θ is a single point orbit we get a monomorphism $\mathcal{R}(G, \mathbb{K}, \theta) \hookrightarrow \mathcal{R}(G, \overline{\mathbb{K}}, \theta)$ which is an isomorphism if θ is split over $\mathbb{K}$. As a (almost) particular case, suppose θ is a point of $\Theta(G, \overline{\mathbb{K}})$ defined over $\mathbb{C}$, then the tensorization $\pi \mapsto \pi \otimes_{\mathbb{C}} \mathbb{K}$ gives an isomorphism $\mathcal{R}(G, \mathbb{C}, \theta) \simeq \mathcal{R}(G, \mathbb{K}, \theta)$. This is a consequence of the absolute irreducibility of elements in $Irr(G, \mathbb{C})$.

v) *Restriction of scalars:* Let $\mathbb{K}' \subset \overline{\mathbb{K}}$ be a finite extension of $\mathbb{K}$, then any finite length $G, \mathbb{K}'$ module may be viewed as a finite length $G, \mathbb{K}$-module. Let $\theta_{\mathbb{K}'}$ be a $\mathbb{K}'$-infinitesimal character and $\theta_{\mathbb{K}}$ be the $\Gamma_{\mathbb{K}}$-orbit containing $\theta_{\mathbb{K}'}$, we get in this fashion a morphism of "restriction of scalars":

$$R_{\mathbb{K}'/\mathbb{K}} : \mathcal{R}(G, \mathbb{K}', \theta_{\mathbb{K}'}) \longrightarrow \mathcal{R}(G, \mathbb{K}, \theta_{\mathbb{K}}) \ .$$

Thanks to the algebra interpretation and Appendix B, this morphism turns out to be of finite cokernel. More precisely for any $\sigma \in Irr_{\mathbb{K}}(G)$ there exists a $\pi \in Irr_{\mathbb{K}'}(G)$ and an integer n_π such that $R_{\mathbb{K}'/\mathbb{K}}[\pi] = n_\pi[\sigma]$.

vi) Properties: It is rather clear that induction and restriction commute with base change and restriction of scalars. In particular, let M be a standard Levi subgroup of G, Lemma 4.7 shows that T_M stabilizes $\mathcal{R}(G, \mathbb{K}, \theta_{\mathbb{K}})$ for any $\theta_{\mathbb{K}}$, and we have $R_{\mathbb{K}'/\mathbb{K}} T_M = T_M R_{\mathbb{K}'/\mathbb{K}}$ and $- \otimes_{\mathbb{K}} \mathbb{K}' \circ T_M = T_M \circ - \otimes_{\mathbb{K}} \mathbb{K}'$.

vii) *Discrete $\mathbb{K}$-infinitesimal characters:* Langlands' parametrization of irreducible representations no longer makes sense here, since square-integrable, unitary, tempered are not defined. However, the word "discrete" can be given a sense following [23] and [6]: $x \in \mathcal{R}(G, \mathbb{K})$ is discrete if and only if it is not induced, *i.e* $x \notin \mathcal{R}_I(G, \mathbb{K})$. Turning to $\mathbb{K}$-infinitesimal characters, we will distinguish between $\mathbb{K}$-*discrete* and $\mathbb{K}$-*quasi-discrete* characters, the definition being the same as in 4.12. Note however that we cannot yet use the criterion of vanishing of the operator $A_{d(G)}$ since 2.6 i) is not known for $\mathbb{K}$-representations. One of our aims here is to determine the set $\Theta^{disc}(G, \mathbb{K})$ of $\mathbb{K}$-discrete infinitesimal characters.

viii) *Linear independence of characters:* Suppose θ is a single-point $\Gamma_{\mathbb{K}}$-orbit (*i.e.* a $\mathbb{K}$-point) of $\Theta(G, \overline{\mathbb{K}})$ and that θ is split over $\mathbb{K}$ (see iii)). Identify $\mathcal{R}(G, \mathbb{K}, \theta)$ with the Grothendieck group of $\mathcal{H}_\theta = \mathcal{H}^s(G) \otimes_{\mathfrak{Z}(G),\theta} \mathbb{K}$ via the equivalence of category of 4.1. We may attach to each irreducible, and hence to each element in $\mathcal{R}(G,\theta,\mathbb{K})$ a trace function on $\mathcal{H}_\theta$. Then by [16, 3.41], we have

$$\forall x \in \mathcal{R}(G,\mathbb{K},\theta),\ \exists f \in \mathcal{H}_\theta,\ \ Tr_x(f) \neq 0$$

ix) *Varieties attached to $\mathbb{K}$-infintesimal characters:* Let $\theta_\mathbb{K} \in \Theta(G,\mathbb{K})$, $\kappa \in \theta_\mathbb{K}$, and $\mathfrak{s}$ the unique inertia class such that $\kappa(\mathfrak{Z}_\mathfrak{s}) \neq 0$. The kernel of κ in $\mathfrak{Z}_\mathfrak{s}$ is a prime ideal which depends only on $\theta_\mathbb{K}$ (and not on the choice of $\kappa \in \theta_\mathbb{K}$), hence is the defining ideal of an irreducible $\mathbb{C}$-subvariety $Y = Y(\theta_\mathbb{K})$ of $\Theta_\mathfrak{s}$.

Many problems arise when we quit the ground field $\mathbb{C}$. Especially for all the statements whose published proofs make heavy use of Langlands' theory or unitary arguments, etc.... Hence the following result, which may appear rather obvious actually requires some attention.

Theorem 6.2 *With all the foregoing notations:*

i) Let $M < G$ and $w \in W_G$ such that $w(M) < G$ then $i^G_{w(M)} \circ w = i^G_M$ on $\mathcal{R}(M,\mathbb{K})$.

ii) For a $\overline{\mathbb{K}}$-point θ of $\Theta(G)$, the following properties are equivalent:

(a) θ is $\overline{\mathbb{K}}$-discrete.

(b) θ is $\mathbb{K}'$-discrete if $\mathbb{K}'$ is a splitting field for θ.

(c) $Y(\theta) \subset \Theta^{disc}(G)$. (see ix)

(d) θ is a $\overline{\mathbb{K}}$-point of $\Theta^{disc}(G)$.

Moreover, if $\theta_\mathbb{K}$ is a discrete $\mathbb{K}$-infinitesimal character, then every $\kappa \in \theta_\mathbb{K}$ enjoys these properties.

Remark: ii) is generally false if we replace "discrete" by quasi-discrete, even if $\theta_\mathbb{K}$ is supposed to be $\mathbb{K}$-split.

Proof: (Begining): i) will be shown in 6.14. In ii) standard arguments give the equivalence of (a) and (b). The implication $(c) \Rightarrow (d)$ is a tautology. The implications $(b) \Rightarrow (c)$ and $(d) \Rightarrow (a)$ will be shown in 6.11. As for the last point, it is a consequence of the surjectivity (after killing the torsion) of restriction of scalars (see v)) and the commutation of this restriction with induction. □

6.3 *Aim:* Let $\theta_\mathbb{K}$ be a $\mathbb{K}$-infinitesimal character, we would like to relate $\mathcal{R}(G,\mathbb{K},\theta_\mathbb{K})$ with the $\mathcal{R}(G,\mathbb{C},\theta)$'s for θ describing the associated variety $Y = Y(\theta_\mathbb{K})$ as in ix). Roughly speaking, we would like to have an imbedding

$$\mathcal{R}(G,\mathbb{K},\theta_\mathbb{K}) \hookrightarrow \prod_{\theta\in Y} \mathcal{R}(G,\mathbb{C},\theta)$$

in order to move known properties of $\mathbb{C}$-modules to $\mathbb{K}$-modules. As is suggested by the injectivity of base change, we will assume that $\theta_\mathbb{K}$ is a single $\mathbb{K}$-point in $\Theta(G)$.

Lemma 6.4 *There exists an étale morphism $Y' \xrightarrow{\phi} Y$ of algebraic varieties such that:*

i) *The extension $\mathbb{C}(Y) \xrightarrow{\phi^*} \mathbb{C}(Y')$ splits the finite dimensional $\mathbb{C}(Y)$-algebra $\mathcal{H}(G) \otimes_{\mathfrak{Z}(G),\theta_{\mathbb{K}}} \mathbb{C}(Y)$.*
ii) *Y' is a smooth variety.*
iii) *The algebra $\mathcal{H}_{Y'} := \mathcal{H}^{\mathfrak{s}}(G) \otimes_{\mathfrak{Z}(G),\phi^*\circ\theta_{\mathbb{K}}} \mathbb{C}[Y']$ is locally free as a $\mathbb{C}[Y']$-module.*

Note that the only difficult thing to get is iii). We postpone the proof and derive some consequences. We fix such an étale morphism and write $\theta_{Y'}$ for the $\mathbb{C}(Y')$-infinitesimal character $\mathfrak{Z}(G) \xrightarrow{\theta_{\mathbb{K}}} \mathbb{C}[Y] \xrightarrow{\phi^*} \mathbb{C}(Y')$. We call $\mathbb{K}'$ the compositum $\mathbb{K}.\mathbb{C}(Y')$, then we have the following properties where all morphisms are given by tensorization (see 6.1 iv)):

- $\mathcal{R}(G, \mathbb{K}, \theta_{\mathbb{K}}) \longrightarrow \mathcal{R}(G, \mathbb{K}', \theta_{\mathbb{K}})$ is a monomorphism.
- $\mathcal{R}(G, \mathbb{C}(Y'), \theta_{Y'}) \overset{\sim}{\longrightarrow} \mathcal{R}(G, \mathbb{K}', \theta_{\mathbb{K}})$.

Concerning the *Aim* above, it is thus sufficient to deal with $\mathbb{K} = \mathbb{C}(Y')$ and $\theta_{\mathbb{K}} = \theta_{Y'}$.

6.5 *Definition of r:* Let $y \in Y'$, we want to define a kind of "reduction modulo y":

$$\mathcal{R}(G, \mathbb{C}(Y'), \theta_{Y'}) \xrightarrow{r_y} \mathcal{R}(G, \mathbb{C}, \phi(y)) .$$

To do that we use classical restriction morphisms in algebraic K-theory; we put $\mathcal{K}(G, Y') := \mathcal{K}(\mathcal{H}_{Y'})$ so that we have the analogue of 4.5

$$G^{d(Y)}\mathcal{K}(G, Y') \simeq \mathcal{R}(G, \mathbb{C}(Y'), \theta_{Y'}) .$$

Recall Y' is regular, thus for any $y \in Y'$, there exists a finite $\mathbb{C}[Y']$-locally free resolution of $\mathbb{C}_y$. Hence, if $V \in Mod(\mathcal{H}_{Y'})$, for any integer $k \geqslant 0$, the k^{th} Tor space $Tor_k^{\mathbb{C}[Y']}(V, \mathbb{C}_y)$ inherits a structure of $\mathcal{H}_{Y'} \otimes \mathbb{C}_y$-module (that doesn't depend on the locally free resolution), hence of G-module and lies in $Mod(G, \mathbb{C}, \phi(y))$. We thus define:

$$(6.6) \qquad \begin{array}{rl} r_y : & \mathcal{K}(G, Y') \to \mathcal{R}(G, \mathbb{C}, \phi(y)) \\ & [V] \mapsto \sum_k (-1)^k [Tor_k^{\mathbb{C}[Y']}(V, \mathbb{C}_y)] \end{array} .$$

Now, from iii) in the Lemma 6.4, we can give a more convenient way of calculating r_y. Let V be any $\mathcal{H}_{Y'}$-module, we can form an exact sequence:

$$\mathcal{H}_{Y'}^{n_{d(Y)}} \xrightarrow{p_{d(Y)}} \mathcal{H}_{Y'}^{n_{d(Y)-1}} \longrightarrow \cdots \longrightarrow \mathcal{H}_{Y'}^{n_0} \xrightarrow{p_0} V \longrightarrow 0 .$$

But then Hilbert's syzygy theorem on the smooth variety Y' implies that *ker* $p_{d(Y)}$ is a locally free $\mathbb{C}[Y']$-module so that we get a $\mathbb{C}[Y']$-locally free $\mathcal{H}_{Y'}$ resolution:

$$0 \longrightarrow L_{d(Y)} \longrightarrow \cdots \longrightarrow L_0 \longrightarrow V \longrightarrow 0 .$$

Hence we get the formula:

$$r_y[V] = \sum_i (-1)^i r_y[L_i] = \sum_i (-1)^i [L_i \otimes \mathbb{C}_y] \ . \tag{6.7}$$

Now we can state:

Lemma 6.8 *In the context above:*

i) The above morphism r_y annihilates $F^{d(Y)-1}\mathcal{K}(G, Y')$ so that we get a morphism:

$$\mathcal{R}(G, \mathbb{C}(Y'), \theta_{Y'}) \xrightarrow{r_y} \mathcal{R}(G, \mathbb{C}, \phi(y)) \ .$$

ii) The product morphism: $r = \prod_{y\in Y'} r_y \ : \ \mathcal{R}(G, \mathbb{C}(Y'), \theta_{Y'}) \longrightarrow \prod_{y\in Y'} \mathcal{R}(G, \mathbb{C}, \phi(y))$ is injective.

Proof: For a $\mathcal{H}_{Y'}$-module V, we fix a resolution $(L_.)$ as above. Fix $y \in Y'$ and choose an open neighborhood U_y of y in Y' such that $L_i^{U_y} = L_i \otimes_{\mathbb{C}[Y]} \mathbb{C}[U_y]$ is free over $\mathbb{C}[U_y]$ for each i. Then one can speak of the trace of a $\mathbb{C}[U_y]$-operator on $L_i^{U_y}$ and define:

$$\begin{aligned} t_V : \quad \mathcal{H}_{Y'}^{U_y} &\to U_y \\ f &\mapsto \textstyle\sum_i (-1)^i \, Tr(f|L_I^{U_y}) \end{aligned} \ .$$

On one hand we get, tensorizing with $\mathbb{C}(Y')$,

$$t_V(f) = Tr_{[V\otimes\mathbb{C}(Y')]}(f) \ .$$

On the other hand, 6.7 implies

$$\forall x \in U_y, \quad x(t_V(f)) = Tr_{r_x[V]}\, x(f) \ .$$

Now, suppose that *Supp* $V \subsetneq Y'$, then $V \otimes \mathbb{C}(Y') = 0$, hence $t_V = 0$. But the the map $\mathcal{H}_{Y'}^{U_y} \xrightarrow{x} \mathcal{H}_{Y'} \otimes \mathbb{C}_x$ is surjective hence $Tr_{r_x[V]} = 0$ and 6.1 viii) implies that $r_x[V] = 0$. This proves i).

Now, suppose $X = \sum_k n_k[V_k] \in \mathcal{K}(G, Y')$ and choose a U_y as above such that each given resolution of the V_k's be free and define t_X by linearity. Then if $r(X) = 0$, in particular for any $f \in \mathcal{H}_{Y'}^{U_y}$, $x(t_X(f)) = 0$ for any $x \in U_y$ hence $t_X(f) = 0$. But $\mathcal{H}_{Y'}^{U_y}$ generates $\mathcal{H}_{Y'} \otimes \mathbb{C}(Y')$ as a $\mathbb{C}(Y')$-space, hence $Tr_{X\otimes\mathbb{C}(Y')} = 0$ and $X \otimes \mathbb{C}(Y')$=0. This shows ii). □

6.9 *Proof of Lemma 6.4: Step* 1: Recall $\mathfrak{s}$ is the inertia class such that $Y \subset \Theta_{\mathfrak{s}}$ and choose $(M, \rho) \in \mathfrak{s}$. The algebra $\mathcal{H}^{\mathfrak{s}}$ of 4.1 has a structure of $\mathbb{C}[M/M^0]$-module and is easily seen to be free for this structure. Indeed

$$\begin{aligned} \mathcal{H}^{\mathfrak{s}}(G) &= End_G(i_M^G(\rho \otimes \mathbb{C}[M/M^0])) \\ &\simeq Hom_M(r_G^M(i_M^G(\rho \otimes \mathbb{C}[M/M^0])), \rho \otimes \mathbb{C}[M/M^0]) \end{aligned}$$

hence by the Geometric Lemma in [2], $\mathcal{H}^{\mathfrak{s}}$ admits a $\mathbb{C}[M/M^0]$ filtration with quotients isomorphic to $Hom_M(w(\rho \otimes \mathbb{C}[M/M^0]), \rho \otimes \mathbb{C}[M/M^0])$ hence free as $\mathbb{C}[M/M^0]$-modules, so that the whole is free over $\mathbb{C}[M/M^0]$.

Step 2: Now recall that $\Theta_{\mathfrak{s}}$ is the algebraic quotient of $\Psi(M)$ by a certain finite group $W_{\mathfrak{s}}$: let π be the projection and Y_M for the inverse image $\pi^{-1}(Y)$: it is a closed subset of $\Psi(M)$ of pure dimension $d(Y)$. Consider the subset of Y_M

$$U_M = \{x \in \Psi(M), \quad \pi \text{ is flat at } x\} .$$

It enjoys the following properties:

- U_M is open (by [21, III ex 9.4.]).
- U_M is non empty. It is a consequence of (the much deeper fact) [21, III.10.5] applied to one of the irreducible components of Y_M.
- U_M is $W_{\mathfrak{s}}$-invariant. In particular, the set $U := \pi(U_M)$ is open and we have $U_M = \pi^{-1}(U)$.

Step 3: Now we can choose an étale morphism $Y' \xrightarrow{\phi} Y$ satisfying i) and ii) in Lemma 6.4 and such that $\phi(Y') \subset U$. We then put $Y'_M := Y_M \times_Y Y'$ and $\pi' : Y'_M \longrightarrow Y'$ the extension of π. By Step 1, $\mathcal{H}_{Y'}$ has a structure of free $\mathbb{C}[Y'_M]$-module. But Step 2 implies that:

$$\pi' : \quad Y'_M = U_M \times_U Y' \longrightarrow Y'$$

is flat and finite, *i.e.* $\mathbb{C}[Y'_M]$ is locally free over $\mathbb{C}[Y']$ (it is a consequence of [21, III.9.2.(e)]). This completes the proof. □

Lemma 6.10 *The morphism r_y of 6.6 commutes with the respective endomorphisms T_N, $N < G$ on $\mathcal{R}(G, \mathbb{C}(Y'), \theta_{Y'})$ and $\mathcal{R}(G, \mathbb{C}, \phi(y))$.*

Proof: The "respective" endomorphisms are those provided by Lemma 4.7. More precisely, $\mathfrak{s}$ being as above, the equivalence of categories of 4.1 provides functors T_N on $\mathcal{H}^{\mathfrak{s}} - Mod$ which commute with the action of $\mathfrak{Z}_{\mathfrak{s}}$, hence which induce functors on $\mathcal{H}_{Y'} - Mod$. Now to get the lemma (and since T_N is exact) it is sufficient to show that a $\mathbb{C}[Y']$-locally free $\mathcal{H}_{Y'}$-module is mapped on a $\mathbb{C}[Y']$-locally free $\mathcal{H}_{Y'}$-module under T_M. In fact it is even sufficient to show the latter for $\mathcal{H}_{Y'}$ itself since applying T_N to the resolution below 6.6 and using again Hilbert's syzygy theorem, we could conclude. Now recall the pro-generator $P_{\mathfrak{s}}$ of $Mod(\mathfrak{s})$ from 4.1. From the Geometric Lemma of [2], we see that $T_N(P_{\mathfrak{s}})$ has a filtration indexed by those $w \in W^{MN_G}$ such that $w(M) \subset N$ and whose quotients are of the form $i^G_{w(M)}(w(\rho \otimes \mathbb{C}[M/M^0]))$ hence isomorphic to $P_{\mathfrak{s}}$ by 4.1.

As a consequence, going back to $\mathcal{H}_{Y'}$ by the equivalence of 4.1 and the map $Y' \xrightarrow{\phi} \Theta_{\mathfrak{s}}$, we see that $T_M(\mathcal{H}_{Y'})$ has a filtration whose quotients are isomorphic to $\mathcal{H}_{Y'}$, hence it is locally free. □

6.11 *Proof of 6.2 ii):* We begin proving the following thing: if $\theta_{\mathbb{K}}$ is a discrete $\mathbb{K}$-infinitesimal character, then $Y(\theta_{\mathbb{K}}) \subset \Theta^{disc}(G)$. From the lemma above, we see that r commutes with the A_d's defined in 2.4. Now suppose that Y is not contained in $\Theta^{disc}(G)$: from the fact that $A_{d(G)}$ is zero on the open subset $Y \setminus \Theta^{disc}(G)$ and the injectivity of r, we see that $A_{d(G)}$ annihilates $\mathcal{R}(G, \mathbb{K}, \theta_{\mathbb{K}})$ hence $\theta_{\mathbb{K}}$ is not discrete. Hence in order for $\theta_{\mathbb{K}}$ to be discrete, we must have $Y(\theta_{\mathbb{K}}) \subset \Theta^{disc}(G)$.

Now the implications $(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (d)$ are clear and it remains to get $(d) \Rightarrow (a)$. But as was already mentioned, the set $\Theta^{disc}(G)$ was shown in [6] to be a finite union of $\Psi(G)$-orbit. This implies, since $\overline{\mathbb{K}}$ is algebraically closed that the set $\Theta^{disc}(G, \overline{\mathbb{K}})$ of $\overline{\mathbb{K}}$-points of $\Theta^{disc}(G)$ is in turn a finite union of $\Psi(G, \overline{\mathbb{K}})$-orbits. In particular, every $\overline{\mathbb{K}}$-point of $\Theta^{disc}(G)$ is conjugate under $\Psi(G, \overline{\mathbb{K}})$ to a $\mathbb{C}$-discrete infinitesimal character hence must be discrete as a $\overline{\mathbb{K}}$-infinitesimal character. □

6.12 *Discrete orbits:* Here are discussed the consequences of the fact that a discrete infinitesimal $\overline{\mathbb{K}}$-character is conjugate under $\Psi(G, \overline{\mathbb{K}})$ to a discrete infinitesimal $\mathbb{C}$-character. Maybe it should be stressed that a $\mathbb{K}$-point of Θ^{disc} need not be conjugate under $\Psi(G, \mathbb{K})$ (*i.e.* unramified characters with values in $\mathbb{K}$) to such a $\mathbb{C}$-point. Actually, assume $\theta_{\mathbb{K}}$ is a $\mathbb{K}$-point of $\Theta^{disc}(G)$ and choose $\psi \in \Psi(G, \overline{\mathbb{K}})$ such that $\psi\theta_{\mathbb{K}}$ is a $\mathbb{C}$-point. Write $\mathbb{K}_\psi$ for the field generated by $\mathbb{K}$ and the image of ψ in $\overline{\mathbb{K}}$. Then the following diagram shows that $\mathbb{K}_\psi$ is a splitting field for $\theta_{\mathbb{K}}$ and that $\mathcal{R}(G, \mathbb{K}, \theta_{\mathbb{K}})$ may be embedded in $\mathcal{R}(G, \mathbb{C}, \psi\theta_{\mathbb{K}})$:

$$\begin{tikzcd}
\mathcal{R}(G, \mathbb{K}, \theta_{\mathbb{K}}) \arrow[r, hook, "\otimes"] \arrow[d, dashed] & \mathcal{R}(G, \mathbb{K}_\psi, \theta_{\mathbb{K}}) \arrow[d, equal, "\otimes\psi"] \\
\mathcal{R}(G, \mathbb{C}, \psi\theta_{\mathbb{K}}) \arrow[r, equal] & \mathcal{R}(G, \mathbb{K}_\psi, \psi\theta_{\mathbb{K}})
\end{tikzcd} .$$

The bottom equality comes from 6.1 iv). The following lemma is used in the proof of 4.18 and is a consequence of Appendix B:

Lemma 6.13 *Let $\mathbb{F}$ be a Galois extension of $\mathbb{K}$ which splits the infinitesimal $\mathbb{K}$-character θ. Then $Gal(\mathbb{F}/\mathbb{K})$ acts on $\mathcal{R}(G, \mathbb{F}, \theta)$ (and permutes the irreducible) and the tensorization $- \otimes_{\mathbb{K}} \mathbb{F}$ gives an isomorphism (over $\mathbb{Q}$)*

$$\mathcal{R}_{\mathbb{C}}(G, \mathbb{K}, \theta) \xrightarrow{\sim} \mathcal{R}_{\mathbb{C}}(G, \mathbb{F}, \theta)^{Gal(\mathbb{F}/\mathbb{K})} .$$

6.14 *Proof of 6.2 i):* We only give a sketch of the proof since this result is not fundamental for our purposes. Actually it requires minor refinements of Lemma 6.4. Let $M < G$ and $w \in W_G$ such that $w(M) < G$. Fix a cuspidal pair (N, ρ) with $N < M$ and write $\mathfrak{s}_M$, resp. $\mathfrak{s}$, for the N-(resp G-)inertia class of (N, ρ). In the sequel the double arrow symbol means that we use either the morphism i_M^G or the morphism $i_{w(M)}^G \circ w$; thus we get two morphisms

$$\mathcal{K}(\mathcal{H}^{\mathfrak{s}_M}(M)) \rightrightarrows \mathcal{K}(\mathcal{H}^{\mathfrak{s}}(G)) \ .$$

Remark: The above morphisms come from corresponding functors carried from $Mod(M)$, $Mod(G)$ to $\mathcal{H}^{\mathfrak{s}_M} - Mod$ and $\mathcal{H}^{\mathfrak{s}} - Mod$ via the equivalence of 4.1. Actually these functors come from morphisms of algebra $\mathcal{H}^{\mathfrak{s}_M} \rightrightarrows \mathcal{H}^{\mathfrak{s}}$. More precisely, recall the notations of 4.1: there exist projective generators $P_{\mathfrak{s}_M}$, (resp. $P_{\mathfrak{s}}$) in $Mod(\mathfrak{s}_M)$, (resp. $Mod(\mathfrak{s})$) such that

- $\mathcal{H}^{\mathfrak{s}_M} = End_M(P_{\mathfrak{s}_M})$ and $\mathcal{H}^{\mathfrak{s}} = End_G(P_{\mathfrak{s}})$
- $\forall w \in W_G$ such that $w(M) < G$, there is an isomorphism $i_{w(M)}^G P_{\mathfrak{s}_M}^w \simeq P_{\mathfrak{s}}$, unique up to composition with an invertible element of $\mathcal{H}^{\mathfrak{s}}$.

This provides a collection of morphisms $\mathcal{H}^{\mathfrak{s}_M} \xrightarrow{i_w} \mathcal{H}^{\mathfrak{s}}$ (defined only up to composition with an inner automorphism of $\mathcal{H}^{\mathfrak{s}}$) and it can be checked using the two adjunction theorems that for any $\mathcal{H}^{\mathfrak{s}_M}$-module V,

$$\left(i_{w(M)}^G \circ w\right)V \simeq \mathcal{H}^{\mathfrak{s}} \otimes_{\mathcal{H}^{\mathfrak{s}_M}, i_w} V.$$

Step 1: Let $\mathcal{A}$ be any $\mathfrak{Z}_{\mathfrak{s}_M}(M)$ algebra, then the remark above (or alternatively and more simply [6, prop 2.4]) insures that i_M^G and $i_{w(M)}^G \circ w$ induce well defined morphisms:

$$\mathcal{K}(\mathcal{H}^{\mathfrak{s}_M} \otimes_{\mathfrak{Z}_{\mathfrak{s}_M}} \mathcal{A}) \rightrightarrows \mathcal{K}(\mathcal{H}^{\mathfrak{s}}(G) \otimes_{\mathfrak{Z}_{\mathfrak{s}}} \mathcal{A})$$

where the RHS tensor product is taken w.r.t the morphism $\mathfrak{Z}_{\mathfrak{s}} \longrightarrow \mathfrak{Z}_{\mathfrak{s}_M} \longrightarrow \mathcal{A}$. (Note that the two morphisms $i_M^G,\ i_{w(M)}^G \circ w : \Theta(M) \rightrightarrows \Theta(G)$ by definition coincide.)

Step 2: Assume $\sigma \in \mathcal{R}(M, \mathbb{K}, \theta_{\mathbb{K},M})$ for some $\mathbb{K}$-infinitesimal character $\theta_{\mathbb{K},M}$ of M and call $\theta_{\mathbb{K},G} = i_M^G(\theta_{\mathbb{K},M})$: this is a $\mathbb{K}$-infinitesimal character of G. We also write $Y_M = Y(\theta_{\mathbb{K},M}) \subset \Theta_{\mathfrak{s}_M}(M)$ for the associated variety as in 6.1 ix) and $Y = i_M^G(Y_M) = i_{w(M)}^G \circ w(Y_M)$. Then we claim

i) We can choose a morphism $Y'_M \xrightarrow{\phi_M} Y_M$ satisfying conditions i) ii) and iii) of Lemma 6.4 for the pair $(M, \theta_{\mathbb{K},M})$, such that the composite $Y'_M \xrightarrow{\phi_M} Y_M \xrightarrow{i_M^G} Y$ also satisfies these conditions for the pair $(G, \theta_{\mathbb{K},G})$ and such that moreover the two $\mathbb{C}[Y'_M]$-modules

$$\mathcal{H}^{\mathfrak{s}} \otimes_{\mathfrak{Z}_{\mathfrak{s}_M}} \mathbb{C}[Y'_M]$$

obtained by tensorizing w.r.t the two morphisms $\mathfrak{Z}_{\mathfrak{s}_M} \underset{i_w}{\overset{i_1}{\rightrightarrows}} \mathcal{H}^{\mathfrak{s}}$ (see the *remark* above) be locally free.

ii) In order to see that i_M^G and $i_{w(M)}^G \circ w$ coincide on $\mathcal{R}(M, \mathbb{K}, \theta_{\mathbb{K},M})$, it is enough to check it for the pair $(\mathbb{K} = \mathbb{C}(Y'_M), \theta_{\mathbb{K},M} = \theta_{Y'_M})$ where $\theta_{Y'_M}$ is the $\mathbb{C}(Y'_M)$-infinitesimal character given by the composite $\mathfrak{Z}(M) \longrightarrow \mathbb{C}[Y_M] \xrightarrow{\phi_M^*} \mathbb{C}[Y'_M]$.

Step 3: We shall apply Step 1 to the algebra $\mathcal{A} = \mathbb{C}[Y'_M]$. Define $\mathcal{H}_{Y'_M} := \mathcal{H}^{\mathfrak{s}_M}(M) \otimes_{\mathfrak{Z}(M)} \mathbb{C}[Y'_M]$ and $\mathcal{H}_{Y'} := \mathcal{H}^{\mathfrak{s}}(G) \otimes_{\mathfrak{Z}(G)} \mathbb{C}[Y'_M]$ and call $\theta_{Y'} = i_M^G \theta_{Y'_M}$: this is the $\mathbb{C}(Y'_M)$-infinitesimal character of G given by the composite $\mathfrak{Z}(G) \longrightarrow \mathfrak{Z}(M) \xrightarrow{\theta_{Y'_M}} \mathbb{C}(Y'_M)$. Then applying (a variant of) 4.5 we get a diagram

$$\begin{tikzcd}
\mathcal{K}(\mathcal{H}_{Y'_M}) \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & \mathcal{K}(\mathcal{H}_{Y'}) \arrow[d] \\
\mathcal{R}(M, \mathbb{C}(Y'_M), \theta_{Y'_M}) \arrow[r, shift left] \arrow[r, shift right] & \mathcal{R}(G, \mathbb{C}(Y'_M), \theta_{Y'})
\end{tikzcd}$$

which is obviously commutative.

Step 4: Now fix $y \in Y'_M$, applying the construction of r_y (see 6.6), we get a diagram:

$$\begin{tikzcd}
\mathcal{K}(\mathcal{H}_{Y'_M}) \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "r_y"'] & \mathcal{K}(\mathcal{H}_{Y'}) \arrow[d, "r_y"] \\
\mathcal{R}(M, \mathbb{C}, \phi_M(y)) \arrow[r, shift left] \arrow[r, shift right] & \mathcal{R}(G, \mathbb{C}, i_M^G \circ \phi_M(y))\ .
\end{tikzcd}$$

To get the commutativity of this diagram we proceed as in the proof of 6.10: it is enough to show that $i_M^G(\mathcal{H}_{Y'_M})$ (resp. $i_{w(M)}^G \circ w(\mathcal{H}_{Y'_M})$) is locally free as a $\mathbb{C}[Y'_M]$-module; but this is exactly what is required in condition i) of Step 2.

Step 5: We have thus obtained the following commutative diagram:

$$\begin{tikzcd}
\mathcal{R}(M, \mathbb{C}(Y'_M), \theta_{Y'_M}) \arrow[r, shift left, "i_M^G"] \arrow[r, shift right, "i_{w(M)}^G \circ w"'] \arrow[d, hook, "\prod r_y"'] & \mathcal{R}(G, \mathbb{C}(Y'_M), \theta_{Y'}) \arrow[d, hook, "\prod r_y"] \\
\prod_{y \in Y'_M} \mathcal{R}(M, \mathbb{C}, \phi_M(y)) \arrow[r, shift left, "i_M^G"] \arrow[r, shift right, "i_{w(M)}^G \circ w"'] & \prod_{y \in Y_M} \mathcal{R}(G, \mathbb{C}, i_M^G \circ \phi_M(y))
\end{tikzcd}$$

which finishes the proof of 6.2 ii) since we know that i_M^G and $i_{w(M)}^G \circ w$ coincide on $\mathcal{R}(M, \mathbb{C})$.

A A short exact sequence for $\overline{\mathcal{H}}$

Our sources will be [19] and [1]. We begin with giving an equivalent definition of $\overline{\mathcal{H}}(\mathcal{A})$. To an abelian category $\mathcal{A}$ we may associate the abelian category $\mathcal{A}^{\mathbb{N}}$ defined by

$$\begin{array}{rcl} \mathcal{A}^{\mathbb{N}} & := & \text{category with objects the pairs } (V,u) \text{ with } V \in \mathcal{A} \text{ and} \\ & & u \in End_{\mathcal{A}}(V) \text{and morphisms are:} \\ & & Hom_{\mathcal{A}^{\mathbb{N}}}((V,u),(V',u')) := \{f \in Hom_{\mathcal{A}}(V,V'),\ \ u'f = fu\} \\ & = & \text{Fonc}(\overline{\mathbb{N}}, \mathcal{A}) \text{ where } \overline{\mathbb{N}} \text{ is the single object category with} \\ & & End_{\overline{\mathbb{N}}}(\bullet) = \mathbb{N}. \end{array}$$

Now assume $\mathcal{A}$ is locally noetherian and let $\mathcal{A}_f$ be the abelian subcategory of noetherian objects. Note that we have a canonical full embedding $(\mathcal{A}_f)^{\mathbb{N}} \hookrightarrow (\mathcal{A}^{\mathbb{N}})_f$ which is not "surjective" at all in general (for example $(\mathbb{Z}[\frac{1}{p}], \frac{1}{p})$ is noetherian in $\mathfrak{Ab}^{\mathbb{N}}$...). We will note $\mathcal{A}_f{}^{\mathbb{N}} := (\mathcal{A}_f)^{\mathbb{N}}$ from now on. We define $\mathcal{F}(\mathcal{A})$ to be the free abelian group with basis the set of isomorphism classes of objects in $\mathcal{A}_f{}^{\mathbb{N}}$. There is a canonical epimorphism $\mathcal{F}(\mathcal{A}) \longrightarrow \mathcal{K}(\mathcal{A}_f{}^{\mathbb{N}})$. Now put

$$\mathcal{E}(\mathcal{A}) := \bigoplus_{V \in Ob(\mathcal{A}_f)/\sim} \overline{End_{\mathcal{A}}(V)}$$

where $\overline{End_{\mathcal{A}}(V)} = End_{\mathcal{A}}(V)/[End_{\mathcal{A}}(V), End_{\mathcal{A}}(V)]$ (note that the definition is licit since $\overline{End_{\mathcal{A}}(V)}$ and $\overline{End_{\mathcal{A}}(V')}$ are canonically isomorphic when V and V' are isomorphic). Once again there is a canonical epimorphism $\mathcal{F}(\mathcal{A}) \longrightarrow \mathcal{E}(\mathcal{A})$. Then we can express $\overline{\mathcal{H}}(\mathcal{A})$ as a fiber coproduct:

Lemma A.1 *We have $\overline{\mathcal{H}}(\mathcal{A}) \simeq \mathcal{E}(\mathcal{A}) \coprod_{\mathcal{F}(\mathcal{A})} \mathcal{K}(\mathcal{A}_f{}^{\mathbb{N}})$ as a $\mathfrak{Z}(\mathcal{A})$-module (letting the latter act on the left hand term of the coproduct).*

Proof: Consider the group $B(\mathcal{A})$ defined as the quotient of the free group on the symbols (V,u) modulo the relations

- $(V,u) = (V_1,u_1) + (V_2,u_2)$ for any short exact sequence

$$0 \longrightarrow (V_1,u_1) \xrightarrow{f} (V,u) \xrightarrow{g} (V_2,u_2) \longrightarrow 0 \text{ in } \mathcal{A}^{\mathbb{N}}$$

- $(V,u) + (V,v) = (V,u+v)$.

Then we have to show that in $B(\mathcal{A})$, the subgroup generated by commutators

$$\left\langle (V,\ fg) - (V',gf) \text{ with } f : V' \longrightarrow V \text{ and } g : V \longrightarrow V' \right\rangle$$

coincides with the subgroup

$$\left\langle (V,u) - (V',u') \text{ with } (V,u) \simeq (V',u') \text{ in } \mathcal{A}^{\mathbb{N}} \right\rangle .$$

The inclusion of the second one in the first one is obvious, so fix a pair of maps $V \xrightarrow{g} V' \xrightarrow{f} V$. We have in $B(\mathcal{A})$:

- $(V, fg) = (ker\ g, 0) + (V/ker\ g, \overline{f\overline{g}}) = (V/ker\ g, \overline{f\overline{g}})$ where $\overline{g} : V/ker\ g \longrightarrow V'$ and $\overline{f} : im\ g \longrightarrow V/ker\ g$ are canonically obtained from g and f.
- $(V', gf) = (im\ g, \overline{g}\overline{f}) + (V'/im\ g, 0) = (im\ g, \overline{g}\overline{f})$.

But $\overline{g}$ realizes an isomorphism $(V/ker\ g, \overline{f\overline{g}}) \simeq (im\ g, \overline{g}\overline{f})$ hence we are done. $\square$

Now assume $\mathcal{S}$ is a localizing subcategory of $\mathcal{A}$, let $T : \mathcal{A} \longrightarrow \mathcal{A}/\mathcal{S}$ be the quotient functor and choose a left adjoint S. From [19, III.prop.10], $\mathcal{S}_f$ is a thick subcategory of $\mathcal{A}_f$ and the canonical functor $\mathcal{A}_f/\mathcal{S}_f \longrightarrow (\mathcal{A}/\mathcal{S})_f$ is an equivalence. On another hand ${\mathcal{S}_f}^{\mathbb{N}}$ is easily seen to be thick inside ${\mathcal{A}_f}^{\mathbb{N}}$ and we have a canonical exact functor ${\mathcal{A}_f}^{\mathbb{N}}/{\mathcal{S}_f}^{\mathbb{N}} \longrightarrow (\mathcal{A}/\mathcal{S})_f^{\mathbb{N}}$ so that we get the following diagram

$$\begin{tikzcd} \mathcal{K}\left({\mathcal{S}_f}^{\mathbb{N}}\right) \arrow[r] & \mathcal{K}\left({\mathcal{A}_f}^{\mathbb{N}}\right) \arrow[r] & \mathcal{K}\left({\mathcal{A}_f}^{\mathbb{N}}/{\mathcal{S}_f}^{\mathbb{N}}\right) \arrow[r] \arrow[d] & 0 \\ & & \mathcal{K}\left((\mathcal{A}/\mathcal{S})_f^{\mathbb{N}}\right) & \end{tikzcd}$$

where the line is exact by [1, VIII.(5.5)]. Note that there is again a canonical map $\mathcal{F}(\mathcal{A}) \longrightarrow \mathcal{K}({\mathcal{A}_f}^{\mathbb{N}}/{\mathcal{S}_f}^{\mathbb{N}})$ so that we can consider the fiber coproduct map:

$$\mathcal{E}(\mathcal{A}) \coprod_{\mathcal{F}(\mathcal{A})} \mathcal{K}\left({\mathcal{A}_f}^{\mathbb{N}}\right) \longrightarrow \mathcal{E}(\mathcal{A}) \coprod_{\mathcal{F}(\mathcal{A})} \mathcal{K}\left({\mathcal{A}_f}^{\mathbb{N}}/{\mathcal{S}_f}^{\mathbb{N}}\right) .$$

From standard properties of coproducts, this map is surjective and its kernel is the image of the map

$$\mathcal{K}\left({\mathcal{S}_f}^{\mathbb{N}}\right) \longrightarrow \mathcal{E}(\mathcal{A}) \coprod_{\mathcal{F}(\mathcal{A})} \mathcal{K}\left({\mathcal{A}_f}^{\mathbb{N}}\right) = \overline{\mathcal{H}}(\mathcal{A})$$

which factors through the map $\overline{\mathcal{H}}(\mathcal{S}) \longrightarrow \overline{\mathcal{H}}(\mathcal{A})$. On another hand, there is canonical map

$$\mathcal{E}(\mathcal{A}) \coprod_{\mathcal{F}(\mathcal{A})} \mathcal{K}\left((\mathcal{A}/\mathcal{S})_f^{\mathbb{N}}\right) \longrightarrow \mathcal{E}(\mathcal{A}/\mathcal{S}) \coprod_{\mathcal{F}(\mathcal{A}/\mathcal{S})} \mathcal{K}\left((\mathcal{A}/\mathcal{S})_f^{\mathbb{N}}\right) = \overline{\mathcal{H}}(\mathcal{A}/\mathcal{S})$$

given by the commutative diagram

$$\begin{tikzcd} \mathcal{F}(\mathcal{A}) \arrow[r] \arrow[d] & \mathcal{F}(\mathcal{A}/\mathcal{S}) \arrow[d] \\ \mathcal{E}(\mathcal{A}) \arrow[r] & \mathcal{E}(\mathcal{A}/\mathcal{S}) . \end{tikzcd}$$

Summing this up, we have obtained the following diagram

$$\begin{tikzcd}
\overline{\mathcal{H}}(\mathcal{S}) \arrow[r] & \overline{\mathcal{H}}(\mathcal{A}) \arrow[r] & \mathcal{E}(\mathcal{A}) \coprod_{\mathcal{F}(\mathcal{A})} \mathcal{K}\left(\mathcal{A}_f{}^{\mathbb{N}}/\mathcal{S}_f{}^{\mathbb{N}}\right) \arrow[r] \arrow[d, "\phi"] & 0 \\
& & \overline{\mathcal{H}}(\mathcal{A}/\mathcal{S}) &
\end{tikzcd}$$

where the line is still exact. Now our goal is to show that ϕ is injective.

Lemma A.2 *The canonical functor $\mathcal{A}_f{}^{\mathbb{N}}/\mathcal{S}_f{}^{\mathbb{N}} \longrightarrow (\mathcal{A}/\mathcal{S})_f{}^{\mathbb{N}}$ is a full embedding of the first category onto a thick subcategory of the second one.*

Proof: Keeping the notations above, there is a natural transformation $T \circ S \longrightarrow \mathrm{Id}_{\mathcal{A}/\mathcal{S}}$ which is an isomorphism of functors (see [19, III.2]). Now consider the functors $T^{\mathbb{N}} : \mathcal{A}^{\mathbb{N}} \longrightarrow (\mathcal{A}/\mathcal{S})^{\mathbb{N}}$ and $S^{\mathbb{N}} : (\mathcal{A}/\mathcal{S})^{\mathbb{N}} \longrightarrow \mathcal{A}^{\mathbb{N}}$. Then $T^{\mathbb{N}}$ is exact and right adjoint to $\mathcal{S}^{\mathbb{N}}$, and the natural transformation above induces an isomorphism of functors $T^{\mathbb{N}} \circ S^{\mathbb{N}} \longrightarrow \mathrm{Id}_{(\mathcal{A}/\mathcal{S})^{\mathbb{N}}}$. Moreover it is clear that the subcategory *ker* $T^{\mathbb{N}}$ identifies with $\mathcal{S}^{\mathbb{N}}$. Now we can apply Proposition 5 of [19, III.2] to deduce that $\mathcal{S}^{\mathbb{N}}$ is a localizing subcategory of $\mathcal{A}^{\mathbb{N}}$ such that $T^{\mathbb{N}}$ induces an equivalence

$$\mathcal{A}^{\mathbb{N}}/\mathcal{S}^{\mathbb{N}} \simeq (\mathcal{A}/\mathcal{S})^{\mathbb{N}}.$$

We may, as is easily checked from the definitions, identify $\mathcal{A}_f{}^{\mathbb{N}}/\mathcal{S}_f{}^{\mathbb{N}}$ as a full and thick subcategory of $\mathcal{A}^{\mathbb{N}}/\mathcal{S}^{\mathbb{N}}$ and the canonical functor of the lemma with the restriction of $T^{\mathbb{N}}$, which concludes the proof. □

From now on we assume that $(\mathcal{A}/\mathcal{S})$ is a locally finite category, so that $(\mathcal{A}/\mathcal{S})_f$ is the subcategory of finite length object in $\mathcal{A}/\mathcal{S}$. Let $\mathcal{I}$ be the set of isomorphism classes of simple objects in $\mathcal{A}/\mathcal{S}$. The categories $(\mathcal{A}/\mathcal{S})_f{}^{\mathbb{N}}$ and $\mathcal{A}_f{}^{\mathbb{N}}/\mathcal{S}_f{}^{\mathbb{N}}$ are finite and we have the following description of simple objects:

Lemma A.3 *The simple objects of $(\mathcal{A}/\mathcal{S})_f{}^{\mathbb{N}}$ are of the form $(\mathcal{V}^n, u)$ with $\mathcal{V}$ simple in $\mathcal{A}/\mathcal{S}$ and $u \in End_{\mathcal{A}}(\mathcal{V}^n) = \mathcal{M}_n(End_{\mathcal{A}}(\mathcal{V}))$ leaving no subobject of $\mathcal{V}^n$ stable. The subcategory $\mathcal{A}_f{}^{\mathbb{N}}/\mathcal{S}_f{}^{\mathbb{N}}$ identifies with the category of finite length objects in $(\mathcal{A}/\mathcal{S})_f{}^{\mathbb{N}}$ whose simple subquotients are of the form $(\mathcal{V}^n, u)$ with $(\mathcal{V}^n, u) \simeq (TV, Tv)$ for some $(V, v) \in Ob(\mathcal{A}_f{}^{\mathbb{N}})$.*

Proof: The assertion about $\mathcal{A}_f{}^{\mathbb{N}}/\mathcal{S}_f{}^{\mathbb{N}}$ follows readily from the first assertion and the previous lemma, so we only deal with the first assertion. Let X be an object in $(\mathcal{A}/\mathcal{S})_f$ and $Soc(X)$ be the maximal semi-simple subobject of X (its socle). Then it is clear that any $u \in End_{\mathcal{A}/\mathcal{S}}(X)$ leaves $Soc(X)$ stable so that $(Soc(X), u_{|Soc(X)})$ is a subobject of (X, u) in $(\mathcal{A}/\mathcal{S})_f{}^{\mathbb{N}}$. Now suppose $X = Soc(X)$ and let $X_{\mathcal{V}}$ be the $\mathcal{V}$-isotypic part of X for any simple object $\mathcal{V}$ of $\mathcal{A}/\mathcal{S}$, then obviously $X_{\mathcal{V}}$ is stable under u and isomorphic to some power $\mathcal{V}^n$. This implies the first assertion. □

Going on with this discussion, we see that for any finite length object X in $\mathcal{A}/\mathcal{S}$ there are morphisms

$$(p_{Soc(X)}, p^{Soc(X)}) : \quad End_{\mathcal{A}/\mathcal{S}}(X) \longrightarrow \\ End_{\mathcal{A}/\mathcal{S}}(Soc(X)) \oplus End_{\mathcal{A}/\mathcal{S}}(X/\,Soc(X))$$

and for any semi-simple finite length object

$$(p_{\mathcal{V}})_{\mathcal{V}\in\mathcal{I}} : \quad End_{\mathcal{A}/\mathcal{S}}(X) \longrightarrow \bigoplus_{\mathcal{V}\in\mathcal{I}} End_{\mathcal{A}/\mathcal{S}}(X_{\mathcal{V}}) \;.$$

We will use the same symbols ($p_{Soc(X)}$, etc...) for the morphisms induced on the $\overline{End_{\mathcal{A}/\mathcal{S}}(\bullet)}$.

Let $\mathcal{V}$ be any object of $(\mathcal{A}/\mathcal{S})_f$. Define $\overline{E}(\mathcal{V})$ as the subgroup of $\overline{End_{\mathcal{A}/\mathcal{S}}(\mathcal{V})}$ generated by all the images $im\,(\overline{End_{\mathcal{A}}(V)} \longrightarrow \overline{End_{\mathcal{A}/\mathcal{S}}(\mathcal{V})})$ where $TV \simeq \mathcal{V}$:

$$\overline{E}(\mathcal{V}) := \sum_{TV\simeq\mathcal{V}} im\left(\overline{End_{\mathcal{A}}(V)} \longrightarrow \overline{End_{\mathcal{A}/\mathcal{S}}(\mathcal{V})}\right)$$

This only depends, as well as $\overline{End_{\mathcal{A}/\mathcal{S}}(\mathcal{V})}$, on the isomorphism class of $\mathcal{V}$.

Lemma A.4 *There exists a unique system of morphisms* $\Phi_X : \overline{End_{\mathcal{A}/\mathcal{S}}(X)} \longrightarrow \bigoplus_{\mathcal{V}\in\mathcal{I}} \overline{End_{\mathcal{A}/\mathcal{S}}(\mathcal{V})}$ *for X any finite length object in $\mathcal{A}/\mathcal{S}$ satisfying:*

- $\Phi_X = \Phi_{Soc(X)} \circ p_{Soc(X)} + \Phi_{X/Soc(X)} \circ p^{Soc(X)}$ *for any X.*
- $\Phi_X = \sum_{\mathcal{V}\in\mathcal{I}} \Phi_{X_{\mathcal{V}}} \circ p_{\mathcal{V}}$ *for any semi-simple object X.*
- $\Phi_{\mathcal{V}^n}$ *is the usual trace map*
  $$\Phi_{\mathcal{V}^n} : \overline{End_{\mathcal{A}/\mathcal{S}}(X)} \simeq \overline{\mathcal{M}_n(End_{\mathcal{A}/\mathcal{S}}(\mathcal{V}))} \to \overline{End_{\mathcal{A}/\mathcal{S}}(\mathcal{V})}$$
  $$u \mapsto Tr(u)$$

Moreover, for any X we have $\Phi_X(\overline{E}(X)) \subset \bigoplus_{\mathcal{V}\in\mathcal{I}} \overline{E}(\mathcal{V})$.

Proof: The existence of this system is obvious by induction on the length of X. As for the last assertion, fix a noetherian object W of $\mathcal{A}$ and define $S(W) := W \times_{S\circ T(W)} S(Soc(TW))$ where the morphism $W \longrightarrow S\circ T(W)$ is the adjunction morphism (see proof of lemma A.2). Then $S(W)$ is stable under any $w \in End_{\mathcal{A}}(W)$ since $Soc(TW)$ is stable under Tw. This implies that for any X as in the lemma, $p_{Soc(X)}(\overline{E}(X)) \subset \overline{E}(Soc(X))$ and $p^{Soc(X)}(\overline{E}(X)) \subset \overline{E}(X/\,Soc(X))$. Now returning to W, and assuming TW to be semi-simple, we may define for each $\mathcal{V} \in \mathcal{I}$ the subobject $W_{\mathcal{V}} := W \times_{S\circ T(W)} S((TW)_{\mathcal{V}})$. Again each $W_{\mathcal{V}}$ is stable under $End_{\mathcal{A}}(W)$ and this implies that for any X as in the lemma, $p_{\mathcal{V}}(\overline{E}(X)) \subset \overline{E}(X_{\mathcal{V}})$. Now return to W and assume there exists (and fix) a decomposition $TW \simeq \mathcal{V}^n = \bigoplus_{i=1}^n \mathcal{V}_i$ where $\mathcal{V}_i$ is a copy of $\mathcal{V}$. Then define $W_i := W \times_{S\circ T(W)} S(\mathcal{V}_i)$; we have $T(W_i) \simeq \mathcal{V}$. Now fix $w \in End_{\mathcal{A}}(W)$ and let $(Tw)_{ii} \in End_{\mathcal{A}/\mathcal{S}}(\mathcal{V}_i)$ be the (i,i) matrix coefficient of Tw in the previously fixed decomposition of TW, then $w \times S((Tw)_{ii})$ defines an endomorphism of W_i and $(Tw)_{ii} = T(w \times S((Tw)_{ii}))$ so that we get: $\Phi_{\mathcal{V}^n}(\overline{E}(\mathcal{V}^n)) \subset \overline{E}(\mathcal{V})$. $\square$

The following result has nothing to do with localization and is true for any finite category, but we express it for $\mathcal{A}/\mathcal{S}$:

Proposition A.5 *There is an isomorphism* $\overline{\mathcal{H}}(\mathcal{A}/\mathcal{S}) \simeq \bigoplus_{\mathcal{V}\in\mathcal{I}} \overline{End_{\mathcal{A}/\mathcal{S}}(\mathcal{V})}$.

Proof: We first define a morphism $\delta : \overline{\mathcal{H}}(\mathcal{A}/\mathcal{S}) \longrightarrow \bigoplus_{\mathcal{V}\in\mathcal{I}} \overline{End_{\mathcal{A}/\mathcal{S}}(\mathcal{V})}$ thanks to the universal property of coproducts and the following diagram:

$$\begin{tikzcd}
\mathcal{F}(\mathcal{A}/\mathcal{S}) \arrow[r] \arrow[d] & \mathcal{K}\big((\mathcal{A}/\mathcal{S})_f{}^{\mathbb{N}}\big) \arrow[d] \arrow[ddr, "\Psi"] & \\
\mathcal{E}(\mathcal{A}/\mathcal{S}) \arrow[r] \arrow[drr, "\Phi"] & \overline{\mathcal{H}}(\mathcal{A}/\mathcal{S}) \arrow[dr, dotted, "\delta"] & \\
 & & \bigoplus_{\mathcal{V}\in\mathcal{I}} \overline{End_{\mathcal{A}/\mathcal{S}}(\mathcal{V})}
\end{tikzcd}$$

where $\Phi = \bigoplus_X \Phi_X$ and $\Psi([\mathcal{V}^n, u]) = \Phi_{\mathcal{V}^n}(u)$ for any simple object $(\mathcal{V}^n, u)$ in $(\mathcal{A}/\mathcal{S})_f{}^{\mathbb{N}}$. By definition this map is surjective. We could show "by hand" that it is injective (see next proof) but here the morphism $\gamma : \bigoplus_{\mathcal{V}\in\mathcal{I}} \overline{End_{\mathcal{A}/\mathcal{S}}(\mathcal{V})} \longrightarrow \overline{\mathcal{H}}(\mathcal{A}/\mathcal{S})$ induced by the canonical inclusion $\bigoplus_{\mathcal{V}\in\mathcal{I}} \overline{End_{\mathcal{A}/\mathcal{S}}(\mathcal{V})} \hookrightarrow \mathcal{E}(\mathcal{A}/\mathcal{S})$ is clearly inverse to δ. $\square$

We shall say that a locally noetherian category $\mathcal{C}$ is $\mathfrak{Z}$-finite if for any noetherian object X in $\mathcal{C}$, $End_{\mathcal{C}}(X)$ is finitely generated as a $\mathfrak{Z}(\mathcal{C})$-module ($\mathfrak{Z}(\mathcal{C})$ being the center of $\mathcal{C}$). It is said to be $\mathbb{Q}$-linear if $\mathfrak{Z}(\mathcal{C})$ is a $\mathbb{Q}$-algebra.

Proposition A.6 *There is a commutative diagram*

$$\begin{tikzcd}
\mathcal{E}(\mathcal{A}) \coprod_{\mathcal{F}(\mathcal{A})} \mathcal{K}_{\mathbb{C}}\big(\mathcal{A}_f{}^{\mathbb{N}}/\mathcal{S}_f{}^{\mathbb{N}}\big) \arrow[r, "\phi"] \arrow[d, "\delta'"] & \overline{\mathcal{H}}(\mathcal{A}/\mathcal{S}) \arrow[d, "\delta"] \\
\bigoplus_{\mathcal{V}\in\mathcal{I}} \overline{E}(\mathcal{V}) \arrow[r, hook] & \bigoplus_{\mathcal{V}\in\mathcal{I}} \overline{End_{\mathcal{A}/\mathcal{S}}(\mathcal{V})}
\end{tikzcd}$$

where δ is an isomorphism and δ' is an isomorphism if $\mathcal{A}$ and $\mathcal{A}/\mathcal{S}$ are assumed to be $\mathfrak{Z}$-finite and $\mathbb{Q}$-linear.

Proof: To construct δ' we use similar diagram and universal property as in the previous proof:

$$\begin{tikzcd}
\mathcal{F}(\mathcal{A}) \arrow[r] \arrow[d] & \mathcal{K}\left(\mathcal{A}_f{}^{\mathbb{N}}/\mathcal{S}_f{}^{\mathbb{N}}\right) \arrow[d] \arrow[ddr, bend left, "\Psi'"] & \\
\mathcal{E}(\mathcal{A}) \arrow[r] \arrow[drr, "\Phi'"'] & \mathcal{E}(\mathcal{A}) \coprod_{\mathcal{F}(\mathcal{A})} \mathcal{K}_{\mathbb{C}}\left(\mathcal{A}_f{}^{\mathbb{N}}/\mathcal{S}_f{}^{\mathbb{N}}\right) \arrow[dr, dotted, "\delta'"] & \\
& & \bigoplus_{\mathcal{V}\in\mathcal{I}} \overline{E}(\mathcal{V})
\end{tikzcd}$$

where Ψ' is the restriction of the Ψ of the previous proof and $\Phi' = \bigoplus_{W\in Ob(\mathcal{A}_f)} \Phi_{TW} \circ T$. Note that from Lemma A.4 the image of Ψ' and Φ' are equal to $\bigoplus_{\mathcal{V}\in\mathcal{I}} \overline{E}(\mathcal{V})$ so that δ' is surjective. To see that it is also injective, we have to show that any element in the kernel of Ψ' gives zero in $\mathcal{E}(\mathcal{A}) \coprod_{\mathcal{F}(\mathcal{A})} \mathcal{K}_{\mathbb{C}}(\mathcal{A}_f{}^{\mathbb{N}}/\mathcal{S}_f{}^{\mathbb{N}})$. Note that *ker* Ψ' is generated by elements of the form

$$\begin{aligned} &[\mathcal{V}^{n_1}, u_1] - [\mathcal{V}^{n_2}, u_2] - [\mathcal{V}^{n_3}, u_3], \\ \text{with } &\Phi_{\mathcal{V}^{n_1}}(u_1) = \Phi_{\mathcal{V}^{n_2}}(u_2) + \Phi_{\mathcal{V}^{n_3}}(u_3)\ . \end{aligned} \tag{A.7}$$

Here we use our $\mathfrak{Z}$-finiteness assumption: it implies that $End_{\mathcal{A}/\mathcal{S}}(\mathcal{V})$ is a division algebra $\mathbb{D}$ of finite dimension over its center $\mathbb{K}$ (its degree will be noted n_D). In particular, for $(\mathcal{V}^n, u)$ to be simple in $\mathcal{A}/\mathcal{S}$, it is necessary that $\mathbb{K}[u] \subset \mathcal{M}_n(\mathbb{D})$ be a field. Let $Z \subset \mathbb{K}$ be the image of the canonical morphism $\mathfrak{Z}(\mathcal{A}) \longrightarrow \mathbb{D}$, we have the following criterion

Lemma A.8 *Let $u \in End_{\mathcal{A}/\mathcal{S}}(\mathcal{V}^n)$, then $(\mathcal{V}^n, u)$ is an object of $\mathcal{A}_f{}^{\mathbb{N}}/\mathcal{S}_f{}^{\mathbb{N}}$ if and only if u is an integral element of $\mathcal{M}_n(\mathbb{D})$ over Z.*

Proof: That the condition is necessary is obvious from the $\mathfrak{Z}$-finiteness hypothesis on $\mathcal{A}$. For the sufficicency, take any noetherian subobject W of $S\mathcal{V}^n$ such that $TW \simeq \mathcal{V}^n$. Since $Z[Su] \subset End_{\mathcal{A}}(S\mathcal{V}^n)$ is finitely generated over $\mathfrak{Z}(\mathcal{A})$, the subobject $Z[Su](W)$ of $S\mathcal{V}^n$ is noetherian and stable under Su so that $(\mathcal{V}^n, u) \simeq (TW, TSu)$. $\square$

We go back to the situation of A.7 and we identify the $u_i's$ with elements of $\mathcal{M}_{n_i}(\mathbb{D})$. Note that there is an isomorphism $\mathbb{K} \simeq \overline{\mathbb{D}} \simeq \overline{End_{\mathcal{A}/\mathcal{S}}(\mathcal{V})}$ such that the maps $\Phi_{\mathcal{V}^{n_i}}$ coincide with the corresponding reduced trace maps $\mathcal{M}_{n_i}(\mathbb{D}) \longrightarrow \mathbb{K}$. Now we use the following fact from the theory of central simple algebras: there exists a Galois extension E of $\mathbb{K}$ with degree $N.n_D$ such that each n_i divides N and:

i) For each i there is a commutative diagram:

$$\begin{tikzcd} \mathbb{K}[u_i] \arrow[r, hook] \arrow[d, hook] & \mathcal{M}_{n_i}(\mathbb{D}) \arrow[d, hook] \\ E \arrow[r, hook] & \mathcal{M}_N(\mathbb{D}) \end{tikzcd}$$

ii) Let $\Gamma = Gal(E/\mathbb{K})$, there exists a E-base $(e_\gamma)_{\gamma\in\Gamma}$ of $\mathcal{M}_N(\mathbb{D})$ such that the multiplication law is given by:

$$(\sum x_\gamma e_\gamma)(\sum y_\gamma e_\gamma) = \sum_{\gamma,\delta} k_{\gamma,\delta}.x_\gamma.\gamma(y_\delta).e_{\gamma\delta}$$

for some family $(k_{\gamma,\delta}) \in E^{*\Gamma^2}$. (See [22, 2.6])

iii) For each $(\gamma,\delta) \in \Gamma^2$, $k_{\gamma,\delta}$ is a root of unity. (See [22, 2.13.14]: here we use the $\mathbb{Q}$-linearity which implies that Char $\mathbb{K} = 0$)

The diagram of i) above enables to identify the u_i's with elements of E (noted v_i...) which are integral over Z by the previous lemma. Let E_Z be the sub-Z-algebra of E generated by the v_i's, the $k_{\gamma,\delta}$'s and all their conjugates under Γ. By properties of integral elements, it is finitely generated as a Z-module. Moreover, from the multiplication law of ii), the submodule $A_Z = \bigoplus_{\gamma\in\Gamma} E_Z e_\gamma$ is a subring of $\mathcal{M}_N(\mathbb{D})$ which is finitely generated as a Z-module. Note that for each $\gamma \in \Gamma$, the inverse of e_γ^{-1} is $\gamma^{-1}(k^{-1}_{\gamma,\gamma^{-1}} k^{-1}_{e,e}) e_{\gamma^{-1}}$, hence lies in A_Z.

Now consider the element $v = n_1 v_1 - n_2 v_2 - n_3 v_3 \in E_Z \subset A_Z \subset \mathcal{M}_N(\mathbb{D})$, the assumption on the traces in A.7 implies that the trace of v as an element of $\mathcal{M}_N(\mathbb{D})$ (or as an element of E) is zero, that is: $\sum_{\gamma\in\Gamma} \gamma(v) = 0$. As a consequence of the multiplication law of ii), we have $e_\gamma v(e_\gamma)^{-1} = \gamma(v)$, thus we get (recall $\mathbb{Q} \subset Z$)

$$v = \frac{1}{|\Gamma|} \sum_{\gamma\in\Gamma} (v - e_\gamma v (e_\gamma)^{-1}) \; .$$

that is: $v \in [A_Z, A_Z]$

Now we view the v_i's as endomorphisms of $\mathcal{V}^N$ and fix a noetherian subobject W of $S\mathcal{V}^N$ stable under $S(A_R)$. Since $Sv = 0$ in $\overline{End_{\mathcal{A}}(W)}$, and $(TW, TSv_i) = (\mathcal{V}^N, v_i)$, we get $n_1[\mathcal{V}^N, v_1] - n_2[\mathcal{V}^N, v_2] - n_3[\mathcal{V}^N, v_3] = 0$ in $\mathcal{E}(\mathcal{A}) \coprod_{\mathcal{F}(\mathcal{A})} \mathcal{K}_{\mathbb{C}}(\mathcal{A}_f{}^{\mathbb{N}}/\mathcal{S}_f{}^{\mathbb{N}})$ and, as we have $[\mathcal{V}^N, v_i] = \frac{N}{n_i}[\mathcal{V}^{n_i}, u_i]$ in $\mathcal{K}(\mathcal{A}_f{}^{\mathbb{N}}/\mathcal{S}_f{}^{\mathbb{N}})$, we have proven:

$$[\mathcal{V}^{n_1}, u_1] - [\mathcal{V}^{n_2}, u_2] - [\mathcal{V}^{n_3}, u_3] = 0 \text{ in } \mathcal{E}(\mathcal{A}) \coprod_{\mathcal{F}(\mathcal{A})} \mathcal{K}_{\mathbb{C}}\big(\mathcal{A}_f{}^{\mathbb{N}}/\mathcal{S}_f{}^{\mathbb{N}}\big).$$

□

Now we have completed the proof of Proposition 5.6:

Corollary A.9 *With the foregoing assumptions, the sequence* $\overline{\mathcal{H}}(\mathcal{S}) \longrightarrow \overline{\mathcal{H}}(\mathcal{A}) \longrightarrow \overline{\mathcal{H}}(\mathcal{A}/\mathcal{S})$ *is exact.*

B A little review of finite dimensional algebras

This is intended to collect the numerous references on the subject that are needed in the text. The reference book will be [16]. Let $\mathbb{K}$ be a field, $\mathbb{K}'$ a separable finite extension and A a finite dimensional $\mathbb{K}$-algebra. Put $A' := A \otimes_{\mathbb{K}} \mathbb{K}'$ and $M' = M \otimes_{\mathbb{K}} \mathbb{K}' = M \otimes_A A'$ for any A-module M. Also $Rad(A)$ for the Jacobson radical of A and $\overline{A} := A/Rad(A)$. Then we recall the following facts:

i) $Rad(A') = Rad(A)'$ and $\overline{A'} = \overline{A}'$. See [16, 7.9].
ii) M is A-semi-simple if and only if M' is A' semisimple. See [16, 7.8].
iii) Let $\{M_1, \cdots, M_s\}$ be a set of representatives of isomorphism classes of irreducible A-modules and write $M_i' = \bigoplus_j N_{ij}$ the decomposition of M_i' into simple A'-modules. Then every simple A'-module is isomorphic to some N_{ij} and if $N_{ij} = N_{i'j'}$ then $i = i'$. See [16, 7.9]
iv) We can precise the latter point in the case where $\mathbb{K}'$ is Galois extension of $\mathbb{K}$ and splits the algebra A. In this case we can write

$$M_i' = (\bigoplus_k N_{ik}')^{m_i}$$

where the N_{ik}' are non-isomorphic simple A'-modules which are permuted by the action of $Gal(\mathbb{K}'/\mathbb{K})$ on the classes of isomorphism of A-modules.

Consider now the restriction of scalars; for a A'-module M the composition

$$A \simeq A \otimes 1 \hookrightarrow A \otimes_{\mathbb{K}} \mathbb{K}' \longrightarrow End_{\mathbb{K}'}(M) \hookrightarrow End_{\mathbb{K}}(M)$$

provides a structure of A-module on M which is noted $R(M)$.

Lemma B.1 *Let M be a simple A'-module, then $R(M)$ is a multiple of some simple A-module. Moreover any simple A-module occurs in the restriction of some simple A'-module.*

Proof: Note first that from point i) above, $Rad(A) \subset Rad(A')$ acts as zero on $R(M)$, so that the latter is semi-simple. Now if N is a simple A-module occurring in $R(M)$ then

$$0 \neq Hom_A(N, R(M)) = Hom_{A'}(N', M)$$

so that M occurs in N'. Now from point iii), this fixes the isomorphism class of N. $\square$

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