Syntomic complexes of regular rings
(joint with B. Bhatt)

Let X be a smooth scheme over a field k.

$\rightsquigarrow K(X)$ K-theory spectrum

$\rightsquigarrow$ Analog of AHSS for $K(X)$

$\rightsquigarrow \mathbb{Z}(i)^{mot}_X \in \mathcal{D}(X_{zar}),\ i \geq 0$

- constructed by Bloch
- codim i algebraic cycles on $X \times \mathbb{A}^n$

<u>Theorem</u> (Friedlander–Suslin, Levine, Voevodsky)

There is a spectral sequence

$$\mathbb{H}^p_{zar}(X, \mathbb{Z}(q)^{mot}_X) \Rightarrow K_{2q-p}(X).$$

There is a filtration on $K(X)$,

$\mathrm{Fil}^{\geq \bullet} K(X)$ w/

$$\mathrm{gr}^i K(X) = R\Gamma_{\mathrm{Zar}}(X, \mathbb{Z}(i)^{\mathrm{mot}}_X)[2i].$$

If $\ell \neq \mathrm{char}(k)$, then

$$\mathbb{Z}/\ell(i)^{\mathrm{mot}}_X = \tau^{\leq i} Rr_*(\mu_\ell^{\otimes i})$$

$$r : X_{\text{ét}} \to X_{\mathrm{Zar}}$$

$$\in \mathcal{D}(X_{\mathrm{Zar}})$$

(Voevodsky–Rost).

If $p = \mathrm{char}(k)$,

$$\mathbb{Z}/p(i)^{\mathrm{mot}}_X = \Omega^i_{\log}[-i]$$ (Geisser–Levine)

Here $\Omega^i_{\log} \subseteq \Omega^i_{X/\mathbb{F}_p}$ generated by

$\frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_i}{x_i}$ $x_1, \ldots, x_i$ local units

Above, X smooth / field.
Such constructions are not yet known for arbitrary schemes.
Analog for $K^{\acute{e}t}(X; \mathbb{Z}_p)$.

<u>Def</u>: $K^{\acute{e}t}(X)$ = étale sheafification of $K_{\geq 0}$ over X.

For schemes $X/\mathbb{Z}[1/p]$,

$$K^{\acute{e}t}(X; \mathbb{Z}_p) = L_{K(1)} K(X)$$

on connective covers.

If $X = \mathrm{Spec}(R)$, R p-complete

then $K^{\acute{e}t}(X; \mathbb{Z}_p) = TC(X; \mathbb{Z}_p)$

<u>Construction</u> (Thomason, Bhatt-Morrow-Scholze)
BMS2

Let X be any scheme. There is a natural filtration $\mathrm{Fil}^{\geq\cdot} K^{\acute{e}t}(X; \mathbb{Z}/p^n)$, with

$$\mathrm{gr}^i K^{\acute{e}t}(X; \mathbb{Z}/p^n) = R\Gamma_{\acute{e}t}(X, \mathbb{Z}/p^n(i)_X)[2i]$$

Here $\mathbb{Z}/p^n(i)_X \in \mathcal{D}(X_{\acute{e}t}, \mathbb{Z}/p^n)$ is called the <u>syntomic complex</u>.

Filtration is Postnikov filtration in syntomic topology (X q-syntomic).

Bhatt–Lurie: construct $\mathbb{Z}/p^n(i)_X$ purely algebraically (without algebraic cycles).

Ex) $\mathbb{Z}/p^n(0)_X = \mathbb{Z}/p^n$.

Ex) $\mathbb{Z}/p^n(1)_X = \mathrm{fib}(\mathbb{G}_m \xrightarrow{p^n} \mathbb{G}_m)$.

Ex) $\mathbb{Z}/p^n(i)_X\big|_{X[1/p]} = \mu_{p^n}^{\otimes i}\big|_{X[1/p]}$

Ex) If X regular$/\mathbb{F}_p$,

$$\mathbb{Z}/p(i)_X = \Omega^i_{\log}[-i] \quad \text{(étale top)}$$

Ex) $X = \mathrm{Spec}(R)$, R p-complete ring (*)

(BMS2)

$$R\Gamma(X, \mathbb{Z}/p^n(i)_X) = \mathrm{eq}\Big(N^{\geq i} \Delta_X\{i\} \underset{\mathrm{can}}{\overset{\varphi_i}{\rightrightarrows}} \Delta_X\{i\} \Big) \Big/ p^n$$

Associate to X

$\rightsquigarrow$ $\Delta_X \xrightarrow{\varphi} \Delta_X$ prismatic complex

$\rightsquigarrow$ $N^{\geq *} \Delta_X$ Nygaard filt

$\rightsquigarrow$ $\Delta_X\{i\}$ $i \in \mathbb{Z}$

$\rightsquigarrow$ Divided Frob $\varphi_i : N^{\geq i} \Delta_X\{i\} \to \Delta_X\{i\}$.

Recall $K^{\acute{e}t}(R; \mathbb{Z}_p) = TC(R; \mathbb{Z}_p)$

(*) is analog of Nikolaus-Scholze description of $TC(-; \mathbb{Z}_p)$.

Goal: Describe $\mathbb{Z}/p^n(i)_X$ when X regular & p-torsion free.

$j : X[1/p] \subseteq X$

Construction Have map

$$\mathbb{Z}/p^n(i)_X \longrightarrow Rj_*(\mu_{p^n}^{\otimes i}).$$

since $j^* \mathbb{Z}/p^n(i)_X = \mu_{p^n}^{\otimes i}$.

Fact: (Antieau-M-Morrow-Nikolaus),

$$\mathbb{Z}/p^n(i)_X \in \mathcal{D}^{\leq i}(X).$$

Above factors through

$$\mathbb{Z}/p^n(i)_X \xrightarrow{f} \tau_{\leq i} Rj_*(\mu_{p^n}^{\otimes i}).$$

Theorem (Bhatt–M)

If X regular noeth. scheme,

a) then the map f is an iso in cohomological degrees $< i$.

b) and an injection in degree i, image is subsheaf generated by $(\mathcal{O}_X^\times)^{\otimes i}$.

Here $\mathcal{O}_X^\times[-1] \longrightarrow \mathbb{Z}/p^n(1)_X$

Gives $(\mathcal{O}_X^\times)^{\otimes i}[-i] \to \mathbb{Z}/p^n(i)_X$

Construction is due to Schneider for X smooth over DVR,

Geisser — relates to étale motivic complex

Sato — studies in semistable case

"p-adic étale Tate twists"

Cases are proved by Kurihara, Kato, Tsuji, Colmez–Niziol. (in low weights, or up to isogeny)

Proof:

$$K^{\acute{e}t}(R; \mathbb{Z}_p) \simeq TC(R; \mathbb{Z}_p)$$

R p-complete.

Result describes the associated graded of $TC(R; \mathbb{Z}_p)$ R regular.

Studied extensively in literature,
e.g.
Hesselholt–Madsen: R smooth/DVR.

An essential ingredient is the
<u>Segal conjecture</u>:

$$\varphi : THH(R; \mathbb{Z}_p) \to THH(R; \mathbb{Z}_p)^{tC_p}$$

is an iso in large degrees.

We observe that for R regular,
the Segal conjecture holds on
associated graded terms.

<u>Def</u> A ring R is called <u>F-smooth</u> if

$\mathrm{fib}\left(\mathrm{gr}^i_{\mathrm{BMS}}(\varphi)\right) \in \mathcal{D}(R)$

has p-complete Tor-amplitude in degrees $\leq i-2$.

This class of rings includes perfectoid rings, smooth algs over perfectoids.

For an $\mathbb{F}_p$-alg R, R is F-smooth $\iff$ R is <u>Cartier smooth</u> (Kelly-Morrow, Kerz-Strunk-Tamme).

1) $L_{R/\mathbb{F}_p}$ flat

2) $\Omega^i_R \xrightarrow[\sim]{C^{-1}} H^i(\Omega^\bullet_{R/\mathbb{F}_p})$

Studied for R/perfectoid by Bouis.

<u>Theorem</u>: A p-complete noeth. ring is F-smooth if & only if regular.

<u>Theorem</u>: The description of syntomic cx for p-torsionfree regular scheme holds for any p-torsionfree F-smooth scheme.

Étale comparison theorem

The other ingredient in the proof is the comparison between syntomic complex of any scheme X & $X[1/p]$.

Cons: $v_1 \in H^0_{syn}(\mathbb{F}_p(p-1)(\mathbb{Z}))$.

Theorem (Étale comparison, Bhatt-Scholze)

For any scheme X, the Bhatt-Lurie

weq

$$\bigoplus_{i\geq 0} R\Gamma(X, \mathbb{F}_p(i)_X) \longrightarrow \bigoplus_{i\in\mathbb{Z}} R\Gamma(X[1/p], \mu_p^{\otimes i})$$

$\quad = $ syntomic cx of $X[1/p]$

exhibits target as some $[1/v_1]$.

In K-theory,

$$L_{K(1)} K(X) = L_{K(1)} K(X[1/p])$$

(Bhatt-Clausen-M
Land-M-Meier-Tamme).

Proof is a bit of linear alg.:
Identify v_1 in prismatic coh.

Analog:

$$THH(\mathbb{Z}_p; \mathbb{F}_p) = \mathcal{E}(\alpha) \otimes P(\theta)$$

$$|\alpha| = 2p-1$$
$$|\theta| = 2p.$$

v_1 is detected in $TC^-(\mathbb{Z}_p; \mathbb{F}_p)$
In HFPSS, detected by class
θ (Bökstedt–Madsen).