# SEMINAR ON SEMI-SIMPLE LIE ALGEBRAS AND THEIR REPRESENTATIONS

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## 1. Summary

Lie algebras are ubiquitous in algebra, but also in many other areas of mathematic. They usually arise as linearization (i.e. as the tangent space) of Lie groups or algebraic groups, but can also be studied as objects of intrinsic interest. By definition they are given by a vector space $\mathfrak{g}$ over a field $k$ together with a Lie bracket $[-,-] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ that behaves like the commutator $XY - YX$ on $\mathfrak{gl}_n = \mathrm{Mat}_{n\times n}(k)$. In the first part of the seminar we will study Lie algebras and in particular semi-simple Lie algebras. It turns out that these objects admit a beautiful classification in terms of root systems.

In the second part of the seminar we will study representations of semi-simple Lie algebras. The category of finite dimensional representations turns out to be semi-simple and there is a complete classification of the irreducible representations. The last three talks will present an introduction to further and more advanced topics about certain infinite dimensional representations.

Most of the seminar will follow closely Humphreys' book [1]. The last talks about infinite dimensional representations will follow [2].

## 2. Talks

**Talk 1: Basics of Lie algebras**
Define Lie algebras and introduce basic notions about them (homomorphisms, ideals, solvable and nilpotent Lie algebras).
Discuss many examples ($\mathfrak{gl}_n, \mathfrak{sl}_\mathfrak{n}, \mathfrak{sp}_{2n}, \mathfrak{so}_n$,...). Prove Engel's theorem. [1, Ch. I]

**Talk 2: Semi-simple Lie algebras**
Discuss the Jordan–Chevalley decomposition and introduce simple and semi-simple Lie algebras [1, Ch. II]. Prove Cartan's criterion [1, Ch. II, 4.3.]. Define the Killing form and discuss semi-simplicity using the Killing form [1, Ch. II, 5.].

**Talk 3: Representations of Lie algebras, I**
Introduce representations of Lie-algebras [1, Ch. II, 6.]. Prove that every finite dimensional representation of a semi-simple Lie algebra is semi-simple (Weyl's theorem) [1, Ch. II, 6.3.] and show that representations preserve the Jordan decomposition [1, Ch. II, 6.4.]. Present the $\mathfrak{sl}_2$-case in detail [1, Ch. II, 7.].

**Talk 4: Structure theory of semi-simple Lie algebras**
Introduce root spaces and the root space decomposition of a semi-simple Lie algebra [1, Ch. II, 8.].

**Talk 5: Root systems**
Define the abstract notion of root systems and present their classification [1, Ch. III, 9.-11.].

**Talk 6: Classification of semi-simple Lie algebras**
The aim of this talk is to survey the classification of semi-simple Lie algebras in terms of root systems. In the first part of the talk, explain the isomorphism theorem [1, Ch. IV, 14.2]. Then prove that all Cartan subalgebras are conjugate [1, Ch. IV, 16.2.]. Explain the statement of the Existence theorem [1, Ch. V, 18.4.]. If time permits, sketch (parts of) its proof.

**Talk 7: Universal enveloping algebras and the PBW theorem**
Define tensor algebras and symmetric algebras [1, Ch V., 17.1] and the universal enveloping algebra of a Lie algebra [1, Ch. V, 17.2.]. State and prove the Poincaré-Birkhoff-Witt Theorem (PBW theorem) [1, Ch. V,17.3., 17.4.].

**Talk 8: Representations of Lie algebras, II**
Define weight spaces of Lie algebra representations and cyclic modules [1, Ch. VI, 20.1., 20.2.]. Introduce the standard cyclic modules $Z(\lambda)$ and prove the existence and uniqueness of irreducible cyclic modules of highest weight $\lambda$ [1, Ch. VI, 20.3.]. Deduce the classification of finite dimensional representations [1, Ch. VI, 21.1., 21.2]. See also [2, I, Ch. 1, 1.2-1.6.].

**Talk 9: The center of the enveloping algebra and Harish-Chandra's theorem**
Define the Harish-Chandra homomorphism [2, I, Ch. 1,1.9] and discuss the proof of Harish-Chandra's theorem [2, I, Ch. 1,1.10]. The proof is only sketched in [2, I, Ch. 1,1.10]. Details can be found in [1, Ch. VI, 23.].

**Talk 10: The BGG-category $\mathcal{O}$**
Define the BGG category $\mathcal{O}$ [2, I, Ch. 1, 1.1.]. Prove that the category $\mathcal{O}$ is Artinian [2, I, Ch. 1, 1.11.]. Study the action of the center of the enveloping algebra [2, I, Ch. 1, 1.7, 1.8.] and deduce the block decomposition [2, I, Ch. 1, 1.12, 1.13.].

**Talk 11: Characters of finite dimensional modules**
Introduce the notions of formal characters for finite dimensional representations as well as for modules in the category $\mathcal{O}$ [2, I, Ch. 1, 1.14-1.16.]. Prove the fundamental theorems of Weyl and Kostant about characters of finite dimensional representations [2, I, Ch. 2, 2.4.], see also [1, Ch. VI, 24.]. If time permits also state and prove Steinberg's multiplicity formula [1, VI, 24.4.].

**Talk 12: The Kazhdan-Lusztig conjecture (optional)**
Give a survey about the Kazhdan–Lusztig conjecture, see [2, I, Ch. 8.].

## References


[1] J.E. Humphreys, *Introduction to Lie Algebras and Representation Theory*, Graduate Texts in Mathematics **9**, Springer, 1974.

[2] J.E. Humphreys, *Representations of Semisimple Lie Algebras in the BGG Category $\mathcal{O}$*, Graduate Studies in Mathematics **94**, AMS 2008.