# Oberseminar: Higher Siegel–Weil formula for unitary groups: the non-singular terms

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## 1 Introduction

The goal of this Oberseminar is to get some insight into the recent development in the Kudla program for function fields [FYZ21a, FYZ21b, FYZ23]. Kudla's original program is formulated for Shimura varieties and special cycles on them whose intersection numbers are (conjecturally) expressed as derivatives of Eisenstein series.

### 1.1 Why should we learn [FYZ21a] ?

Classically, remarkable relations between Eisenstein series and theta series lead to deep results in number theory. For example the equality $\theta(\tau)^2 = E(\tau)$ of the square of the *Jacobi theta series*

$$\theta(\tau) \;:=\; \sum_{n\in\mathbb{Z}} q^{n^2} \qquad \text{with} \quad q := e^{2\pi i\tau} \quad \text{for} \quad \tau\in\mathbb{C},\ \Im(\tau)>0$$

with the *Eisenstein series* for the non-trivial Dirichlet character $\chi\colon (\mathbb{Z}/4\mathbb{Z})^\times \to \{\pm 1\}$

$$E(\tau) \;:=\; 1 + c_1\cdot \sum_{n\geq 1}\Big(\sum_{\text{odd } d|n}\chi(d)\Big)q^n,$$

yields $c_1 = 4$ and Jacobi's formula for the *representation number* of $n$ as a sum of two squares

$$\#\{\,(x,y)\in\mathbb{Z}^2\colon n = x^2+y^2\,\} \;=\; 4\sum_{\text{odd } d|n}\chi(d),$$

because $\theta(\tau)^2 = \sum_{(x,y)\in\mathbb{Z}^2} q^{x^2+y^2}$. This was vastly generalized, first by Siegel and Weil, followed by contributions of many authors. In particular, Kudla [Kud97] discovered[1] a relation between an "arithmetic theta function" — a generating series of arithmetic cycles on an integral model of a Shimura curve — and the first central derivative of a Siegel–Eisenstein series on $\mathrm{Sp}_4$. The *Kudla program* was further developed by Kudla, Rapoport and others. It is an exciting success story in arithmetic geometry.

Equally exciting is the recent formulation of its function field analog by Feng, Yun and Zhang [FYZ21a, FYZ21b, FYZ23]. In the seminar we will study the first of the three articles, where special cycles on moduli stacks of unitary shtukas are constructed and their degree is expressed as the derivative of a Fourier coefficient of a Siegel-Eisenstein series. In the Oberseminar from last semester we already wandered through the world of function field shtukas. We will now discover another fascinating area in this world.

[1] Cited from [FYZ21a, page 1] where more on the history of the Kudla program can be found.

## 1.2 The main result

Let $X$ be a proper, smooth, geometrically connected curve over a finite field $\mathbb{F}_q$ with $q$ elements and characteristic $\neq 2$. Let $X' \to X$ be an étale cover of degree 2 with non-trivial automorphism $\sigma \in \mathrm{Aut}_X(X')$. Let $F$ and $F'$ be the function fields of $X$ and $X'$, respectively.

On the arithmetic side, the stack $\mathrm{Sht}^r_{U(n)}$ of "Hermitian shtukas" of rank $n$ with $r$ legs parameterizes chains of vector bundles with $\sigma$-Hermitian forms

$$\mathcal{F}_0 \overset{f_0}{\dashrightarrow} \mathcal{F}_1 \overset{f_1}{\dashrightarrow} \dots \overset{f_{r-1}}{\dashrightarrow} \mathcal{F}_r \xrightarrow{\ \sim\ } {}^\tau\mathcal{F}_0 \tag{1.1}$$

where the dashed arrows are isomorphisms of Hermitian vector bundles outside the graph of the corresponding leg, and ${}^\tau\mathcal{F}_0$ denotes the Frobenius pullback of $\mathcal{F}_0$. Feng, Yun, Zhang [FYZ21a] define special cycles $\mathcal{Z}^r_{\mathcal{E}}(a)$ on $\mathrm{Sht}^r_{U(n)}$ indexed by a vector bundle $\mathcal{E}$ of rank $m \in \{1, \dots, n\}$ and a *Hermitian map* $a\colon \mathcal{E} \to \sigma^*\mathcal{E}^\vee$ satisfying $\sigma^* a^\vee = a$, where $\mathcal{E}^\vee := \mathcal{H}om_{X'}(\mathcal{E}, \omega_{X'})$ and $\omega_{X'}$ is the sheaf of differential forms on $X'$. More precisely, $\mathcal{Z}^r_{\mathcal{E}}(a)$ classifies shtukas (1.1) together with maps $t_i\colon \mathcal{E} \to \mathcal{F}_i$ compatible with the isomorphisms $f_i$ such that $a$ is induced from the Hermitian forms on the $\mathcal{F}_i$. Forgetting $\mathcal{E}$ and the $t_i$ defines a morphism $\mathcal{Z}^r_{\mathcal{E}}(a) \to \mathrm{Sht}^r_{U(n)}$. When $m = 1$ these cycles are analogous to the Kudla-Rapoport cycles on unitary Shimura varieties. When $m = n$ and $a$ is injective, [FYZ21a] prove that $\mathcal{Z}^r_{\mathcal{E}}(a)$ is proper over $\mathbb{F}_q$ and defines a zero cycle $[\mathcal{Z}^r_{\mathcal{E}}(a)] \in \mathrm{Ch}_0(\mathcal{Z}^r_{\mathcal{E}}(a))$ whose degree is a well defined number $\deg[\mathcal{Z}^r_{\mathcal{E}}(a)] \in \mathbb{Q}$. The main result of [FYZ21a] expresses $\deg[\mathcal{Z}^r_{\mathcal{E}}(a)]$ as the $r$-th derivative of a Fourier coefficient of a Siegel-Eisenstein series, which we introduce next.

Let $W = (F')^{2n}$ be the standard skew-Hermitian space with the skew-Hermitian form $(v, w) := v^\top \cdot J \cdot \sigma(w)$ for $J = \left(\begin{smallmatrix} 0 & \mathrm{Id}_n \\ -\mathrm{Id}_n & 0 \end{smallmatrix}\right)$. Let $H_n = U(W)$ be the unitary group of $W$ over $F$, which is defined for any $F$-algebra $R$ by

$$H_n(R) := \left\{ g \in \mathrm{GL}_{2n}(F' \otimes_F R)\colon g^\top \cdot J \cdot (\sigma \otimes \mathrm{id}_R)(g) = J \right\}.$$

Consider the standard Siegel parabolic subgroup

$$P_n(R) := \left\{ g = \left(\begin{smallmatrix} \alpha & \beta \\ 0 & \delta \end{smallmatrix}\right) \in H_n(R)\colon \alpha \in \mathrm{GL}_n(F' \otimes_F R),\ \sigma(\alpha)^\top = \delta^{-1},\ \sigma(\beta)^\top = \beta \right\}.$$

Let $\mathbb{A}_F$ be the adèles and $\mathbb{O}_F$ be the integral adèles of $F$. Let $\eta\colon F^\times \backslash \mathbb{A}_F^\times / \mathbb{O}_F^\times \to \{\pm 1\} \subset \mathbb{C}^\times$ be the quadratic character from class field theory and let $\chi\colon (F')^\times \backslash \mathbb{A}_{F'}^\times / \mathbb{O}_{F'}^\times \to \mathbb{C}^\times$ be such that $\chi|_{\mathbb{A}_F^\times} = \eta^n$. Extend $\chi$ and the absolute value $|\bullet|_{F'}\colon (F')^\times \backslash \mathbb{A}_{F'}^\times / \mathbb{O}_{F'}^\times \to \mathbb{C}^\times$ to functions $P_n(F) \backslash P_n(\mathbb{A}_F) / P_n(\mathbb{O}_F) \to \mathbb{C}^\times$ by setting $\left(\begin{smallmatrix} \alpha & \beta \\ 0 & \delta \end{smallmatrix}\right) \mapsto \chi(\det \alpha)$ and $\left|\left(\begin{smallmatrix} \alpha & \beta \\ 0 & \delta \end{smallmatrix}\right)\right|_{F'} := |\det \alpha|_{F'}$. By the Iwasawa decomposition we have $H_n(\mathbb{A}_F) = P_n(\mathbb{A}_F) \cdot H_n(\mathbb{O}_F)$. For $s \in \mathbb{C}$ let

$$\Phi(\bullet, s)\colon\ H_n(\mathbb{A}_F) \longrightarrow \mathbb{C}^\times, \quad p \cdot k \longmapsto \Phi(p \cdot k, s) := \chi(p) \cdot |p|_{F'}^{s + n/2},$$

where $p \in P_n(\mathbb{A}_F)$ and $k \in H_n(\mathbb{O}_F)$. Thus, $\Phi$ is unramified and induced from $P_n$. On the automorphic side we define the *Siegel-Eisenstein series* (which depends on the choice of $\chi$) as

$$E(\bullet, s, \Phi)\colon\ H_n(\mathbb{A}_F) \longrightarrow \mathbb{C}, \quad E(g, s, \Phi) := \sum_{\gamma \in P_n(F) \backslash H_n(F)} \Phi(\gamma g, s), \tag{1.2}$$

which converges for $\Re(s) \gg 0$. It has a *Fourier expansion* as follows. An element $p = \left(\begin{smallmatrix} \alpha & \beta \\ 0 & \delta \end{smallmatrix}\right) \in P_n(F) \backslash P_n(\mathbb{A}_F) / P_n(\mathbb{O}_F)$ induces by projection onto

$$\alpha \in \mathrm{GL}_n(F') \backslash \mathrm{GL}_n(\mathbb{A}_{F'}) / \mathrm{GL}_n(\mathbb{O}_{F'}) = \mathrm{Bun}_{\mathrm{GL}_n, X'}(\mathbb{F}_q)$$

a vector bundle $\mathcal{E} \in \mathrm{Bun}_{\mathrm{GL}_n, X'}(\mathbb{F}_q)$ of rank $n$ on $X'$. Since the function $\Phi$ of $p \cdot k$ only depends on $\alpha$, we write $m(\mathcal{E})$ for the argument of $\Phi(\bullet, s)$, and also of $E(\bullet, s, \Phi)$, that is $E(m(\mathcal{E}), s, \Phi)$. Then

$$E(m(\mathcal{E}), s, \Phi) = \sum_{a\colon \mathcal{E} \dashrightarrow \sigma^* \mathcal{E}^\vee} E_a(m(\mathcal{E}), s, \Phi)$$

is the expansion into *Fourier coefficients* $E_a(m(\mathcal{E}), s, \Phi)$ indexed by Hermitian rational maps $a\colon \mathcal{E} \otimes_{\mathcal{O}_{X'}} F' \to \sigma^*\mathcal{E}^\vee \otimes_{\mathcal{O}_{X'}} F'$ satisfying $\sigma^* a^\vee = a$. The Fourier coefficient is called *regular* if $a\colon \mathcal{E} \hookrightarrow \sigma^*\mathcal{E}^\vee$ is defined on all of $X'$ and is injective. The main theorem of [FYZ21a] is the following

**Theorem.** *Let $n \geq 1$ and $r \geq 0$. Let $\mathcal{E}$ be a rank $n$ vector bundle on $X'$ and $a\colon \mathcal{E} \hookrightarrow \sigma^*\mathcal{E}^\vee$ be an injective Hermitian map, that is $\sigma^* a^\vee = a$. Then*

$$\frac{1}{(\log q)^r}\Big(\frac{\mathrm{d}}{\mathrm{d}s}\Big)^r\Big|_{s=0}\Big(q^{ds}\widetilde{E}_a(m(\mathcal{E}), s, \Phi)\Big) \;=\; \deg\big[\mathcal{Z}^r_{\mathcal{E}}(a)\big], \tag{1.3}$$

*where $d = -\deg(\mathcal{E}) + n\deg\omega_X = -\chi(X', \mathcal{E})$, and where $\widetilde{E}_a$ is a suitable normalization of the regular Fourier coefficient $E_a$.*

For the purpose of introducing the content of the talks of this seminar, let us explain the strategy of the proof[2]. It uses and generalizes the geometrization technique originally developed by Yun and Zhang [YZ17]. For the left, respectively right side of the equation (1.3) one constructs a perverse sheaf $\mathcal{K}^{\mathrm{Eis}}_d(T)$, respectively $\mathcal{K}^{\mathrm{Int}}_d(T)$ on the Artin stack $\mathrm{Herm}_{2d}$ of Hermitian torsion sheaves on $X'$. The pair $(\mathcal{E}, a)$ defines a point $\mathcal{Q} := \mathrm{coker}(a) \in \mathrm{Herm}_{2d}$. Then $\widetilde{E}_a(m(\mathcal{E}), s, \Phi)$, respectively $\deg[\mathcal{Z}^r_{\mathcal{E}}(a)]$ can be computed as the trace of Frobenius at $\mathcal{Q}$ on the sheaf $\mathcal{K}^{\mathrm{Eis}}_d(T)$, respectively on the $r$-th derivative of the sheaf $q^{ds}\mathcal{K}^{\mathrm{Int}}_d(T)$. In particular, both sides only depend on $\mathcal{Q}$ and not on $\mathcal{E}$. The proof is then completed by identifying these two perverse sheaves using a Hermitian variant of Springer theory.

More precisely, on the arithmetic side, the connection between $\mathcal{Z}^r_{\mathcal{E}}(a)$ and Hermitian Springer theory comes via the geometry of a "Hitchin" fibration. The degree of $[\mathcal{Z}^r_{\mathcal{E}}(a)]$ is essentially an intersection number of cycles on $\mathrm{Sht}^r_{U(n)}$. The ambient space $\mathrm{Sht}^r_{U(n)}$ can itself be realized as an intersection of a Hecke correspondence with the graph of the Frobenius endomorphism on the stack $\mathrm{Bun}_{U(n)}$ of $\sigma$-Hermitian vector bundles of rank $n$ on $X'$. The authors use this to "unfold" all the intersections, and then redo them in a different order, performing the linear intersections (i.e., those not involving the Frobenius map) first, and leaving the Frobenius semi-linear intersection till the last step. In this process, a Hitchin-type moduli stack $\mathcal{M}_d$ appears naturally as they perform the linear intersections. The degree of the special cycle $[\mathcal{Z}^r_{\mathcal{E}}(a)]$ can be expressed as a weighted counting of $\mathbb{F}_q$-points on the fiber of the "Hitchin" fibration $f_d\colon \mathcal{M}_d \to \mathcal{A}_d$ over the point $(\mathcal{E}, a) \in \mathcal{A}_d(\mathbb{F}_q)$.

The cokernel $\mathcal{Q} = \mathrm{coker}(a)$ is a torsion sheaf on $X'$ with a Hermitian structure inherited from $a$. This motivates the introduction of the moduli stack $\mathrm{Herm}_{2d}$ that parameterizes torsion coherent sheaves on $X'$ of length $2d$ together with a Hermitian structure, so that $\mathcal{Q}$ is an $\mathbb{F}_q$-point of $\mathrm{Herm}_{2d}$. One shows that the fiber of $f_d\colon \mathcal{M}_d \to \mathcal{A}_d$ over $(\mathcal{E}, a)$ depends only on $\mathcal{Q} = \mathrm{coker}(a)$, and hence the degree of $[\mathcal{Z}^r_{\mathcal{E}}(a)]$ depends only on the point $\mathcal{Q}$ of $\mathrm{Herm}_{2d}$. There is a smooth map $\mathcal{A}_d \to \mathrm{Herm}_{2d}$, and the direct image complex $\mathrm{R}f_{d,*}\overline{\mathbb{Q}}_\ell$ on $\mathcal{A}_d$ descends to a perverse sheaf $\mathcal{K}^{\mathrm{Int}}_d$ on $\mathrm{Herm}_{2d}$.

On the automorphic side, the Fourier coefficients $\widetilde{E}_a(m(\mathcal{E}), s, \Phi)$ of the Eisenstein series $E(g, s, \Phi)$ are expressed by local density formulas for Hermitian lattices of Cho-Yamauchi. The Eisenstein series (1.2) can be written as a product of local terms, which are representation density functions for Hermitian lattices. These density functions again only depend on the torsion sheaf $\mathcal{Q} \in \mathrm{Herm}_{2d}$ together with its Hermitian structure. They can be computed with Hermitian Springer theory. Classically, starting with a reductive Lie algebra $\mathfrak{g}$, Springer theory outputs a perverse sheaf $\mathrm{Spr}_{\mathfrak{g}}$ on $\mathfrak{g}$, defined as the direct image complex of the Grothendieck-Springer resolution $\pi_{\mathfrak{g}}\colon \widetilde{\mathfrak{g}} \to \mathfrak{g}$. This was globalized by Laumon, who developed a version of Springer theory for the moduli stack $\mathrm{Coh}_d$ of torsion coherent sheaves of length $d$ on $X$. In the article,

[2]The following paragraphs are quoted from [FYZ21a, § 1.2].

$\mathrm{Herm}_{2d}$ plays the role of $\mathfrak{g}$. It has a Springer resolution $\pi_{2d}^{\mathrm{Herm}}\colon \widetilde{\mathrm{Herm}}_{2d} \to \mathrm{Herm}_{2d}$ that gives rise to the Hermitian Springer sheaf $\mathrm{Spr}_{2d}^{\mathrm{Herm}} := R\pi_{2d,*}^{\mathrm{Herm}}\overline{\mathbb{Q}}_\ell$. Both sheaves $\mathcal{K}_d^{\mathrm{Eis}}$ and $\mathcal{K}_d^{\mathrm{Int}}$ are linear combinations of direct summands of $\mathrm{Spr}_{2d}^{\mathrm{Herm}}$. The two versions of Springer theory for $\mathrm{Herm}_{2d}$ and $\mathrm{Coh}_d$ are used throughout the article.

# 2 Program

All unexplained references are to [FYZ21a].

**0. Overview (October 16).** The organizer will handle this.

**1. Background on algebraic stacks (October 23).**
Goals: Give an introduction to Artin and Deligne-Mumford stacks and to smooth and proper morphisms of Artin stacks.

**2. Fourier coefficients of Eisenstein series (October 30).** Cover all of Section 2, except for the proof Theorem 2.2.
Goals: Expand the Siegel-Eisenstein series $E(g, s, \Phi)$ into Fourier coefficients $E_a(m(\mathcal{E}), s, \Phi)$ and express the latter by local density functions for Hermitian lattices.

**3. Springer theory for torsion coherent sheaves (November 6).** Cover all of Section 3.
Goals: Introduce the stack $\mathrm{Coh}_d$ of torsion coherent sheaves and explain Springer theory for it, consisting of the Springer resolution $\pi_d^{\mathrm{Coh}}\colon \widetilde{\mathrm{Coh}}_d \to \mathrm{Coh}_d$ and the induced perverse sheaf $\mathrm{Spr}_d := R\pi_{d,*}^{\mathrm{Coh}}\,\overline{\mathbb{Q}}_\ell$ on $X$. All of Section 3 will be needed in Sections 4 and 5.

**4.(a) Springer theory for Hermitian torsion sheaves (November 13).** Cover §§ 4.1–4.4 of Section 4.
Goals: Introduce the stack $\mathrm{Herm}_{2d}$ of Hermitian torsion sheaves on $X'$ and explain Springer theory for it. The goal is Proposition 4.13 in the next Talk 4.(b).

**4.(b) Springer theory for Hermitian torsion sheaves (November 20).** Cover §§ 4.5, 4.6 of Section 4.
Goals: Proposition 4.13 which compares the stalks and the action of Frobenius on $\mathrm{Spr}_d$ and $\mathrm{HSpr}_d$. The latter is defined as a suitable subsheaf of $\mathrm{Spr}_{2d}^{\mathrm{Herm}}$. This comparison will allow to express the Frobenius trace on $\mathrm{HSpr}_d$ and $\mathcal{K}_d^{\mathrm{Eis}}(T)$ in terms of local densities in the next talk.

**5. Geometrization of local densities (November 27).** Cover Section 5.
Goals: Theorem 5.3, which expresses the local densities as trace of Frobenius on the perverse sheaf $\mathcal{K}_d^{\mathrm{Eis}}(T)$. Together with Theorem 2.7 it gives a formula for the Fourier coefficient $E_a(m(\mathcal{E}), s, \Phi)$ in terms of the trace of Frobenius on $\mathcal{K}_d^{\mathrm{Eis}}(q^{-2s})$ at the point $\mathcal{Q} = \operatorname{coker} a$.

**6. Moduli of Hermitian shtukas and special cycles (December 4).** Cover all of Section 6 and §§ 7.1, 7.2 including all proofs except for Lemma 6.8.
Goals: Define the stacks $\mathrm{Bun}_{U(n)}$ of Hermitian bundles, $\mathrm{Hk}_{u(n)}^r$ of Hermitian Hecke data with $r$ legs and modifications of length 1, and $\mathrm{Sht}_{U(n)}^r$ of Hermitian shtukas, and prove their properties in § 6.4. Define the special cycles $\mathcal{Z}_{\mathcal{E}}^r(a)$.

**7.(a) Special cycles: basic properties (December 11).** Cover §§ 7.3–7.6 of Section 7.
Goals: Prove that $\mathcal{Z}_{\mathcal{E}}^r(a) \to \mathrm{Sht}_{U(n)}^r$ is finite and prove the functoriality properties of $\mathcal{Z}_{\mathcal{E}}^r(a)$ with respect to $\mathcal{E}$. Mention the results about $\operatorname{rk}\mathcal{E} = 1$ and $\operatorname{rk}\mathcal{E} = n$ from §§ 7.5, 7.6.

**9. Special cycles of corank 1 (December 18).** Cover all of Section 9 including all proofs, maybe except of Lemma 9.3 and in § 9.2.
Goals: For $m = 1$ construct a stratification of the cycle $\mathcal{Z}^r_{\mathcal{L}}(a)$ and compute its dimension. This will be used in the next talk to show that $[\mathcal{Z}^r_{\mathcal{E}}(a)] \in \mathrm{Ch}_0(\mathcal{Z}^r_{\mathcal{E}}(a))$ when $\mathcal{E} = \mathcal{L}_1 \oplus \ldots \oplus \mathcal{L}_n$ is a direct sum of line bundles.

**7.(b) Intersection theory on stacks (January 8).** Cover §§ 7.7–7.9 of Section 7.
Goals: Explain intersection theory on Deligne-Mumford stacks and apply it to $\mathcal{Z}^r_{\mathcal{E}}(a)$ in order to define the class $[\mathcal{Z}^r_{\mathcal{E}}(a)] \in \mathrm{Ch}_0(\mathcal{Z}^r_{\mathcal{E}}(a))$ when $\mathcal{E}$ is a direct sum of line bundles. More precisely, define the Chow group $\mathrm{Ch}_{c,i}(X)$ of proper cycles on an Artin stack $X$, Gysin maps for l.c.i. morphisms, the intersection product, see [Kre99, Ful98], and the degree map $\deg\colon \mathrm{Ch}_{c,0}(X) \to \mathbb{Q}$ of a Deligne-Mumford stack $X$, see [YZ17, Appendix A] and [Kre99, § 3.3].

**8. Hitchin type moduli spaces (January 15).** Cover all of Section 8 including all proofs.
Goals: Introduce the Hitchin stack $\mathcal{M}_d$, base $\mathcal{A}_d$ and fibration $f_d\colon \mathcal{M}_d \to \mathcal{A}_d$, and in the special case $m = n$ also Hecke and Shtuka stacks for Hitchin spaces. Prove properties of the Hitchin fibration $f_d$ and of the map $\mathcal{A}_d \to \mathrm{Herm}_{2d}$. Define the 0-cycle class $[\mathrm{Sht}^r_{\mathcal{M}_d}]$ (Def. 8.16).

**11. Local intersection number and trace formula (January 22).** Cover §§ 11.1–11.3 from Section 11 in the case that $\mathcal{E}$ is a direct sum of line bundles.
Goals: Show that $[\mathcal{Z}^r_{\mathcal{E}}(a)]$ only depends on the Hermitian torsion sheaf $\mathcal{Q} = \mathrm{coker}(a)$ and prove a formula which relates its degree to the $r$-th derivative at $s = 0$ of the Frobenius trace of $\mathcal{K}^{\mathrm{Int}}_d(q^{-2s})$ at $\mathcal{Q}$. Avoid Theorem 10.1 by taking $\mathcal{E} = \oplus_i \mathcal{L}_i$ and use Proposition 10.9 as a black box.

**12. Matching of sheaves (January 29).** Cover all of Section 12.
Goals: Finish the proof of the higher Siegel–Weil formula by proving that $\mathcal{K}^{\mathrm{Eis}}_d(T) = \mathcal{K}^{\mathrm{Int}}_d(T)$.

## References


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