OBERSEMINAR: SHTUKAS FOR REDUCTIVE GROUPS AND GLOBAL LANGLANDS PARAMETRIZATION

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1. Introduction

The aim of this Oberseminar is to understand the spectral decomposition theorem of V. Lafforgue [Laf18] and the ingredients that go into its proof. This theorem fulfills, in complete generality, the "automorphic-to-Galois" direction of the Langlands program for function fields.

1.1. Why should we learn [Laf18]**?** The reciprocity part of the global Langlands program conjectures that automorphic forms attached to a reductive group G are parametrized by Galois representations valued in its dual group $\check{\mathrm{G}}$. The conjecture can be formulated both over a number field and over a function field, the function field version being more amenable to tools from algebraic geometry.

In the function field context, Langlands reciprocity is known for $\mathrm{G} = \mathrm{GL}_2$ by the works of Drinfeld, and for $\mathrm{G} = \mathrm{GL}_n$ by the works of L. Lafforgue. In both cases, one first establishes the Langlands parametrization of automorphic forms and then deduces their existence from the "converse theorem", which is special to GL_n.

In [Laf18], the Langlands parametrization is constructed uniformly for all reductive groups. Although [Laf18] imports the notion of Shtukas introduced by Drinfeld, its method is radically new (even for GL_n) and produces the action of a new commutative algebra $\mathscr{B}$ on the space of automorphic forms. The action of $\mathscr{B}$ implies the "automorphic-to-Galois" direction of Langlands reciprocity. The "Galois-to-automorphic" direction is still unknown for groups other than GL_n, for a lack of tools to construct automorphic forms.

Besides its generality, the method of [Laf18] is remarkably self-contained: it relies on the geometric Satake equivalence and essentially nothing else (no trace formulas, no compactification of Shtukas, etc.) For the same reason, this construction is extremely flexible and triggered an avalanche of results of "spectral action type" in various other contexts of the Langlands program, including the topological case of Nadler–Yun and the p-adic local case of Fargues–Scholze.

1.2. The main theorem. We will now give a more precise statement of the main theorem of [Laf18].

1.2.1. Let k be a finite field with q elements. Let X be a smooth, proper, geometrically connected curve over k. Its field of fractions is denoted by F. Associated to F are the rings of adèles $\mathbb{A}$ and of integral adèles $\mathbb{O}$. We will also fix an algebraic closure $\bar{\mathrm{F}}$ of F.

Let G be a *split* reductive group.[1] We write $\mathrm{Z} \subset \mathrm{G}$ for its center and fix a cocompact lattice $\Xi \subset \mathrm{Z}(\mathrm{F})\backslash\mathrm{Z}(\mathbb{A})$.

Let $\mathrm{N} \subset \mathrm{X}$ be a closed subscheme finite over k. It determines an open compact subgroup $\mathrm{K}_{\mathrm{N}} \subset \mathrm{G}(\mathbb{O})$ as the kernel of the projection onto $\mathrm{G}(\mathscr{O}_{\mathrm{N}})$.

[1] The article [Laf18] also treats nonsplit reductive groups, but we will not worry about them.

Finally, our field of coefficients is a fixed algebraic closure $\overline{\mathbb{Q}}_\ell$ of $\mathbb{Q}_\ell$, where ℓ is a prime not dividing q.[2] For the proof, we will actually consider finite extensions $\mathrm{E} \supset \mathbb{Q}_\ell$ contained in $\overline{\mathbb{Q}}_\ell$ instead, and the Langlands dual group of G will be regarded as a pinned split reductive group $\check{\mathrm{G}}$ over E.

To invoke the Satake isomorphism, we fix a square root $q^{1/2}$ in $\overline{\mathbb{Q}}_\ell$ and the finite extension E is supposed to contain it.

1.2.2. With the notations of §1.2.1, we define the space of automorphic forms to be the vector space of $\overline{\mathbb{Q}}_\ell$-valued compactly supported functions on $\mathrm{G}(\mathrm{F})\backslash\mathrm{G}(\mathbb{A})/\mathrm{K}_\mathrm{N}\Xi$:

$$\mathrm{C}_c(\mathrm{G}(\mathrm{F})\backslash\mathrm{G}(\mathbb{A})/\mathrm{K}_\mathrm{N}\Xi, \overline{\mathbb{Q}}_\ell). \tag{1.1}$$

The vector space (1.1) contains a finite-dimensional subspace:

$$\mathrm{C}_c^{\mathrm{cusp}}(\mathrm{G}(\mathrm{F})\backslash\mathrm{G}(\mathbb{A})/\mathrm{K}_\mathrm{N}\Xi, \overline{\mathbb{Q}}_\ell) \tag{1.2}$$

consisting of functions f whose constant term:

$$f_\mathrm{P} : \mathrm{G}(\mathbb{A}) \ni g \mapsto \int_{\mathrm{U}(\mathrm{F})\backslash\mathrm{U}(\mathbb{A})} f(ug)$$

vanishes for all proper parabolic subgroups $\mathrm{P} \subset \mathrm{G}$ with unipotent radical U.

1.2.3. For each closed point v of $\mathrm{X}\backslash\mathrm{N}$ and (finite-dimensional, algebraic) representation V of $\check{\mathrm{G}}$, the Satake isomorphism produces an element in the unramified Hecke algebra:

$$h_{\mathrm{V},v} \in \mathrm{Fun}_c(\mathrm{G}(\mathrm{O}_v)\backslash\mathrm{G}(\mathrm{F}_v)/\mathrm{G}(\mathrm{O}_v), \overline{\mathbb{Q}}_\ell),$$

where O_v, F_v stand for the completed local ring, respectively the local field at v.

The operators $h_{\mathrm{V},v}$ act on (1.1), preserving the subspace of cusp forms (1.2).

The Langlands parametrization of cusp forms amounts to diagonlizing (1.2) simultaneously for all the Hecke operators.

1.2.4. The main result of [Laf18] is the construction of a commutative algebra $\mathscr{B}$ acting on (1.2). Consequently, we obtain a decomposition:

$$\mathrm{C}_c^{\mathrm{cusp}}(\mathrm{G}(\mathrm{F})\backslash\mathrm{G}(\mathbb{A})/\mathrm{K}_\mathrm{N}\Xi, \overline{\mathbb{Q}}_\ell) \cong \bigoplus_{\nu:\mathscr{B}\to\overline{\mathbb{Q}}_\ell} \mathbf{H}_\nu, \tag{1.3}$$

where each $\mathbf{H}_\nu$ is the generlized eigenspace for the character ν.

Furthermore, each character ν determines a $\check{\mathrm{G}}(\overline{\mathbb{Q}}_\ell)$-conjugacy class of continuous morphisms $\sigma : \mathrm{Gal}(\bar{\mathrm{F}}/\mathrm{F}) \to \check{\mathrm{G}}(\overline{\mathbb{Q}}_\ell)$ which are defined over a finite extension of $\mathbb{Q}_\ell$, semi-simple, and unramified outside N. Each Hecke operator $h_{\mathrm{V},v}$ acts on $\mathbf{H}_\nu$ as multiplication by the scalar $\mathrm{Tr}(\sigma(\mathrm{Fr}_v) \mid \mathrm{V})$, where Fr_v is a lift of the geometric Frobenius element at v.

The description of the Hecke action on these summands shows that (1.3) fulfills the "automorphic-to-Galois" direction of Langlands reciprocity.

1.3. Suggestions to the speakers. There is an English version of the introduction `https://arxiv.org/abs/1404.6416` which might be helpful.

The notations of [Laf18] are sometimes heavy, and we recommend that you drop some of the indices, especially when they are constant during your talk.

The program below is not meant to be followed strictly. To the contrary, you are highly encouraged to present your own understanding of the material instead.

2. Program

2.1. Overview (April 14). The organizers will handle this.

[2] The article [Laf18] also treats the case with $\overline{\mathbb{F}}_\ell$-coefficients. This will also be omitted from the seminar.

2.2. Geometric players (April 21). Introduce the moduli stack of G-bundles, the iterated Hecke stacks, as well as the Beilinson–Drinfeld affine Grassmannian. All the objects of [Laf18, §1] are supposed to appear in this talk, excluding Theorem 1.17.

Two essential things to explain are the Weil uniformization and the finite-dimensionality of the space of cusp forms [Har74, Theorem 1.2.1].

We recommend that you prove as many properties of these geometric objects as possible (Are they smooth? Are they ind-schematic? Are they quasi-compact? What are the natural maps among them?). Other references include [Zhu17, §3-4] and [Xue20a, §1].

2.3. Geometric Satake I (April 28). Define the Satake category at a $\bar{k}$-point of X and understand it on the level of abelian category, *i.e.* prove that it is semisimple and describe its irreducible objects.

Ultimately, we will need the Satake category over multiple copies of X, but this talk plays an important pedagogical role as it gives us concrete objects to play with.

Please define ℓ-adic perverse sheaves and give some examples. You will also need to define them on ind-schemes of ind-finite type and deal with some quotient situations, as explained in [Zhu17, §5.1, Appendix A.1]. The proof of semisimplicity is documented in [BR18, §4], the key ingredient being Lusztig's parity vanishing.

2.4. Geometric Satake II (May 5). Equip the Satake category with the structure of a symmetric monoidal category.

You may first define the convolution monoidal structure [Zhu17, §5.1], but it is fine to leave out Lusztig's theorem that convolution preserves perversity [Zhu17, Proposition 5.1.4]. (This important fact will follow from "convolution = fusion", as explained there).

The main topics in this talk are the construction of the fusion symmetric monoidal structure [Zhu17, §5.4] (or the original [MV07, §5]), and the proof of "convolution = fusion". These results involve the Beilinson–Drinfeld affine Grassmannian in the most essential way.

The argument in [Zhu17] uses the notion of universal local acyclicity, which is a wonderful concept to explain. You may also systematically substitute perversity by relative perversity [HS21], as you see fit.

2.5. Geometric Satake III (May 12). Construct the fiber functor and prove that the Satake category is equivalent to $\mathrm{Rep}(\check{G})$ via Tannaka duality.

You may follow [BR18, §5, 8-9] for example. It is not necessary to introduce the functor of total cohomology: what *op.cit.* calls "weight functor" is a more natural candidate for the fiber functor. (It is the constant term functors for the Satake category.) Please explain why it is symmetric monoidal (after tweaking the commutativity constraint), conservative, and perverse t-exact. The t-exactness uses Braden's hyperbolic localization theorem, which is a nice thing to explain.

The reconstruction of $\check{G}$ is not so important[3], and you may say as much or as little about it as you like. It is perfectly fine to do everything over a $\bar{k}$-point of X, but if you are more ambitious, you could also do things over X^I and reconstruct $\check{G}^I$ directly, as done in [FS21, VI.10-11].

2.6. Cohomology of Shtukas (May 19). Introduce the moduli stack of (global, iterated) Shtukas ([Laf18, §2], [Var04, §2]) and construct the family of functors indexed by finite sets I, as in [Laf18, Définition 4.7, Proposition 4.12]:

$$\mathrm{Rep}(\check{G}^I) \to \mathrm{Ind}(\mathrm{Shv}_c(X^I)), \quad W \mapsto \mathscr{H}_{I,W}. \tag{2.1}$$

[3] As a wise man once said: geometric Satake is the definition of $\check{G}$.

Here, $\mathrm{Shv}_c(\mathrm{X}^{\mathrm{I}})$ stands for the category of constructible E-sheaves, and $\mathrm{Ind}(\mathrm{Shv}_c(\mathrm{X}^{\mathrm{I}}))$ its ind-completion. What we denote by $\mathscr{H}_{\mathrm{I},\mathrm{W}}$ here is the direct limit of Lafforgue's (4.9) over the Harder–Narasimhan strata. For our purposes, only the degree-0 piece plays a role.

To construct (2.1), you will first have to bootstrap the geometric Satake equivalence to the Satake functors of [Laf18, Théorème 1.17]. An alternative to Lafforgue's way of doing it appears in [NY19, Remark 6.1.2].

The point of this talk is to really get into the geometry of the moduli stack of Shtukas and justify why (2.1) is well-defined. In particular, the role of this lattice Ξ will be explained. You are also encouraged to give some examples with GL_n-Shtukas.

2.7. Structures on $\mathscr{H}_{\mathrm{I},\mathrm{W}}$ (May 26). Equip the sheaves $\mathscr{H}_{\mathrm{I},\mathrm{W}}$ with the following pieces of structure:

(1) actions of partial Frobenii [Laf18, §3, §4.3];
(2) actions of the "trace-of-Frobenius" operators $\mathrm{S}_{\mathrm{V},v}$ [Laf18, §6.1];
(3) action of the Hecke algebra [Laf18, Construction 2.20, §4.4];
(4) the constant term morphism [Xue20a, §3.5];

It is nice to make explicit the notion of cohomological correspondences [Laf18, §4.1], as they will come in handy in during the next two talks.

Note that (4) uses the compatibility of the geometric Satake equivalence with the constant term functors, but this follows from the way we built the fiber functor in Talk 2.5.

Finally, we ask you to define the cuspidal cohomology $\mathscr{H}^{\mathrm{cusp}}_{\mathrm{I},\mathrm{W}}$ of the moduli of Shtukas [Xue20a, §3.5] and explain how it recovers the space of cusp forms when the Hecke modification is trivial [Xue20a, Example 3.5.15].

2.8. "S = T" I (June 9). This talk and the next are devoted to the proof of the "S = T" Theorem [Laf18, Proposition 6.2]. It shows that $\mathrm{S}_{\mathrm{V},v}$ extends the Hecke operator $\mathrm{T}(h_{\mathrm{V},v})$ to points where the latter is not defined (at the paws).

The goal of this talk is to reduce [Laf18, Proposition 6.2] to [Laf18, Lemme 6.11].

We recommend that you start by explaining the Grothendieck–Lefschetz trace formula [Laf18, §6.2] (and see the referenced parts of [Var07].)

2.9. "S = T" II (June 16). Prove [Laf18, Lemme 6.11] and state the Eichler–Shimura relation [Laf18, Proposition 7.1].

According to [Laf18, Remarque 6.19], the proof of Lemme 6.11 is not optimal. We suspect that the "arguments généraux" alluded to in Remarque 6.19 are closely related to [Var07] and we invite the speaker to try and figure them out with us. Ideally, this talk would then deal with a general framework of cohomological correspondences. Regardless, looking at the minuscule case [Laf18, 0.16] first could help to clarify the ideas and the principal difficulties.

The Eichler–Shimura relation is rather formal (especially after one recognizes the analogy with Cayley–Hamilton), so we do not anticipate too much explanation here.

2.10. Drinfeld's Lemma (June 23). Using the partial Frobenii equivariance of the sheaves $\mathscr{H}_{\mathrm{I},\mathrm{W}}$, equip their geometric generic fibers with an action of $\mathrm{Weil}(\eta,\overline{\eta})^{\mathrm{I}}$ [Xue20c, Proposition 1.3.4].

The proof of the proposition combines two ingredients: Drinfeld's Lemma and the Eichler–Shimura relation (to salvage the non-constructibility of $\mathscr{H}_{\mathrm{I},\mathrm{W}}$).

The proof of Drinfeld's Lemma is documented in [Xue20b, §3.2] and we think it is worthwhile to both present the proof and to explain why the partial Frobenii equivariance is necessary (see [HRS20, Example 1.13].)

This talk and the next are essentially a study of Xue's work [Xue20b] [Xue20c], which we will use to replace some arguments in [Laf18, §8-9].

2.11. Smoothness (June 30). The goal of this talk is to prove that the $\mathrm{Weil}(\eta, \bar{\eta})^{\mathrm{I}}$-action constructed in the last talk factors through $\mathrm{Weil}(\mathrm{X}\backslash\mathrm{N}, \bar{\eta})^{\mathrm{I}}$ [Xue20c, Proposition 5.0.4].

In doing so, you will have to prove that $\mathscr{H}_{\mathrm{I,W}}$ is an ind-lisse sheaf over $(\mathrm{X}\backslash\mathrm{N})^{\mathrm{I}}$ [Xue20c, Theorem 4.2.3]. This theorem uses much of the material of [Xue20c, §2-3].

Finally, deduce the analogous results for the cuspidal cohomology sheaves $\mathscr{H}_{\mathrm{I,W}}^{\mathrm{cusp}}$, following [Xue20c, §7].

2.12. Excursion (July 7). Define the excursion operator:

$$\mathrm{S}_{\mathrm{I,W},x,\xi,(\gamma_i)_{i\in\mathrm{I}}} \in \mathrm{End}\, \mathrm{C}_c^{\mathrm{cusp}}(\mathrm{Bun}_{\mathrm{G,N}}(k)/\Xi, \mathrm{E})$$

of [Laf18, Définition-Proposition 9.1]. Note that, thanks to Xue's smoothness theorem for $\mathscr{H}_{\mathrm{I,W}}^{\mathrm{cusp}}$, you may define these operators without referring to Hecke-finiteness and also directly taking $(\gamma_i)_{i\in\mathrm{I}}$ from $\pi_1(\mathrm{X}\backslash\mathrm{N}, \bar{\eta})^{\mathrm{I}}$.

Then you could explain some properties of excursion operators in [Laf18, §10], the most important one being their pairwise commutativity. These are formal consequences of the definition, so you could be brief about them.

Your goal from there will be to prove the main theorem [Laf18, Théorème 11.11]. Only one new ingredient is needed now: [Laf18, Proposition 11.7]. It involves a nice discussion of geometric invariant theory.

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