

Introduction

The goal of this Oberseminar is threefold: (1) introduce the BGG category $\mathcal{O}$ and discuss its basic theory, (2) prove the Beilinson–Bernstein localization theorem and develop the operations on algebraic D -modules, and (3) use de Rham stacks and the six-functor formalism to obtain equivariant localization for $(\mathfrak{g}, B)$ -modules.

Let k be a field of characteristic zero, let G/k be a split, connected, simply connected semisimple algebraic group, and fix a Borel subgroup B and a maximal torus $T \subset B$. Write $X = G/B$ for the flag variety, $\mathfrak{g} = \text{Lie}(G)$, $\mathfrak{h} = \text{Lie}(T)$ and $\mathfrak{b} = \text{Lie}(B)$. Let $\Phi^+ \subseteq \Phi(G, T)$ be the positive roots corresponding to B , and put $\rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha$. Denote by W the Weyl group, by $P = X^*(T)$ the weight lattice, and by P_+ the dominant weights. Put $\mathfrak{n}^\pm = \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_{\pm\alpha}$.

The first part studies highest weight modules and the category $\mathcal{O}$ of finitely generated $U(\mathfrak{g})$ -modules which are direct sums of $\mathfrak{h}$ -weight spaces and locally finite under $\mathfrak{b}$. Starting with the theorem of the highest weight, we discuss the Harish–Chandra homomorphism, the block decomposition and BGG duality. The existence of projective covers and their Verma filtrations will allow us to view the blocks of $\mathcal{O}$ as highest weight categories; we then study translation functors ([SoO, §§2.1–2.5]; [Hum, Chapters 1, 3 and 7]). Finally, we prove Borel–Weil–Bott following [Lur], construct the BGG resolution from the Cousin complex and its local-cohomology description ([Kum, Theorems 40 and 42]), and deduce Weyl’s character and dimension formulas.

For a smooth variety Y and a vector bundle $\mathcal{V}$ on Y , let $\mathcal{D}_Y^\mathcal{V}$ denote the sheaf of differential operators acting on $\mathcal{V}$, equipped with its filtration by order ([HTT, §11.1]). To an integral weight η we associate the G -equivariant line bundle $\mathcal{L}_\eta = G \times^B k_\eta$ on X , and set

$$\mathcal{D}_\lambda = \mathcal{D}_X^{\mathcal{L}_{\lambda+\rho}}.$$

We normalize the Harish–Chandra isomorphism so that a Verma module of highest weight μ has central character $\chi_{\mu+\rho}$, and write $U_\lambda = U(\mathfrak{g})/U(\mathfrak{g}) \ker \chi_\lambda$. An integral weight λ is antidominant if $\langle \lambda, \alpha^\vee \rangle \leq 0$ for all $\alpha \in \Phi^+$, and regular if these pairings are nonzero.

Theorem 1 (Beilinson–Bernstein, [HTT, Theorems 11.2.2–11.2.4 and Corollary 11.2.6]). *For regular integral antidominant λ , the infinitesimal G -action induces an isomorphism $U_\lambda \simeq \Gamma(X, \mathcal{D}_\lambda)$. Global sections and localization give inverse equivalences*

$$\mathcal{D}_\lambda\text{-Mod}_{\text{qc}} \rightleftarrows U_\lambda\text{-Mod}, \quad \mathcal{M} \longmapsto \Gamma(X, \mathcal{M}), \quad V \longmapsto \mathcal{D}_\lambda \otimes_{U_\lambda} V.$$

Here quasi-coherence refers to the underlying $\mathcal{O}_X$ -module.

The second part is devoted to the proof of this theorem. We also discuss its relation with translation functors and state the derived localization theorem for regular weights ([Mil, Chapter 3, Theorems 1.12 and 1.21]). The description of the fibers of derived localization in terms of Lie algebra homology gives another interpretation of Borel–Weil–Bott.

The third part introduces the de Rham prestack

$$Y_{\text{dR}}(A) = Y(\text{Spec } A_{\text{red}})$$

for a commutative k -algebra A . For smooth Y , there is an equivalence between the derived category of quasi-coherent $\mathcal{D}_Y$ -modules and $\text{QCoh}(Y_{\text{dR}})$ ([RC25, Theorem 10.17]). These categories admit

the six-functor formalism constructed in [Sch, Theorem 8.24]; its exceptional functors require the normalization of [Sch, Remarks 8.31, 8.37]. In particular, Kashiwara’s theorem takes the form of excision ([Sch, Proposition 8.32]).

The same construction gives a geometric description of representations. If $\widehat{G}$ denotes the formal completion of G at the identity, then

$$\mathrm{QCoh}(B\widehat{G}) \simeq \mathrm{D}(U(\mathfrak{g})\text{-Mod})$$

by [RC, Proposition 5.2.15]. More generally, quasi-coherent sheaves on the classifying stack of $G \times_{G_{\mathrm{dR}}} B_{\mathrm{dR}}$ describe $(\mathfrak{g}, B)$ -modules ([RC, Proposition 5.3.15]). These are $\mathfrak{g}$ -modules with a compatible algebraic B -action whose differential agrees with the restricted $\mathfrak{b}$ -action.

We conclude by applying localization to these representations. Let $\chi_{\mathrm{triv}} = \chi_{-\rho}$, and let $\mathcal{O}_{\mathrm{triv}}^{\mathrm{exact}}$ denote the full subcategory of $\mathcal{O}$ on which $\ker \chi_{\mathrm{triv}}$ acts by zero.

Theorem 2 (Equivariant localization, [HTT, Theorem 11.5.3 and Proposition 12.2.1]). *Localization and global sections induce inverse equivalences*

$$\mathcal{O}_{\mathrm{triv}}^{\mathrm{exact}} \simeq \mathcal{D}_X\text{-Mod}_{\mathrm{coh}}^B,$$

where the right-hand side is the category of coherent, strongly B -equivariant $\mathcal{D}_X$ -modules.

By descent, this category is the full subcategory of $\mathrm{QCoh}([B \backslash G/B]_{\mathrm{dR}})^{\heartsuit}$ whose pullback to X is coherent as a $\mathcal{D}_X$ -module ([RC25, Proposition 5.28 and Theorem 10.17]; [HTT, Definition 11.5.2]). Excision gives a filtration by support on closed unions of Bruhat cells. The example of SL_2 relates this filtration to the Verma modules of the first part.

Program

I. Highest weight theory and category $\mathcal{O}$

(I) The theorem of the highest weight.

Recall the root decomposition of $\mathfrak{g}$, simple roots and coroots, the Weyl group, the order on weights and the finite-dimensional representation theory of $\mathfrak{sl}_2$ ([HTT, §§9.1–9.3]). Introduce the universal enveloping algebra and prove the Poincaré–Birkhoff–Witt theorem ([SoHL, Theorem 3.3.11]; [HTT, Theorem 9.1.1]). Define Verma modules

$$\Delta(\mu) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} k_{\mu}$$

and prove their universal property and the existence of a unique simple quotient $L(\mu)$ ([SoHL, Proposition 3.4.3, Lemma 3.4.6 and Theorem 3.4.7]; [Hum, Theorems 1.2–1.3]). Prove that $L(\mu)$ is finite-dimensional precisely when μ is dominant integral, obtaining the classification of finite-dimensional simple $\mathfrak{g}$ -modules ([SoHL, Proposition 3.4.10 and Lemma 3.4.13]; [Hum, Theorem 1.6]). Discuss the example of $\mathfrak{sl}_2$ throughout, including the sequence

$$0 \longrightarrow \Delta(-2) \longrightarrow \Delta(0) \longrightarrow L(0) \longrightarrow 0$$

([SoHL, Example 3.6.29]; [Hum, §1.5]).

(II) The center and category $\mathcal{O}$.

Construct the Harish–Chandra homomorphism and prove the Harish–Chandra isomorphism, taking Chevalley’s restriction theorem as an input ([SoO, Theorems 1.3.1 and 1.4.8]; [Hum, §§1.9–1.10]). Introduce the dot action $w \cdot \mu = w(\mu + \rho) - \rho$ and determine the central character of $\Delta(\mu)$; two Verma modules have the same central character precisely when their highest weights lie in the same dot orbit ([SoO, Proposition 1.5.5]; [Hum, Theorem 1.10]).

Define category $\mathcal{O}$ and establish its basic properties: stability under subquotients and tensor products with finite-dimensional modules, finite-dimensional weight spaces, finite length, and the classification of simple objects ([SoO, Definition 2.1.1 and Proposition 2.1.6]; [Hum, §§1.1–1.3 and Theorem 1.11]). Introduce formal characters and compute $\text{ch } \Delta(\mu) = e^\mu / \prod_{\alpha \in \Phi^+} (1 - e^{-\alpha})$ from PBW ([Hum, §§1.15–1.16]). Discuss the action of the Casimir element for $\mathfrak{sl}_2$ ([SoO, Example 2.1.8]; [Hum, §1.9, Example]).

(III) Blocks, duality and projective objects.

Describe the decompositions by generalized central character and by weight coset modulo the root lattice, and state the resulting block decomposition, proving indecomposability for integral central characters by embeddings of Verma modules ([SoO, Lemmas 2.2.2, 2.2.4 and Theorem 2.2.6]; [Hum, §§1.12–1.13 and Theorem 4.9]; the required simple-root embeddings are in [SoHL, Lemma 3.4.13] and [Hum, §4.3]). Construct BGG duality using the anti-involution fixing $\mathfrak{h}$ and interchanging positive and negative simple root vectors. Define $\nabla(\mu) = d\Delta(\mu)$; this duality preserves weights ([SoO, Exercises 2.1.16–2.1.17]; [Hum, Theorem 3.2 and §3.3]).

Prove the existence of enough projectives using truncated induction from $\mathfrak{h}$ to $\mathfrak{b}$ and then to $\mathfrak{g}$, as in [SoO, Proposition 2.3.5]; an English treatment of the projective objects and their filtrations is [Hum, §§3.8–3.10]. Prove that direct summands of modules with Verma filtrations again have such filtrations. In particular, construct a projective module surjecting onto $\Delta(\mu)$ whose kernel has a filtration with subquotients $\Delta(\nu)$ for $\nu > \mu$ ([SoO, Theorem 2.3.4 and Propositions 2.3.5–2.3.6]; [Hum, Theorems 3.8–3.10 and the remark following Theorem 3.10]). Describe projective covers and injective envelopes. State the finite highest weight category axioms and verify them for these projective covers. Deduce that each block is a finite highest weight category, with standard objects $\Delta(\mu)$ and costandard objects $\nabla(\mu)$, and is equivalent to the category of finite-dimensional modules over a quasi-hereditary algebra ([SoO, §2.3, especially 2.3.8]; [Hum, §§3.9–3.10 and 13.7–13.8]; [CPS, Example 3.3(c) and Theorem 3.6]). Describe the two indecomposable projectives in the principal $\mathfrak{sl}_2$ block ([SoO, Examples 2.1.8–2.1.14]; [Hum, §3.12]).

(IV) Translation functors.

Prove the tensor identity and describe the Verma filtration on $E \otimes \Delta(\mu)$ for a finite-dimensional $\mathfrak{g}$ -module E ([SoO, Proposition 2.4.1 and Lemma 2.5.9]; [Hum, Theorem 3.6 and its proof]). Define translation functors by tensoring with a finite-dimensional representation and projecting to a block. Prove their exactness, the adjunctions between translation in opposite directions, and compatibility with BGG duality ([SoO, Definition 2.5.3 and Lemma 2.5.5]; [Hum, §7.1 and Proposition 7.2]).

Compute translation on Verma modules and prove the equivalence between regular integral blocks ([SoO, Proposition 2.5.7 and Theorem 2.5.13]; [Hum, Theorems 7.6 and 7.8]). Discuss translation onto and out of a wall in the example of $\mathfrak{sl}_2$, and define translation through a wall ([SoO, 2.5.8, Definition 2.6.4 and Lemma 2.6.6]; [Hum, §7.7, Example, and §§7.14–7.15]).

(V) Flag varieties and Borel–Weil–Bott.

Introduce the flag variety, the Bruhat decomposition and the projections $G/B \rightarrow G/P_\alpha$ for minimal parabolic subgroups P_α ([HTT, Theorems 9.9.2, 9.9.4–9.9.5]; [Lur, Lemma 4]). Describe equivariant vector bundles as associated bundles and define $\mathcal{L}_\eta = G \times^B k_\eta$ ([HTT, §9.11, pp. 255–256]).

Prove Borel–Weil and Borel–Weil–Bott following [Lur, Theorems 2–3, Lemma 4 and Theorem 5]: use the cohomology of line bundles on $\mathbf{P}^1$, the minimal-parabolic fibrations and Serre duality. For $\mu \in P$, use the dot action $w \cdot \mu = w(\mu + \rho) - \rho$. If $\mu + \rho$ is singular, all cohomology of $\mathcal{L}_{-\mu}$ vanishes; otherwise, for the unique w such that $w \cdot \mu \in P_+$,

$$H^i(X, \mathcal{L}_{-\mu}) = \begin{cases} L(w \cdot \mu)^*, & i = \ell(w), \\ 0, & i \neq \ell(w). \end{cases}$$

This is [HTT, Theorem 9.11.2(iii)]; Lurie’s notation L^λ corresponds to $\mathcal{L}_{\rho-\lambda}$. The passage from complex coefficients to k follows by flat base change from a split model over $\mathbf{Q}$ ([Stacks, Tag 02KH]). Also record the global generation and ampleness criteria for $\mathcal{L}_\eta$ ([HTT, Theorem 9.11.2(i)–(ii)]).

(VI) The Cousin complex and the BGG resolution.

Introduce cohomology with locally closed support and the Cousin complex for the decreasing filtration by closed unions of opposite Bruhat cells of codimension at least p . Apply the exactness criterion of [Kum, Theorem 40] to $\mathcal{L}_{-\mu}$ for $\mu \in P_+$. Its term in degree p is

$$\bigoplus_{\ell(w)=p} \Delta(w \cdot \mu)^\vee,$$

where $M^\vee = \bigoplus_\eta (M_\eta)^*$ has the contragredient action $(x\varphi)(m) = -\varphi(xm)$ ([Kum, Theorem 42]). The local-cohomology identification may be stated in general, with a proof for SL_2 . Borel–Weil–Bott identifies the degree-zero cohomology with $L(\mu)^*$; taking the restricted contragredient dual gives the BGG resolution

$$0 \longrightarrow \Delta(w_0 \cdot \mu) \longrightarrow \cdots \longrightarrow \bigoplus_{\ell(w)=1} \Delta(w \cdot \mu) \longrightarrow \Delta(\mu) \longrightarrow L(\mu) \longrightarrow 0$$

for $\mu \in P_+$ ([Kum, Theorem 37 and §17]).

Deduce Weyl’s character formula by taking the alternating sum of the characters in this resolution, and derive the dimension formula ([Kum, §15]; [SoHL, Theorems 3.6.8 and 3.6.2]; [Hum, Theorems 2.4–2.5]).

II. Differential operators and localization

(VII) Differential operators and twisted differential operators.

Define the sheaf $\mathcal{D}_Y$ on a smooth variety Y and its filtration by order. In local coordinates, describe its sections as finite sums $\sum_\alpha a_\alpha \partial^\alpha$, with the relations $[\partial_i, a] = \partial_i(a)$ and $[\partial_i, \partial_j] = 0$. Identify $\mathrm{gr} \mathcal{D}_Y$ with $\mathrm{Sym}_{\mathcal{O}_Y} \mathcal{T}_Y$ ([HTT, Proposition 1.1.3]). For $Y = \mathbf{A}^1$, compute the left modules D , $D/D\partial \simeq k[x]$ and $D/Dx \simeq k[x, x^{-1}]/k[x]$, where $D = k\langle x, \partial \rangle / (\partial x - x\partial - 1)$ ([HTT, §1.2]; [Sch, Example 8.35]). Explain the equivalence between quasi-coherent left $\mathcal{D}_Y$ -modules and quasi-coherent sheaves with integrable connection ([HTT, Lemma 1.2.1]).

Define differential operators acting on a vector bundle and, for a line bundle, explain the equivalence of module categories induced by tensoring with that line bundle ([HTT, §11.1, equations (11.1.1)–(11.1.5)]; [Mil, Chapter 2, §2, p. 21]). Introduce $\mathcal{D}_\lambda$, the homomorphism $U(\mathfrak{g}) \rightarrow \Gamma(X, \mathcal{D}_\lambda)$ and the localization adjunction ([HTT, §11.2, equations (11.2.1)–(11.2.5)]). Compute the operators and their gluing on $\mathbf{P}^1$ ([HTT, Example 11.2.1]) and state the localization theorem.

(VIII) Beilinson–Bernstein I: global differential operators and vanishing.

Prove $\Gamma(X, \mathcal{D}_\lambda) \simeq U_\lambda$ for integral λ ([HTT, Theorem 11.2.2 and §11.3]). Use the Springer resolution $T^*X \rightarrow \mathcal{N}$, together with the normality and defining equations of the nilpotent cone ([HTT, Proposition 10.3.3 and Theorems 10.3.7–10.3.8]). These geometric results may be stated. Prove the assertions about the associated graded homomorphism and the action of the center in [HTT, Propositions 11.3.1–11.3.2], and deduce the result by induction on the filtration.

For $\nu \in -P_+$, put $E_\nu = \Gamma(X, \mathcal{L}_\nu)$. The dual of evaluation gives a map

$$\mathcal{M} \longrightarrow (\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{L}_\nu) \otimes_k E_\nu^*.$$

Prove that, for integral antidominant λ and a quasi-coherent $\mathcal{D}_\lambda$ -module $\mathcal{M}$, this map is split injective as a morphism of sheaves of k -vector spaces ([HTT, Proposition 11.4.1(ii)]). Use the central-character decomposition in its proof and Serre vanishing, applied to coherent $\mathcal{O}_X$ -submodules of $\mathcal{M}$, to deduce $H^i(X, \mathcal{M}) = 0$ for $i > 0$ ([HTT, proof of Theorem 11.2.3, pp. 282–283]).

(IX) Beilinson–Bernstein II: localization and translation.

For regular integral antidominant λ , prove that evaluation induces a split surjection of sheaves of k -vector spaces

$$\mathcal{M} \otimes_k E_\nu \longrightarrow \mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{L}_\nu, \quad \nu \in -P_+,$$

where $E_\nu = \Gamma(X, \mathcal{L}_\nu)$ ([HTT, Proposition 11.4.1(i)]). Deduce generation by global sections and finish the proof of the localization equivalence, including its restriction to coherent $\mathcal{D}_\lambda$ -modules and finitely generated U_λ -modules ([HTT, Theorem 11.2.4, its proof on p. 283, and Corollary 11.2.6]).

Explain the geometric interpretation of translation. For regular integral antidominant λ and $\lambda + \nu$, tensoring with $\mathcal{L}_\nu$ and applying localization gives an equivalence

$$U_\lambda\text{-Mod} \longrightarrow U_{\lambda+\nu}\text{-Mod}, \quad M \longmapsto \Gamma(X, \mathcal{L}_\nu \otimes_{\mathcal{O}_X} \text{Loc}_\lambda(M))$$

([Mil, Chapter 2, §2, p. 21; Chapter 3, Corollary 1.3]). For $\nu \in -P_+$, identify it with $M \mapsto \text{pr}_{\chi_{\lambda+\nu}}(E_\nu \otimes_k M)$ using [HTT, Proposition 11.4.1(i) and equations (11.4.9)–(11.4.15)]. Compare with the restriction of the translation functors of Talk IV to modules with exact central character.

(X) Derived localization and Lie algebra homology.

Describe the construction of $\mathcal{D}_\lambda$ for arbitrary $\lambda \in \mathfrak{h}^*$ ([Mil, Chapter 1, Proposition 2.3], with parameter $\lambda + \rho$ as in Chapter 2, §1) and state the derived localization theorem

$$\text{L Loc}_\lambda : \text{D}(U_\lambda\text{-Mod}) \rightleftarrows \text{D}(\mathcal{D}_\lambda\text{-Mod}_{\text{qc}}) : \text{R}\Gamma$$

for regular λ , including its restriction to bounded complexes with finitely generated, respectively coherent, cohomology ([Mil, Chapter 3, Theorems 1.12 and 1.21]). Explain the proof for bounded complexes from the following inputs: conservativity of $\text{R}\Gamma$, acyclicity of $\mathcal{D}_\lambda$ and finite homological dimension of U_λ ([Mil, Chapter 2, Theorems 4.1 and 6.1; Chapter 3, Theorem 1.4 and Lemmas 1.7–1.11]). These inputs and the extension to unbounded complexes are stated without proof.

Define Lie algebra homology by the Chevalley–Eilenberg complex, using the left-module convention obtained from the antipode $x \mapsto -x$ ([Wei, Definition 7.7.1, Theorem 7.7.2 and Corollary 7.7.3]). Deduce Kostant’s formula

$$H_i(\mathfrak{n}^-, L(\mu)) \simeq \bigoplus_{\ell(w)=i} k_{w \cdot \mu}$$

from the BGG resolution ([Kum, Theorem 39]). State the identification of the derived fiber of $\text{L Loc}_\lambda(M)$ at eB with the $(\lambda + \rho)$ -weight space in the $\mathfrak{n}^+$ -homology of M ([Mil, Chapter 3,

Theorem 2.5 and Corollary 2.6]). For $\mu \in P$ with $\mu + \rho$ regular, let $w \cdot \mu \in P_+$, $q = \ell(w)$ and $E = L(w \cdot \mu)^*$. Kostant's formula gives

$$H_i(\mathfrak{n}^+, E)_{-\mu} = \begin{cases} k_{-\mu}, & i = q, \\ 0, & i \neq q. \end{cases}$$

The derived-fiber formula and G -equivariance therefore identify

$$\mathrm{L} \mathrm{Loc}_{-\mu-\rho}(E) \simeq \mathcal{L}_{-\mu}[q], \quad \mathrm{R}\Gamma(X, \mathcal{L}_{-\mu}) \simeq E[-q]$$

([Kum, Theorem 39]; [Mil, Chapter 3, Theorems 1.12 and 2.5, Corollary 2.6]). Explain the passage from the derived fiber to the line bundle using G -equivariance and the description of equivariant bundles by their B -fibers ([HTT, §9.11, pp. 255–256]). If $\mu + \rho$ is singular, each cohomology group of $\mathcal{L}_{-\mu}$ is a finite-dimensional G -module with central character $\chi_{-\mu-\rho}$ and hence vanishes, since no finite-dimensional simple module has singular infinitesimal character ([HTT, Theorem 11.2.2]; [Hum, Theorems 1.6 and 1.10]). This gives the derived-localization interpretation of the formulas of Talk V. Compute the example of $\mathbf{P}^1$.

III. De Rham stacks and equivariant localization

(XI) De Rham stacks and crystals.

Recall descent for quasi-coherent complexes and define $Y_{\mathrm{dR}}(A) = Y(\mathrm{Spec} A_{\mathrm{red}})$ and $\mathrm{Crys}(Y) = \mathrm{QCoh}(Y_{\mathrm{dR}})$ ([Sch, Definition 8.25]). Describe quasi-coherent sheaves on a prestack as the limit of the categories $D(A)$ over its affine points ([RC25, Definition 5.26 and Remark 5.27]). Test algebras may be taken to be commutative k -algebras of finite type ([Sch, p. 78, footnote 14, and Remark 8.26]). For smooth Y , identify the Čech nerve of $Y \rightarrow Y_{\mathrm{dR}}$ with the formal completions of the diagonals $Y \hookrightarrow Y^{n+1}$, and describe a crystal by descent on this infinitesimal groupoid ([RC25, Proposition 7.24]).

Prove that descent on the formal diagonal is equivalent to an integrable connection, using Taylor expansion ([RC25, Exercises 1.5–1.7 and the proof of Theorem 10.17, Step 2]). State the derived extension

$$\mathrm{Crys}(Y) \simeq D(\mathcal{D}_Y\text{-Mod}_{\mathrm{qc}})$$

and describe the monadicity argument, taking the required descent and adjoint-existence results as inputs ([RC25, Theorem 10.17, Step 1]). Apply the Taylor-expansion description to the descent datum of the structure sheaf, obtaining $(\mathcal{O}_Y, d)$ ([RC25, proof of Theorem 10.17, Step 2]). Work out the description for $Y = \mathbf{A}^1$ ([RC, proof of Proposition 5.2.3]).

(XII) Six operations and Kashiwara's theorem.

State the construction and extension to stacks of the six-functor formalism for quasi-coherent sheaves ([Sch, Definitions 2.4–2.5 and Theorem 5.19]). Describe the adjunctions, base change and the projection formula in this formalism. Apply the formalism to de Rham stacks and describe the normalization of the exceptional functors used in the D -module formalism ([Sch, Theorems 8.24, 8.27 and Remarks 8.31, 8.37]). In this normalization, for a proper morphism f , the functor $f_!$ is left adjoint to f^* .

For a smooth variety Y with structure map $p : Y_{\mathrm{dR}} \rightarrow \mathrm{Spec} k$, identify direct image with de Rham cohomology. For a crystal corresponding to a quasi-coherent left $\mathcal{D}_Y$ -module $\mathcal{M}$, the identification is

$$\begin{aligned} p_* \mathcal{M} &\simeq \mathrm{R} \mathrm{Hom}_{\mathcal{D}_Y}(\mathcal{O}_Y, \mathcal{M}) \\ &\simeq \mathrm{R}\Gamma(Y, [\mathcal{M} \xrightarrow{\nabla} \Omega_Y^1 \otimes \mathcal{M} \longrightarrow \cdots \longrightarrow \Omega_Y^{\dim Y} \otimes \mathcal{M}]), \end{aligned}$$

with $\mathcal{M}$ in degree zero. The first equivalence follows from the global-sections adjunction and [RC25, Theorem 10.17]; the second follows by applying $\mathrm{R}\mathrm{Hom}_{\mathcal{D}_Y}(-, \mathcal{M})$ to the Spencer resolution ([HTT, Lemma 1.5.27]). For the constant crystal this is precisely [RC25, Theorem 9.15]. The classical direct-image formula is [HTT, Proposition 1.5.28(i)], where the de Rham complex is shifted by $[\dim Y]$.

Prove Kashiwara's theorem in the form of excision: for a closed immersion $i : Z \hookrightarrow Y$ with complement $j : U \hookrightarrow Y$, the functor $i_!$ identifies $\mathrm{Crys}(Z)$ with the subcategory supported on Z , and there is an exact triangle

$$i_! i^* \mathcal{M} \longrightarrow \mathcal{M} \longrightarrow j_* j^* \mathcal{M}$$

([Sch, Proposition 8.32]). Explain the proof by reduction to formal completion ([Sch, Propositions 8.28–8.29]). For $\{0\} \subset \mathbf{A}^1$, identify the excision triangle for the constant crystal with

$$\delta_0[-1] \longrightarrow k[x] \longrightarrow k[x, x^{-1}], \quad \delta_0 = k[x, x^{-1}]/k[x],$$

under the comparison with left $\mathcal{D}$ -modules ([Sch, Example 8.35]). For a k -rational point $p \in \mathbf{P}^1$, define

$$\mathcal{O}_{\mathbf{P}^1}(*p) = \varinjlim_{n \geq 0} \mathcal{O}_{\mathbf{P}^1}(np), \quad \delta_p = \mathcal{O}_{\mathbf{P}^1}(*p)/\mathcal{O}_{\mathbf{P}^1}.$$

Here $\mathcal{O}_{\mathbf{P}^1}(*p)$ is the sheaf of rational functions with poles allowed only at p , of arbitrary finite order, equipped with its natural $\mathcal{D}_{\mathbf{P}^1}$ -action. The quotient δ_p is the delta $\mathcal{D}$ -module supported at p . Obtain the exact sequence $0 \rightarrow \mathcal{O}_{\mathbf{P}^1} \rightarrow \mathcal{O}_{\mathbf{P}^1}(*p) \rightarrow \delta_p \rightarrow 0$ from the corresponding excision triangle, using the preceding local calculation.

(XIII) Formal groups and $(\mathfrak{g}, B)$ -modules.

Recall the interpretation of quasi-coherent sheaves on a classifying stack as representations. For the formal completion $\widehat{G}$ of G at the identity, state the derived equivalence

$$\mathrm{QCoh}(B\widehat{G}) \simeq \mathrm{D}(U(\mathfrak{g})\text{-Mod})$$

as in [RC, Proposition 5.2.15], and prove the equivalence on hearts by formal Taylor expansion, beginning with the explicit formal additive-group calculation in the proof of [RC, Proposition 5.2.3].

Define $(\mathfrak{g}, B)$ -modules and the formal completion $G_B^\wedge = G \times_{G_{\mathrm{dR}}} B_{\mathrm{dR}}$. Describe its representations using

$$1 \longrightarrow \widehat{B} \longrightarrow \widehat{G} \rtimes B \longrightarrow G_B^\wedge \longrightarrow 1,$$

and prove the equivalence $\mathrm{QCoh}(BG_B^\wedge)^\heartsuit \simeq (\mathfrak{g}, B)\text{-Mod}$ ([RC, Definition 5.3.14 and Proposition 5.3.15]). Identify the left adjoint to restriction along $BB \rightarrow BG_B^\wedge$ with

$$V \longmapsto U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} V,$$

and recover Verma modules by taking $V = k_\mu$ for $\mu \in P$ ([RC, Exercise 5.3.16(1)]). Explain the relation between finitely generated $(\mathfrak{g}, B)$ -modules and the integral part of category $\mathcal{O}$ ([HTT, Definition 11.5.1 and §12.2, pp. 295–296]).

(XIV) Equivariant localization and category $\mathcal{O}$.

Define strongly equivariant $\mathcal{D}$ -modules ([HTT, Definition 11.5.2]) and describe crystals on $[X/B]$ by descent along the action groupoid ([RC25, Definition 5.26, Proposition 5.28 and Theorem 10.17]). Relate the resulting abelian category to strongly B -equivariant $\mathcal{D}_X$ -modules. Prove that localization and global sections preserve equivariance, including compatibility of the differentiated B -action with the $\mathfrak{b}$ -action, obtaining

$$\mathcal{O}_{\mathrm{triv}}^{\mathrm{exact}} \simeq \mathcal{D}_X\text{-Mod}_{\mathrm{coh}}^B$$

([HTT, Theorem 11.5.3 and Proposition 12.2.1]). The central ideal acts by zero in this statement; the generalized principal block is a larger category ([Hum, §§1.12 and 3.12]). Using the same descent description, identify the right-hand side with the objects in $\mathrm{QCoh}([B \backslash G/B]_{\mathrm{dR}})^{\heartsuit}$ whose pullback to X is $\mathcal{D}_X$ -coherent.

Put $C_w = BwB/B$, and let $X_{\leq r}$ be the closed union of the cells C_w with $\ell(w) \leq r$, with its reduced scheme structure. Its successive strata are $S_r = X_{\leq r} \setminus X_{\leq r-1} = \coprod_{\ell(w)=r} C_w$ ([HTT, Theorems 9.9.4–9.9.5]). For the inclusions $i_r : X_{\leq r-1} \hookrightarrow X_{\leq r}$ and $j_r : S_r \hookrightarrow X_{\leq r}$, apply excision after passage to quotient de Rham stacks:

$$i_{r,!} i_r^* \mathcal{M} \longrightarrow \mathcal{M} \longrightarrow j_{r,*} j_r^* \mathcal{M}$$

([Sch, Proposition 8.32]; descent as in [RC25, Proposition 5.28]). Thus the full stable subcategories of $\mathrm{Crys}([X/B])$ supported on $[X_{\leq r}/B]$ form a finite filtration. Restriction identifies the quotient of two consecutive stable subcategories with

$$\mathrm{Crys}([S_r/B]) \simeq \prod_{\ell(w)=r} \mathrm{Crys}(BB_w), \quad B_w = B \cap wBw^{-1},$$

using the presentation $C_w = B/B_w$ of each orbit ([HTT, §9.9, proof of Theorem 9.9.4]) and the preceding excision triangle. For coherent objects in the heart, relate support on these closed unions to subcategories of $\mathcal{O}_{\mathrm{triv}}^{\mathrm{exact}}$ by equivariant localization.

For $G = \mathrm{SL}_2$, the filtration consists of the B -fixed point $p = eB$ and its complement. Apply global sections to the exact sequence of Talk XII,

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}^1} \longrightarrow \mathcal{O}_{\mathbf{P}^1}(*p) \longrightarrow \delta_p \longrightarrow 0$$

and identify the resulting sequence with

$$0 \longrightarrow L(0) \longrightarrow \nabla(0) \longrightarrow \Delta(-2) \longrightarrow 0$$

([HTT, Example 11.2.1 and §12.3, pp. 299–301]; [Sch, Example 8.35]). Compute these identifications with $p = \infty$: the global sections are k , $k[z]$ and $k[z]/k$, with $e = -\partial_z$, $h = -2z\partial_z$ and $f = z^2\partial_z$ ([HTT, Example 11.2.1]). Relate the sequence to the BGG dual of the Verma-module sequence in Talk I.

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