Oberseminar p-adische Arithmetik, Wintersemester 2017/18

Derived Hecke algebras and the cohomology of locally symmetric spaces

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In this Oberseminar we want to understand Venkatesh's paper [16] concerning the action of derived Hecke algebras on the cohomology of locally symmetric spaces associated to reductive groups over $\mathbb{Q}$. Given a reductive group G over $\mathbb{Q}$ and a compact open subgroup $K = \prod_p K_p \subset G(\mathbb{A}_f)$, one considers the locally symmetric space

$$Y(K) = G(\mathbb{Q})\backslash\big(G(\mathbb{R}) \times G(\mathbb{A}_f)\big)/K_\infty \times K,$$

where $K_\infty \subset G(\mathbb{R})$ is a maximal compact subgroup. It is known that (the tempered part of) the cohomology of $Y(K)$ only occurs in certain degrees $[q, q+\delta]$ that can be described explicitly in terms of the group G.

Classically, the Hecke algebra

$$\mathcal{H}_0(G(\mathbb{A}_f)/\!\!/K, \mathbb{Z}) = \mathrm{End}_{G(\mathbb{A}_f)}(\mathbb{Z}[G(\mathbb{A}_f)/K])$$

of functions on $G(\mathbb{A}_f)$ that are biinvariant under K acts on the cohomology of $Y(K)$. It is convenient to replace this Hecke algebra by the tensor product of spherical Hecke algebras at good primes p. The characters of this Hecke algebra that show up as systems of eigenvalues of cohomology classes are conjecturally linked to representations of the Galois group $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ via the Langlands program.

In this seminar we want to relace the spherical Hecke algera at good primes p by its derived version

$$\mathcal{H}(G(\mathbb{Q}_p)/\!\!/K_p, \mathbb{Z}/\ell^n) = \bigoplus_i \mathrm{Ext}^i(\mathbb{Z}/\ell^n[G(\mathbb{Q}_p)/K_p], \mathbb{Z}/\ell^n[G(\mathbb{Q}_p)/K_p])$$

with torsion coefficients which naturally acts on the cohomology $H^*(Y(K), \mathbb{Z}/\ell^n)$. Putting together these Hecke algebras for the various good primes p and passing to the limit over the inverse system of coefficients $(\mathbb{Z}/\ell^n)_n$ one obtains an action of a big graded algebra on the cohomology $H^*(Y(K), \mathbb{Q}_\ell)$. The image of this big algebra in the endomorphisms of the cohomology is the derived Hecke algebra of Venkatesh and is denoted by $\tilde{\mathbb{T}}$. Its zeroth graded piece $\tilde{\mathbb{T}}_0$ is the $\mathbb{Q}_\ell$-subalgebra of $\mathrm{End}(H^*(Y(K), \mathbb{Q}_\ell))$ generated by the 'usual' Hecke operators in $\mathcal{H}_0(G(\mathbb{Q}_p)/\!\!/K_p, \mathbb{Z})$ for good primes p.

The first main result[1] (which we will prove in Talk 9) is the statement that (the cuspidal part of) the cohomology is generated as a $\tilde{\mathbb{T}}$-module by $H^q(Y(K), \mathbb{Q}_\ell)$, i.e. the canonical map

$$\tilde{\mathbb{T}} \otimes H^q(Y(K), \mathbb{Q}_\ell) \longrightarrow H^*(Y(K), \mathbb{Q}_\ell)$$

becomes surjective after localizing at a suitable maximal ideal of classical Hecke algebra $\tilde{\mathbb{T}}_0$. This statement can be thought of as follows: the action of the derived object $\tilde{\mathbb{T}}$ 'explains' that (the cuspidal part of) the cohomology is not concentrated in a single degree.

To explain the second main result (which we will prove in Talk 11 and 12) we fix a character χ of $\tilde{\mathbb{T}}_0$ of the classical Hecke algebra which occurs as a system of eigenvalues of a cohomology class in (the cuspidal part of) $H^q(Y(K), \mathbb{Q}_\ell)$. Then the second main result asserts that one can describe the χ-part $\tilde{\mathbb{T}}_\chi$ of the derived Hecke algebra terms of the Galois representation ρ_χ that is (in some cases only conjecturally) associated to χ. It turns out that $\tilde{\mathbb{T}}_\chi$ is isomorphic to the exterior algebra on the dual of the Bloch-Kato Selmer group

$$V = H^1_f(\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}), \mathrm{Ad}^*\rho_\chi(1)).$$

In particular this means that one can define an action of V^* on $H^*(Y(K), \mathbb{Q}_\ell)$ by endomorphisms of degree 1.

[1] The main results are 'only' proven for special groups G and certain level K.

In the last two talks we want to adress a conjecture of Venkatesh which asserts that this action is 'of motivic nature'.

1) **Locally symmetric spaces I: Definitions**
   Introduce locally symmetric spaces associated to reductive groups over $\mathbb{Q}$. The set up is explained in [16, 1.8], see also [17, 2] (in paticular the examples given there). We should mainly focus on the set up presented in [9, 6.1]. Recall that the quotients $Y(K)$ (in the notation of Venkatesh) are smooth manifolds and how their cohomology is linked to the group cohomology of an arithmetic group (see also [11, 3] for some background on notations concerning arithmetic subgroups). Explain Hecke correspondences, and the action of Hecke operators on the cohomology of a locally symmetric space and how this action lifts to the level of complexes. Prove that the resulting Hecke algebras differ by a nilpotent ideal. See [9, 6.2] and Lemma 2.5. of loc.cit. Define the notion of a Galois representations associated to characters of the Hecke algebra, see [16, 1.2.1] and the references cited there. In particular state the known result [15, Theorem I.3]

2) **Locally symmetric spaes II: Examples**
   Discuss further examples for the objects introduced in the first talk. The first fundamental example is the case of SL_2: We want to see how the cohomology of the quotients of the upper half plane is computed in terms of modular forms, i.e. we want to study the Eichler-Shimura isomorphism. This can be done following for example [5, 2] or [13, 6] (it would be nice to include the description of the whole cohomology, not just the interior part).
   In the seminar we are roughly aiming at the stement 'a certain part of the cohomology of a locally symmetric space looks like the cohomology of a torus'. Examples of this kind include the following: The case of anisotropic tori is (briefly) discussed in [16, 9.1,9.2] (see [3, 2] for the classification of real tori needed for loc. cit.), we do not (yet) need the statement about Galois representations or motives. A further nice and interesting example (using Hodge theory) is given in [17, 3.2,3.3].

3) **(Derived) Hecke algebras I: Definition and first properties**
   Define the derived Hecke algebra as in [16, 2.1, Defintion 2.2] and its action on the chain complex of an arithmetic manifold [16, 2.5,2.6]. Then give the concrete description of the derived Hecke algebra in terms of invariant functions and double cosets [16, 2.3, 2.4] (resp. A.8 and A.11 of loc.cit.). Finally make the action of the derived Hecke algebra on derived invariants and on the cohomology of arithmetic manifolds explicit [16, A.9 and 2.10].

4) **(Derived) Hecke algebras II: The spherical Hecke algebra**
   State and prove the derived version of the Satake isomorhpism [16, 3].

5) **(Derived) Hecke algebras III: The Iwahori Hecke algebra**
   Discuss the structure of the Iwahori-Hecke algebra [16, 4.2, 4.3] and its localization as in [16, Lemma 4.5]. Then describe the derived variant of this construction [16, 4.6, Lemma 4.7]. Finally discuss the action of the (underived) Iwahori-Hecke algebra on the corresponding arithmetic manifold $Y_0(q)$ as in [16, 6.5] (without Corollary 6.7 of loc.cit., but including the Remark following it).

6) **Galois representations and Fontaine-Laffaille theory**
   Recall the notion of the universal deformation ring of a residual Galois representation. Recall some basics of p-adic Hodge theory, in particular define the notion of a crystalline representation and state the result on the existence of crystalline deformation rings [10, 1.1, 1.2] (we only focus on the crystalline case, so in particular the Galois type τ in loc. cit. should be assumed to be trivial). See also [1] for some more background on p-adic

Hodge theory. Give some background on Fontaine-Laffaille theory (the material around [2, Theorem 4.3] gives a good summary of what we need, see also [4, 2.4.1]) and explain the construction of the crystalline deformation ring (as well as its formal smoothness) in the Fontaine-Laffaille case, see [4, Lemma 2.4.1]. We moreover need the description of the Selmer group $H^1_f(\mathbb{Q}_p, M) \subset H^1(\mathbb{Q}_p, M)$ in terms of Fontaine-Laffaille theory and (for late use) the fact that $H^1_f(\mathbb{Q}_p, M)$ and $H^1_f(\mathbb{Q}_p, M^*)$ are annihilaters of each other under the local duality pairing, see [16, 8.2].

7) **Taylor-Wiles primes**
Explain the conjecture on the existence of Galois representations over the (underived) Hecke akgebra [16, 6.2], see also [6, Conjecture 6.1]. Introduce Taylor-Wiles primes [16, 6.3] and explain what happens if one adds Iwahori level at a Taylor-Wiles system [16, 6.8,6.9].

8) **Patching and $R = T$ theorems**
Explain the patching process in the case of a definite quaternion algebra [7, 5.5] (where we deal with a single module instead of a complex). Explain further how it is used to prove an $R = T$ theorem, that is deduce [7, Theorem 5.1].

9) **The cohomology is generated by the cohomology in degree q**
Explain the outcome of the patching process in the case of complexes [16, 7.1,7.2] (but we will treat the details as a black box). Then state and prove the first main result [16, Theorem 7.5].

10) **Galois cohomology calculations**
Prove [16, Lemma 8.8]. Introduce the notion of a convergnet Taylor-Wiles datum [16, 8.9]. Then present the computation of tangent spaces [16, 8.11-8.14].

11) **Identification of the derived Hecke algebra I**
Finish the the comparison of tangent spaces with Galois cohomology, see [16, 8.15]. Then state the main result of this section [16, Theorem 8.5] and construct the maps $\iota_{q,n}$ and f_{q_n} of [16, 8.4]. For the construction of $\iota_{q,n}$, see 8.16 of loc.cit.

12) **Identification of the derived Hecke algebra II**
Describe the action of V/p^n on the cohomology and finish the proof of the main result [16, Theorem 8.5]. See 8.18-8.26 of loc.cit.

13) **Motivic Cohomology**
Give a survey about motivic cohomology and Beilinson's conjecture. The material that will be needed in the last talk is summarized in [12, 2.1]. Other good references are [8, 4] or [14].

14) **Venkatesh's conjecture on the motivic cohomology**
Explain the conjectural motivic version of the main results of this seminar following [16, 1.2, 8.6]. Then discuss the Betti realization of this conjecture: give the two definitions of the vector space $\mathfrak{a}^*_G$ in [12, 3.13.2]. Then construct the action of the exterior algebra of $\mathfrak{a}^*_G$ on $(\mathfrak{g}, K_\infty)$-cohomology, see also the example worked out in [17, 6.2]. The link with the étale realization that we discussed in talk 9 is given by the conjecture stated in [12, 5.4].

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