Oberseminar p-adische Arithmetik, Sommersemester 2017

The p-adic analog of the Kazhdan-Lusztig conjecture
Programmvorschlag: Eugen Hellmann

In this Oberseminar we want to understand the formulation [15] and proof [6, §8] of the p-adic analog of the Kazhdan-Lusztig conjecture (KL conjecture). The conjecture concerns smooth parabolically induced representations of p-adic reductive groups $G$ (in [15] only the case $G = \text{GL}_n(F)$ is addressed, while [6, §8] considers the situation for modules over the Iwahori-Hecke algebra). These parabolically induced representations are in general not irreducible and the conjecture describes the multiplicities with which an irreducible representation occurs in a Jordan-H"older series in terms of intersection cohomology on a certain algebraic variety. The varieties that show up in this context parameterize nilpotent (graded) endomorphisms of a given (graded) vector space.

This conjecture parallels the "classical" Kazhdan-Lusztig conjecture which gives a similar description of the multiplicities of irreducible constituents in Verma modules (i.e. induced representations of Lie algebras). In both cases the irreducible objects that occur can be defined as the unique irreducible quotient of another induced representation, and the multiplicities are given by the dimension a certain intersection cohomology group (in the case of representations of Lie algebras one considers intersection complexes of Schubert varieties).

Part I: Perverse sheaves

In the first part of this seminar we will learn the basic facts about intersection cohomology and perverse sheaves that are needed to understand the formulation (and proof) of the conjecture. Whenever it simplifies the situation we consider varieties over $\mathbb{C}$ and middle perversity. Though the book [6] is not the main reference for this part of the seminar is might help to look at [6, §8.3,8.4] which summarizes the contents needed for the last part of our seminar.

1) Constructible sheaves
Discuss the derived category of constructible sheaves and the 6 functor formalism. See for example [8, Chapter 1], [10, §4.5] or [6, §8.3]. See also the beginning of [2, §1.4].

2) Delignes construction of the intersection complex
Define the intersection complex following [8, §3]. One should treat the characterizations [AX1] and [AX2] of loc. cit. (see also [1, Theorem 2.2.2]).

3) Generalized Poincare Duality
Give an example that Poincare duality fails for non-smooth manifolds, see for example the introduction of [7] or [13, §1.1]. Proof the generalized Poincare duality theorem involving intersection cohomology [8, 5.3] simple proof, or [1] (it might also be useful to look at [10, Theorem 8.2.17]). Construct the cup product pairing [8, 5.2].

4) Perverse Sheaves
Define $t$-structures, introduce the perverse $t$-structure and the category of perverse sheaves and discuss its structure, see for example [1, 4.2.3,3, Thm. 4.2.10, 4.3.2], or [10, §8.1.2]. See also [2] and [12] (or [6, p.433-434] for a short summary of the results needed in part III). Define the minimal extension of a pervers sheaf, see e.g. [10, §8.2.2], and relate it to the intersection complex (see e.g. [10, Prop. 8.2.11]).

5) The decomposition theorem
Discuss the decomposition theorem [2, 6.2.5], see also [1, 4.4] and [13, 6.5].
Part II: Representation theory

The second part of the seminar summarizes basic facts from the theory of smooth representations of $p$-adic reductive groups and the relation with modules over the Hecke algebra. In particular we will need the relationship between representations that have a fixed vector under a Iwahori subgroup and modules over the Iwahori-Hecke algebra, as well as the presentation of the latter. We recall the classification of Bernstein-Zelevinski resp. Langlands that every irreducible smooth representation can be written as the quotient of a certain induced representation and the conjectural parametrisation of irreducible smooth representations in terms of Weil-Deligne representations. Finally we formulate the $p$-adic KL conjecture in the case of $GL_n(F)$. 

6) Smooth representations, Hecke algebras and the Langlands quotient
Recall some basic facts about smooth admissible representations and their relation with modules over Hecke algebras (see e.g. [4, 2.1, 2.4, 4.1-4.3], or [3, I, §1-2]). Present the content of [14, §1], in particular Theorem 1.2.2 and Theorem 1.2.5. Then state the conjectural parametrisation of irreducible smooth representation in terms of Weil-Deligne representations, see [14, §4.1,4.2] (we can treat $L$- and $c$-factors as a black box). In particular describe the parameter of a Langlands quotient $Q(\Delta_1,\ldots,\Delta_m)$ in terms of the parameter of its supercuspidal support [14, p.381]. Discuss the examples given in [14, §5.1].

7) The $p$-adic KL conjecture
Present the content of [15, §1-3]. In particular discuss the examples given in section 3.

8) The Iwahori-Hecke algebra
The aim of this talk is to understand that we can rephrase the conjecture from the last talk to a statement about Hecke-algebras, if we restrict attention only to those representations that are generated by their Iwahori fixed vectors. Describe the presentation of the Iwahori-Hecke algebra, see e.g. [3, II,§3] or [9]. Then present the description of parabolic induction in terms of modules over the Hecke algebra, see [5, (7.6), Theorem 7.6.1] (with the notations of that reference we are only interested in the case where $J$ is the Iwahori subgroup and $\lambda$ is the trivial representation. Hence the category $\mathfrak{M}_{\lambda}$ is the category of admissible smooth representations generated by their Iwahori fixed vectors).

Part III: Proof of the $p$-adic KL conjecture for Hecke algebras
In the last part of the seminar we proof the $p$-adic KL conjecture for modules over the Iwahori-Hecke algebra. In the literature some statements are only proved using equivariant $K$-theory. We will have to use these statements as a black box.

9) Geometric description of the Iwahori-Hecke algebra I
Explain the construction of a convolution algebra in Borel-Moore homology in [6, §2.7.40], in particular [6, Cor. 4.7.41, Cor. 4.7.42]. Introduce the Steinberg variety (see the beginning of [6, §3.3]). Introduce the affine Hecke-algebra as defined in [6, Definition 7.1.9] for example, then present the content of [6, p.411-413]. The main aim of the talk is to understand the statement of [6, Prop.8.1.5] (but we have to omit the proof of this proposition).

10) Geometric description of the Iwahori-Hecke algebra II
Given the sheaf-theoretic description of the Hecke algebra in [6, Theorem 8.6.7]. Present as much as possible of its proof (see [6, §8.6.26]). In particular state [6, Proposition 8.6.35].

11) Standard modules and simple modules
Define simple and standard modules using Borel-Moore homology, see [6, p. 414-418] including the sheaf theoretic description in [6, Thm 8.6.12, Prop. 8.6.15, 8.6.16, 8.6.17].
In particular state and proof [6, Lemma 8.5.4] which relates the definition using Borel-Moore homology and the definition using perverse sheaves. Finally state [11, Theorem 6.2] without proof (you also need to translated the statement from equivariant $K$-theory to Borel-Moore homology). Whenever it helps to simplify the exposition we assume that we are dealing with the Hecke algebra for $GL_n(F)$ or $SL_n(F)$.

12) **Proof of the multiplicity formula**

Present the proof of the main theorem [6, Theorem 8.6.23].

**References**


