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WS 2018/2019

Algebriac Geometry
Exercise Sheet 2

To be hand in on 25.10.2018

Exercise 1:

Let $(I, \leq)$ be a partial ordered set which is filtered (i.e. $\forall i, j \in I$, $\exists k \in I$ s.th. $k \geq i, j$). Let $(F_i)_{i\in I}$, $(G_i)_{i\in I}$ and $(H_i)_{i\in I}$ be direct systems of abelian groups indexed by I. Assume that $\{\varphi_i : F_i \to G_i\}$ and $\{\psi_i : G_i \to H_i\}$ are two morphisms between direct systems such that for every $i \in I$, the following sequence:

$$0 \to F(i) \xrightarrow{\varphi(i)} G(i) \xrightarrow{\psi(i)} H(i) \to 0$$

is exact. Show that there is an induced short exact seqeunce:

$$0 \to \varinjlim_{i\in I} F(i) \xrightarrow{\Phi} \varinjlim_{i\in I} G(i) \xrightarrow{\Psi} \varinjlim_{i\in I} H(i) \to 0.$$

Exercise 2:

Let X be a topological space, and let A be an abelian group. Let $x \in X$ be a point, a presheaf $Sky^{(A,x)}$ is called a *skyscrapper sheaf at x* if

$$Sky^{(A,x)}(U) = \begin{cases} A & x \in U \\ 0 & x \notin U \end{cases}$$

where the restriction maps are the obvious ones (i.e. either id_A or 0).

(i) Show that $Sky^{(A,x)}$ is a sheaf.
(ii) Compute the stalks of $Sky^{(A,x)}$.

Exercise 3:

Let X and A be as in exercise 2, and let $Sky^{(A,x)}$ be a skyscraper sheaf at ponit $x \in X$. Let $\underline{A}_X^{\mathrm{pre}}$ be the constant presheaf on X defined by A. Consider the natural morphsim $\varphi_x : \underline{A}_X^{\mathrm{pre}} \to Sky^{(A,x)}$ defined by:

$$\varphi_{x,U} = \begin{cases} \mathrm{id}_A & x \in U \\ 0 & x \notin U \end{cases}$$

(i) Describe the morphism $\varphi_x^+ : \underline{A}_X \to Sky^{(A,x)}$ induced by sheafitication explicitly (i.e. on sections over open subsets).
(ii) Assume X is connected and Hausdoff. Let x_1 and x_2 be two distinct points, consider the morphism:

$$\varphi_{x_1}^+ \oplus \varphi_{x_2}^+ : \underline{A}_X \to Sky^{(A,x_1)} \oplus Sky^{(A,x_2)}$$

Show that $\mathrm{coker}(\varphi_{x_1}^+ \oplus \varphi_{x_2}^+) \neq 0$ in the category of presheaf and that its sheafitication is 0.

Exercise 4:

Let $\mathbb{C}$ be the complex plane, and let $\mathcal{O}_{\mathbb{C}}$ denote the sheaf of holomorphic functions on $\mathbb{C}$. Consider the derivation map $D : \mathcal{O}_{\mathbb{C}} \to \mathcal{O}_{\mathbb{C}}$, $f \mapsto \frac{\mathrm{d}f}{\mathrm{d}z}$.

(i) Describe the kernel of D.
(ii) Show that $\mathrm{coker}(D) \neq 0$ in the category of presheaf and that its sheafitication is 0.