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SoSe23

Selected Topics in Differential Geometry
Sheet 6

Due on **15 June 2023**

Exercise 1: A Basic Covering Theorem

Let X be a metric space, $R > 0$, and $\mathcal{F}$ a family of balls of radius $\leq R$ in X.

(a) Apply Zorn's Lemma to find a subfamily $\mathcal{G}$ of $\mathcal{F}$ which consists of disjoint balls (it is "disjointed") and such that every ball in $\mathcal{F}$ intersects a ball in $\mathcal{G}$ whose radius is at least half the radius of the ball in $\mathcal{F}$.

This is the version of Zorn's Lemma you can use: If every chain in a nonempty partially ordered set P has an upper bound, then P has a maximal element.

(b) Infer that this disjointed subfamily $\mathcal{G}$ satisfies

$$\bigcup_{B\in\mathcal{F}} B \subset \bigcup_{B'\in\mathcal{G}} 5B'.$$

(c) Give an example of a family $\mathcal{F}$ of balls of *unbounded* radius such that there is no disjointed subfamily satisfying the condition in (b).

Exercise 2: Doubling Measures on Metric Spaces

A Borel measure μ on a metric space X is called *doubling* if balls have finite positive measure and there is a constant $C \geq 1$ such that for every ball B in X,

$$\mu(2B) \leq C\mu(B).$$

(a) Show that the volume measure on any compact Riemannian manifold is doubling.

(b) Show that Euclidean $\mathbb{R}^n$ is doubling, but hyperbolic $\mathbb{H}^n$ is not.

(c) Argue that any subfamily $\mathcal{G}$ as in Exercise 1(b) in a metric space X carrying a doubling measure μ has to be countable.

Exercise 3: Besicovitch Covering Theorem

Study the proof of the Besicovitch Covering Theorem in Mattila's *Geometry of Sets and Measures in Euclidean Spaces*, pp. 28–34, where 2.7(2) is the main result. Describe the complete construction (without proofs) of the families of balls in (2) in your own words. Mark every detail in the proof that you do not fully understand (if any) for discussion.