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SoSe23

Selected Topics in Differential Geometry

Sheet 5

Due on **25 May 2023**

Exercise 1: Warped Product

We consider a warped product $(M \times_f N, g = g_M + f^2 g_N)$, for a function $f : M \to \mathbb{R}^+$.

(a) Show that the special case $\mathbb{R} \times_r S^1$ with $(M, g_M) = (\mathbb{R}, \mathrm{d}s^2)$, $(N, g_N) = (S^1, \mathrm{d}\varphi^2)$ generalizes surfaces of revolution in $\mathbb{R}^3$, with $r(s)$ denoting the distance to the z-axis and s the arc length parameter of a curve in the half space $\{(x, 0, z) \mid x > 0\}$ to be rotated around the z-axis. Draw a picture.

(b) In the general case, let X, Y be vector fields tangent to M and U, V tangent to N (i.e., $\mathrm{d}\pi_N(X) = 0$, where $\pi_N : M \times N \to N$ is the natural projection, ...). Assume that all commutators of these vector fields vanish, and that $X.(g_N(U, V)) = 0$, $U.(g_M(X, Y)) = 0$, ... Show that on these vector fields, the covariant derivative ∇ is given by

$$\nabla_X Y = \nabla^M_X Y \qquad \nabla_U V = \nabla^N_U V - f g^N(U, V)\,\mathrm{grad}^M f$$
$$\nabla_X V = \frac{X.f}{f} V \qquad \nabla_U Y = \frac{Y.f}{f} U.$$

Exercise 2: Geodesics on Surfaces of Revolution

Consider a surface $\mathbb{R} \times_r S^1$ as in the previous exercise.

(a) Deduce the geodesic equations for a curve $\gamma(t) = (s(t), \varphi(t))$ in $\mathbb{R} \times_r S^1$:

$$\ddot{s} = r r' \dot{\varphi}^2, \quad \ddot{\varphi} = -\frac{2r'}{r} \dot{s}\dot{\varphi}.$$

Keep in mind that part (a) of the previous exercise holds for coordinate vector fields ∂_s, ∂_φ, but not for arbitrary vector fields with coefficients depending on s and φ!

(b) Check that you obtain the trivial geodesics: The meridians $\varphi = \text{const}$, and the circles $s = \text{const}$ if and only if $\dot{\varphi} = 0$.

(c) Introducing the orthonormal frame $e_s = \partial_s$, $e_\varphi = \frac{1}{r}\partial_\varphi$, we can write the velocity of a curve γ as

$$\dot\gamma = \dot s e_s + r\dot\varphi e_\varphi = |\dot\gamma|(\cos(\alpha)e_s + \sin(\alpha)e_\varphi),$$

where $\alpha(t)$ is the angle between γ and the meridian curves ($\varphi = \text{const}$).

Prove *Clairaut's Theorem*: Along a geodesic γ, the quantity $r^2\dot\varphi = r \cdot \sin\alpha$ is constant. Conversely, all curves satisfying this condition which are parametrized by arc length and have $\dot\varphi \neq 0$ somewhere are geodesics.

(d) We now investigate how far a geodesic can travel in s direction.

Assuming that a geodesic $\gamma(t) = (s(t), \varphi(t))$ is parametrized by arc length, show that

$$\dot s^2 = 1 - \frac{C^2}{r^2},$$

where $C = r(t) \cdot \sin\alpha(t)$ is the constant from Clairaut's theorem. Under what conditions can $\dot s$ change sign, and how does its motion continue after that? Which regions are forbidden for geodesics, depending on the value of C? What happens if the curve $s = \text{const}$ at $r = C$ is also a geodesic?

Exercise 3: Curvature of Warped Products

Starting with the results of Exercise 1 (b), one can show that the scalar curvature on a warped product $M \times_f N^n$ satisfies

$$\operatorname{Scal} = \operatorname{Scal}_M + f^{-2}\operatorname{Scal}_N - 2n\frac{\Delta^M f}{f} - n(n-1)\frac{|\operatorname{grad}^M f|^2}{f^2}.$$

(a) Specialize this to the case of Gauß curvature $K = \operatorname{Scal}/2$ of surfaces $\mathbb{R} \times_r S^1$, and construct such a complete surface with $K \equiv -1$, the *pseudo-sphere*.

(b) Notice that you can realize only a part of this surface as an embedded surface of revolution in $\mathbb{R}^3$. What is the precise condition on the function $r(s)$, where s is the arc length parameter, for this to be possible?

(c) Describe the behavior of geodesics on the pseudo-sphere, taking into account the results of Exercise 2.