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SoSe23

Selected Topics in Differential Geometry

Sheet 4

Due on **11 May 2023**

Exercise 1: Conformal Deformations and Scalar Curvature

Let (M^n, g) be a closed connected Riemannian manifold and $\tilde{g} = u^{4/(n-2)} g$ a conformal deformation of g. Show:

(a) the Laplacian transforms as

$$\Delta^{\tilde{g}} f = u^{-4/(n-2)} (\Delta^g f + 2\, g(\nabla^g \ln u, \nabla^g f)) \quad \text{for any } f \in C^\infty(M).$$

(b) The conformal Laplacian

$$L^g = -\Delta^g + \frac{n-2}{4(n-1)} \operatorname{Scal}^g$$

is covariant under conformal deformations in the sense that

$$L^{\tilde{g}} f = u^{-\frac{n+2}{n-2}} L^g (uf) \quad \text{for any } f \in C^\infty(M).$$

(c) If g has $\operatorname{Scal}^g \geq 0$, then any conformal metric $\tilde{g}$ has integrated scalar curvature $\int_M \operatorname{Scal}^{\tilde{g}} \, \mathrm{d}V^{\tilde{g}} \geq 0$. Moreover, the only way to obtain $\int_M \operatorname{Scal}^{\tilde{g}} \, \mathrm{d}V^{\tilde{g}} = 0$ is to already have $\operatorname{Scal}^g \equiv 0$ and scale by a constant factor ($u \equiv \text{const}$). Hint: Consider integrals of type $\int_M f \, L^g f \, \mathrm{d}V^g$.

(d) If $\operatorname{Scal}^g < 0$ and $\operatorname{Scal}^{\tilde{g}}$ is constant, then $\operatorname{Scal}^{\tilde{g}} < 0$.