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SoSe23

Selected Topics in Differential Geometry

Sheet 3

Due on **4 May 2023**

Exercise 1: *h*-Transform

Let L be a uniformly elliptic operator of second order with smooth coefficients on a manifold M^n without boundary, i.e., locally it can be written as

$$L = -\sum_{i,j} a^{ij}\partial_i\partial_j + \sum_i b^i\partial_i + V$$

with smooth coefficients and there is a constant $\mu \geq 1$ such that $\mu^{-1}|\xi|^2 \leq \sum a^{ij}\xi_i\xi_j \leq \mu|\xi|^2$ for all $\xi \in \mathbb{R}^n$.

(a) Let $h \in C^\infty(M, \mathbb{R}^+)$ be a positive function. Show that the operator given by $L^h u := \frac{1}{h}L(hu)$ is uniformly elliptic with smooth coefficients.

(b) Prove the following version of the minimum principle without restrictions on V:

> Assume that there is a smooth positive function $h > 0$ on M which satisfies $Lh \geq 0$. If $u \in C^2(M)$ is a solution of $Lu \geq 0$, then it cannot attain a local non-positive minimum.

In particular, if such a solution is non-negative on the boundary of a domain $D \Subset M$, then it is non-negative in D.

(c) Prove the uniqueness of the solution of the Dirichlet problem

$$\begin{cases} Lu = 0 & \text{in } D, \\ u = f & \text{on } \partial D, \end{cases}$$

where $D \Subset M$ is a domain and $f \in C(\partial D)$, under the assumption that there is a global positive solution of $Lh \geq 0$.

Exercise 2: The Principal Eigenfunction

Let M^n be a closed Riemannian manifold and $L = -\Delta + V$ a Schrödinger operator on M. Assume there is a minimizer $u \in C^\infty(M)$ of the Rayleigh quotient

$$\mathcal{E}(u) := \frac{\int_M (|\nabla u|^2 + Vu^2)\mathrm{dVol}}{\int_M u^2\,\mathrm{dVol}}$$

among all non-vanishing functions in $H^{1,2}(M)$. This function satisfies the eigenvalue equation $Lu = \lambda_1 u$, where λ_1 is the minimum of the Rayleigh quotient.

(a) Show that either u or $-u$ is strictly positive. Hint: Consider $\mathcal{E}(u^+)$ and $\mathcal{E}(u^-)$, and apply a version of the strong maximum principle.

(b) Show that any other solution $v \in C^\infty(M)$ of $Lv = \lambda_1 v$ is a scalar multiple of u. Hint: Subtract a suitable multiple of u.