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SoSe23

Selected Topics in Differential Geometry

Sheet 2

Due on **27 April 2023**

Exercise 1: Schur's Lemma

Let (M^n, g) be a connected Riemannian (or Lorentzian) manifold of dimension $n \geq 3$. The Einstein field equations from general relativity are

$$G + \Lambda\, g := \operatorname{Ric} - \frac{1}{2}\operatorname{Scal}\, g + \Lambda\, g = c_n T$$

with some $c_n > 0$ depending only on the dimension and the cosmological constant $\Lambda \in \mathbb{R}$. Prove:

(a) $\operatorname{div}(G + \Lambda\, g) = 0$. (Recall that for a symmetric (0,2)-tensor X, the divergence is a (0,1)-tensor, and $(\operatorname{div} X)(V)$ is a defined as the trace of the bilinear form $(\nabla_. X)(\cdot, V)$. Employ the second Bianchi identity $(\nabla_W R)(X,Y) + (\nabla_Y R)(W,X) + (\nabla_X R)(Y,W) = 0$.)

(b) In the vacuum case $T = 0$, the field equations are equivalent to $\operatorname{Ric} = \frac{2}{n-2}\Lambda\, g$. Manifolds with constant Ricci curvature in this sense are called *Einstein manifolds*.

(c) $\operatorname{div}(fg) = \mathrm{d}f$ for $f \in C^\infty(M)$.

(d) If $\operatorname{Ric} = fg$ for an $f \in C^\infty(M)$, then M is already an Einstein manifold.

(e) If for every $p \in M$, the sectional curvature Sec is constant on all planes in T_pM, then Sec is already constant on all of M.

(f) What happens in the case $n = 2$?

Exercise 2: Hölder spaces on manifolds

On a manifold M^n, the space of Hölder continuous functions $C^{k,\alpha}(M)$, $k = 0, 1, \ldots$, $\alpha \in (0, 1]$, is the subset of all functions $f \in C^k(M)$ such that for every $p \in M$ there is a proper chart (U, φ) around p and a $p \in V \Subset U$ such that $f \circ \varphi^{-1} \in C^{k,\alpha}(\varphi(V))$.

(a) Show that $C^{k,\alpha}(M)$ can be given the structure of a locally convex topological vector space by stating appropriate semi-norms.

(b) Argue that it suffices to consider only a fixed cover of M by charts to obtain this topology on $C^{k,\alpha}(M)$.

(c) Deduce that for compact M, $C^{k,\alpha}(M)$ is a Banach space (albeit without a canonical norm).

(d) Write down a natural norm for $C^{k,\alpha}(M)$ on a compact connected *Riemannian* manifold M. You can use the powers of covariant derivatives $\nabla^k f$, norms on tensor spaces, and the Riemannian distance function.

(e) You could try to literally transfer this norm to non-compact Riemannian manifolds. Which $C^{k,\alpha}(M)$ functions would you miss?