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Selected Topics in Differential Geometry

Sheet 1

For discussion on **13 April 2023**

Exercise 1: Famous Theorems, in Two Dimensions

The following statements are true (and hard to prove) in dimension ≥ 3. Prove them in dimension 2 using Gauß–Bonnet or provide a counterexample.

(a) **Bonnet–Myers**: A compact Riemannian manifold with positive Ricci curvature has finite fundamental group.

(b) **Cartan–Hadamard**: A compact manifold with negative sectional curvature has $\mathbb{R}^n$ as universal covering space.

(c) **Synge**: An even-dimensional compact manifold with positive sectional curvature is simply connected if it is orientable, and has fundamental group $\mathbb{Z}/2\mathbb{Z}$ otherwise.

(d) **Negative Ricci curvature**: On every Riemannian manifold there is a complete metric with negative Ricci curvature.

(e) **Ric $<$ 0 Islands**: There is a metric on $\mathbb{R}^n$ which is Euclidean outside a non-empty compact set K and has negative Ricci curvature everywhere on K.

(f) **No Scal $>$ 0 Islands**: If there is a metric on $M = K\#\mathbb{R}^n$ for some compact K which is Euclidean outside a compact set and has scalar curvature ≥ 0 everywhere, then M is already isometric to Euclidean $\mathbb{R}^n$.

(g) **Gromov–Lawson**: On compact manifolds $M\#T^n$ with torus T^n as summand there is no metric with positive scalar curvature.

Exercise 2: Conformal Deformations

Let (M^n, g), $n \geq 3$, be a Riemannian manifold.

(a) For a smooth function $f : M \to \mathbb{R}$, show that the conformally deformed metric $g' = \mathrm{e}^{2f} g$ has Ricci curvature

$$\mathrm{Ric}^{g'} = \mathrm{Ric}^g - (n-2)(\nabla^g \mathrm{d}f - \mathrm{d}f \otimes \mathrm{d}f) + (-\Delta^g f - (n-2)|\mathrm{d}f|_g^2)g.$$

(b) Deduce that this metric has scalar curvature

$$\mathrm{Scal}^{g'} = \mathrm{e}^{-2f}\left(\mathrm{Scal}^g - 2(n-1)\Delta^g f - (n-2)(n-1)|\mathrm{d}f|_g^2\right).$$

(c) For a smooth function $u : M \to (0, \infty)$, show that the metric $g'' = u^{4/(n-2)} g$ has scalar curvature

$$\mathrm{Scal}^{g''} = 4\frac{n-1}{n-2} u^{-\frac{n+2}{n-2}} \left(-\Delta^g + \frac{n-2}{4(n-1)}\mathrm{Scal}^g\right) u.$$

Exercise 3: Blow-up

When enlarging a Riemannian manifold around a fixed point, the metric converges to the flat metric. Make this statement precise using the exponential map and prove it.