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SoSe22

Geometry of Scalar Curvature

Sheet 9

For discussion on **5 July 2022**

Exercise 1: Interpretation of Sectional Curvature

Let M be a Riemannian manifold, $p \in M$ and $v, w \in T_pM$. Show that:

(a) $J(t) := \mathrm{dexp}_{tv}(tw) = t\mathrm{dexp}_{tv}(w)$ (with $\mathrm{dexp}_{tv} : T_{tv}T_pM \cong T_pM \to T_{\exp(tv)}M$) is a Jacobi vector field along the geodesic $\gamma(t) = \exp(tv)$ with $J(0) = 0$ and $\dot{J}(0) = \nabla_{\dot\gamma(0)}J = w$, near $t = 0$.

(b) The Taylor expansion of $|J(t)|^2$ around $t = 0$ is

$$|J(t)|^2 = |w|^2t^2 - \frac{1}{3}g(R(w,v)v,w)t^4 + \mathcal{O}(t^5)\ .$$

To obtain this result, explicitly calculate the derivatives and use the Jacobi field equation $\ddot{J}(t) = R(\dot\gamma(t), J(t))\dot\gamma(t)$.

(c) For the Taylor expansion of the metric $g_{ij} = g(\partial_i, \partial_j)$ in normal coordinates, this implies

$$g_{ij} = \delta_{ij} - \frac{1}{3}R_{ik\ell j}(0)x^kx^\ell + \mathcal{O}(|x|^3)$$

with $\partial_i|_{\exp(tv)} = \mathrm{dexp}_{tv}(e_i)$ for an orthonormal base $(e_i)_i$ of T_pM.

Exercise 2: Interpretation of Ricci and Scalar Curvature

In the same situation as before, prove:

(a) In normal coordinates (x^i) near a point p,

$$\det(g_{ij}) = 1 - \frac{1}{3}\,\mathrm{Ric}_{k\ell}\,x^kx^\ell + \mathcal{O}(|x|^3)$$

with Ricci curvature $\mathrm{Ric}_{k\ell} = g^{ij}R_{ik\ell j}$.

(b) For small balls with center p,

$$\mathrm{Vol}^g(B_r(p)) = \left(1 - \frac{1}{6(n+2)}\,\mathrm{Scal}\,r^2 + \mathcal{O}(r^3)\right)\mathrm{Vol}^{\mathrm{eucl}}(B_r(0))$$

compared to balls with equal radius in Euclidean space.