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SoSe22

Geometry of Scalar Curvature

Sheet 8

For discussion on **21 June 2022 14:00**

Exercise 1: Conformal Invariance of the Conformal Laplacian

On a manifold M^n with metrics g and $\tilde{g} = \varphi^{4/(n-2)} g$, show that the corresponding conformal Laplace operators

$$L = -\Delta + \frac{n-2}{4(n-1)} \operatorname{Scal}$$

and $\tilde{L}$ satisfy

$$\tilde{L}u = \varphi^{\frac{n+2}{n-2}} L(\varphi \cdot u).$$

Exercise 2: The Yamabe Functional

Let (M^n, g) be a closed manifold with Yamabe functional $Q_g(\varphi) = E_g(\varphi)/\|\varphi\|_{g,p}^2$, where $p = 2n/(n-2)$,

$$E_g(\varphi) = \int_M \left(|\nabla\varphi|^2 + \frac{n-2}{4(n-1)} \operatorname{Scal}^g \right) \mathrm{d}V^g \quad \text{and} \quad \|\varphi\|_{g,p} = \left(\int_M |\varphi|^p \mathrm{d}V^g \right)^{1/p}.$$

Prove the variation formula

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} Q_g(\varphi + t\psi) = \frac{2}{\|\varphi\|_{g,p}^2} \int_M \left(L^g\varphi - E(\varphi)\varphi^{p-1}/\|\varphi\|_{g,p}^p \right) \psi \, \mathrm{d}V^g$$

for any $\varphi, \psi \in C^\infty(M)$, showing that the Yamabe equation is the Euler–Lagrange equation for Q_g.

Exercise 3: Stereographic Projection

We write stereographic projection as a map σ from the sphere $S^n \setminus (0, \ldots, 0, 1) \subset \mathbb{R}^{n+1}$ with coordinates $\zeta^1, \ldots, \zeta^n, \xi$ to $\mathbb{R}^n$ with coordinates $(x^1, \ldots, x^n)$ given by

$$x^i = \frac{\zeta^i}{1-\xi} \quad \text{for } i = 1, \ldots, n.$$

(a) Show that the standard metric on the sphere can be written as

$$(\sigma^{-1})^* g_{S^n} = \left(\frac{2}{|x|^2 + 1} \right)^2 g_{\mathrm{Eucl}}.$$

(b) Show that dilations $\delta_\alpha(x) = \alpha^{-1}x$ of $\mathbb{R}^n$ for any $\alpha > 0$ induce conformal diffeomorphisms on S^n, i.e., the metrics $g_\alpha := \kappa_\alpha^* g_{S^n}$ with $\kappa_\alpha := \sigma^{-1} \circ \delta_\alpha \circ \sigma : S^n \to S^n$ satisfy $g_\alpha = t_\alpha^{4/(n-2)} g_{S^n}$ with a function $t_\alpha \in C^\infty(S^n)$. Determine t_α.

Exercise 4: Conformal Diffeomorphisms of the Sphere

The group of conformal diffeomorphisms of the sphere is generated by rotations of the sphere and similarities of $\mathbb{R}^n$ conjugated with stereographic projection. Prove this in dimension two by the following steps:

(a) Compose an arbitrary conformal diffeomorphism with perhaps several of the mappings above to reduce to the case where the north pole is fixed and orientation preserved. Then you have an orientation-preserving conformal diffeomorphism of $\mathbb{C}$ in a stereographic chart, hence an automorphism of $\mathbb{C}$ in the sense of complex analysis.

(b) Show that this automorphism of $\mathbb{C}$ has the form $z \mapsto az + b$ with $a, b \in \mathbb{C}$, $a \neq 0$.

You can use the Casorati–Weierstraß theorem from complex analysis: A holomorphic function $\mathbb{C} \to \mathbb{C}$ is either a polynomial or maps any neighbourhood of ∞ to a dense subset of $\mathbb{C}$.

Exercise 5: An Upper Bound for the Yamabe Invariant

We will prove that $\lambda(M^n) \leq \lambda(S^n)$ for any closed Riemannian manifold M^n, $n \geq 3$.

(a) Starting on $\mathbb{R}^n$, multiply the functions

$$u_\alpha(x) = \left(\frac{\alpha}{|x|^2 + \alpha^2}\right)^{\frac{n-2}{2}}$$

with a radial cutoff function supported in $B_\varepsilon(0)$ to get a test function φ and show an estimate of the form

$$\int_{\mathbb{R}^n} |\nabla\varphi|^2 \mathrm{d}V \leq \lambda(S^n)\|\varphi\|_p^2 + O(\alpha^n).$$

Recall that the u_α satisfy $\|\nabla u_\alpha\|_2^2 = \lambda(S^n)\|u_\alpha\|_p^2$ on $\mathbb{R}^n$.

(b) Transfer this test function to a closed Riemannian manifold (M^n, g) using normal coordinates. Estimate the Yamabe quotient of this function φ as

$$Q_g(\varphi) \leq (1 + C\varepsilon)(\lambda(S^n) + C\alpha)$$

with C depending only on global curvature bounds on (M, g) and the dimension. You may use that $\mathrm{d}V^g = (1 + O(r))\mathrm{d}V^{g_{\mathrm{Eucl}}}$ in normal coordinates.