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Geometry of Scalar Curvature

Sheet 7

For discussion on **3 June 2022**

Exercise 1: Spreading Scal > 0 locally

Assume x_0 is a point in a Riemannian manifold (M, g) such that the exponential map at x_0 is defined and injective on $B_3(0) \subset T_{x_0}M$, $\mathrm{Scal}^g \geq 0$ on $B_3(x_0)$, and $\mathrm{Scal}^g > 0$ on $B_1(x_0)$. Construct a conformally deformed metric $g' = \mathrm{e}^{2f} g$ on M such that

- $\mathrm{Scal}^{g'} > 0$ on $B_2(x_0)$,
- $g' \equiv g$ outside $B_2(x_0)$, and
- g' is arbitrarily close to g in C^k norm, for some fixed k.

Hint: Use the formula

$$\mathrm{e}^{-2f}\,\mathrm{Scal}^{\mathrm{e}^{2f} g} - \mathrm{Scal}^g = -2(n-1)\Delta f - (n-2)(n-1)|\mathrm{d}f|^2$$

and choose a function $f = f(r)$ only depending on $r = \mathrm{dist}(\cdot, x_0)$. You can start with the ansatz $f = -s \cdot \mathrm{e}^{-d/(2-r)}$ on $B_2 \setminus B_1$ and estimate coefficients in the Laplacian very crudely.
Don't worry too much about the last point, you just have to make f smooth and scale it down to 0.

Exercise 2: Spreading Scal > 0 globally

Let U be a connected open set in a Riemannian manifold M with $\mathrm{Scal}^g|_U \geq 0$ and $\mathrm{Scal} > 0$ at a point in U. Find a conformal deformation g' of g such that

- $\mathrm{Scal}^{g'} > 0$ on U,
- $g' \equiv g$ on $M \setminus U$ and
- g' is arbitrarily close to g in compact C^k topology, for some fixed k.