Prof. Dr. Joachim Lohkamp
Dr. Matthias Kemper

WWU Münster
SoSe22

Geometry of Scalar Curvature

Sheet 6

For discussion on **27 May 2022**

Exercise 1: Scalar Curvature of Hypersurfaces

Take traces of the Gauß formula to obtain an expression for the scalar curvature of a hypersurface in terms of the scalar curvature of the exterior space, the mean curvature, and the norm of the second fundamental form.

Exercise 2: Gauß–Bonnet

(a) Show that on a general Riemannian manifold (M^n, g), the differential of the volume element $\mathrm{d}V_g$ with respect to the metric is given by

$$\mathrm{d}V_g'(h) = -\frac{1}{2}(\mathrm{tr}_g\, h)\mathrm{d}V_g,$$

i.e., for a one-parameter family g_t of metrics with $g_0 = g$ and $\frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} g_t = h$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \mathrm{d}V_{g_t} = -\frac{1}{2}(\mathrm{tr}_g\, h)\mathrm{d}V_g.$$

You can use the expression $\mathrm{d}V_g = \sqrt{\det g}\, \mathrm{d}x^1 \cdots \mathrm{d}x^n$ in local coordinates.

(b) Combine (a) with exercise 5.2 to show that if M is compact, the integrated Scalar curvature functional

$$\mathbf{S}_g := \int_M \mathrm{Scal}^g\, \mathrm{d}V_g$$

has derivative

$$\mathbf{S}_g'(h) = -\int_M g(\mathrm{Ric}^g - \frac{1}{2}\,\mathrm{Scal}^g\, g, h)\mathrm{d}V_g.$$

This shows that the vacuum Einstein equation is the Euler–Lagrange equation for this functional.

(c) Deduce that on a compact two-dimensional manifold, the value of the functional $\mathbf{S}_g$ is independent of the metric.

This proves more or less the Gauß–Bonnet theorem. All that is left is to calculate the value for an arbitrary metric on every surface, e.g., on the sphere, $\mathbb{R}P^2$, and for attaching a handle.