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SoSe22

Geometry of Scalar Curvature

Sheet 4

For discussion on **13 May 2022**

Exercise 1: Riemannian Schwarzschild metric I

We consider a spherically symmetric Riemannian metric

$$g = \frac{\mathrm{d}r^2}{V(r)} + r^2 \cdot g_{S^{n-1}}$$

on the product of an interval with the $(n-1)$-dimensional sphere S^{n-1}. Here $g_{S^{n-1}}$ denotes the standard metric on S^{n-1}.

(a) Find a formula for the scalar curvature Scal of g. This should be an ODE in V.

You may use the following formula for the scalar curvature of a warped product $(M \times_{u^{2/(n+1)}} N^n, g = g_M + u^{4/(n+1)} g_N)$, $u \in C^\infty(M, \mathbb{R}^+)$:

$$\mathrm{Scal}_g = -\frac{4n}{n+1}\Delta_M u + \mathrm{Scal}_M\, u + \mathrm{Scal}_N\, u^{\frac{n-3}{n+1}}.$$

(b) Solve the ODE in (a) for $\mathrm{Scal} \equiv 0$ to obtain the family of Riemannian Schwarzschild metrics. What is the maximal interval where r can be defined?

Exercise 2: Riemannian Schwarzschild metric II

(a) For the n-dimensional Riemannian Schwarzschild metric $g = \left(1 - \frac{2m}{r^{n-2}}\right)^{-1} \mathrm{d}r^2 + r^2 g_{S^{n-1}}$ with $m > 0$, find a new radial coordinate $\rho = \rho(r)$ such that g can be written as

$$g = u(\rho)^{\frac{4}{n-2}} \left(\mathrm{d}\rho^2 + \rho^2 g_{S^{n-1}}\right) = u(\rho)^{\frac{4}{n-2}} g_{\mathrm{eucl}}$$

for some function $u = u(\rho)$. You should be able to observe that this metric can be defined for $\rho \in (0, \infty)$.

(b) Find an isometry of g that inverts the radial coordinate ρ and keeps the spherical coordinates fixed. Use this to show that g is complete.

(c) Explain why the sphere fixed by the isometry in (b) is totally geodesic and calculate its $(n-1)$-dimensional volume.

(d) Describe the Riemannian Schwarzschild metric for $m < 0$.

Exercise 3: Field Equations and Einstein Manifolds

Let (M^n, g) be a connected Lorentzian (or Riemannian) manifold of dimension $n \geq 3$. The field equations are

$$G + \Lambda g := \mathrm{Ric} - \frac{1}{2}\,\mathrm{Scal} \cdot g + \Lambda \cdot g = c_n T$$

with some $c_n > 0$ depending only on the dimension and the cosmological constant $\Lambda \in \mathbb{R}$. Prove:

(a) $\mathrm{div}(G + \Lambda g) = 0$. (Recall that for a symmetric (0,2)-tensor X, the divergence $\mathrm{div}\, X(V)$ is defined as the trace of the bilinear form $(\nabla . X)(\cdot, V)$. Employ the second Bianchi identity $(\nabla_W R)(X,Y) + (\nabla_Y R)(W,X) + (\nabla_X R)(Y,W) = 0$.)

(b) In the vacuum case $T = 0$, the field equations are equivalent to $\mathrm{Ric} = \frac{2}{n-2}\Lambda g$. Manifolds with this property are generally called *Einstein manifolds*.

(c) $\mathrm{div}(fg) = \mathrm{d}f$ for $f \in C^\infty(M)$.

(d) If $\mathrm{Ric} = fg$ for an $f \in C^\infty(M)$, then M is already an Einstein manifold.

(e) If for every $p \in M$ the sectional curvature Sec is constant on all nondegenerate planes in T_pM, then Sec is constant on all of M.

(f) What happens in the case $n = 2$?