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# Geometry of Scalar Curvature

## Sheet 3

For discussion on **6 May 2022**

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**Exercise 1: Fibrations with Positive Scalar Curvature**

Let $E \to M$ be a fiber bundle of Riemannian manifolds over a compact manifold $M$ with totally geodesic fibers. Assuming the induced metric on every fiber has $\operatorname{Scal} > 0$, show that the total space $E$ admits a metric with positive scalar curvature.

**Exercise 2: Second Fundamental Form under Conformal Deformations**

Let $(M^n, g)$ be a Riemannian manifold, $S$ an immersed submanifold with induced metric, and $f : M \to \mathbb{R}$ a smooth function.

(a) Show that in the conformally deformed metric $g' = \mathrm{e}^{2f} g$, the second fundamental form $A$ of this immersion satisfies

$$A^{g'}(X, Y) = A^g(X, Y) - g(X, Y)(\operatorname{grad}^g f)^\perp$$

for vector fields $X$, $Y$ tangential to $S$.

(b) Deduce a formula for the mean curvature $\operatorname{tr} A$ of $S$ after conformal deformation.

**Exercise 3: Scalar Curvature under Conformal Deformations**

Let $(M^n, g)$, $n \geq 3$, be a Riemannian manifold.

(a) For a smooth function $f : M \to \mathbb{R}$, show that the conformally deformed metric $g' = \mathrm{e}^{2f} g$ has scalar curvature

$$\operatorname{Scal}^{g'} = \mathrm{e}^{-2f} \left( \operatorname{Scal}^g - 2(n-1)\Delta^g f - (n-2)(n-1)|\mathrm{d}f|_g^2 \right).$$

(b) For a smooth function $u : M \to (0, \infty)$, show that the metric $g' = u^{4/(n-2)} g$ has scalar curvature

$$\operatorname{Scal}^{g'} = 4\frac{n-1}{n-2} u^{-\frac{n+2}{n-2}} \left( -\Delta^g + \frac{n-2}{4(n-1)} \operatorname{Scal}^g \right) u.$$