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Geometry of Scalar Curvature

Sheet 1

For discussion in the first exercise class

Exercise 1: Ricci and Scalar Curvature as Averages

(a) Show that the trace of a symmetric bilinear form B on $\mathbb{R}^n$ satisfies

$$\frac{1}{n}\operatorname{tr} B = \frac{1}{\operatorname{vol}(S^{n-1})}\int_{S^{n-1}} B(x,x)\,\mathrm{d}V(x).$$

(b) Formalize and prove the statement that on a Riemannian manifold M^n, the normalized Ricci curvature $\frac{1}{n-1}\operatorname{Ric}(v,v)$ for a unit vector v is the sectional curvature averaged over all planes containing v.

(c) Similarly, show that the (normalized) scalar curvature is the averaged Ricci curvature.

Exercise 2: Second Fundamental Form

The *second fundamental form* A of a submanifold S in a Riemannian manifold (M,g) is

$$A(X,Y) := (\nabla^g_X Y)^\perp$$

for vector fields X, Y tangential to S, where $^\perp$ denotes the component perpendicular to $TS \subset TM$ (i.e., the projection on the normal bundle $NS \subset TM$). Prove:

(a) $A(X,Y) = A(Y,X)$ for X, Y sections of TS, and in particular, A is well-defined,

(b) A is a $(0,2)$-tensor on S with values in the normal bundle of S, i.e., a section of $T^*S \otimes T^*S \otimes NS$,

(c) the *Gauß equation*

$$\langle R^M(X,Y)V,W\rangle = \langle R^S(X,Y)V,W\rangle + \langle A(X,V),A(Y,W)\rangle - \langle A(X,W),A(Y,V)\rangle$$

for vector fields tangential to S, and

(d) the *Codazzi–Mainardi equation*

$$(R^M(X,Y)Z)^\perp = (\nabla_X A)(Y,Z) - (\nabla_Y A)(X,Z)$$

for vector fields tangential to S.

Exercise 3: Total Geodesic Submanifolds

Prove that a for a closed submanifold S of a Riemannian manifold (M,g), the second fundamental form A vanishes identically if and only if each geodesic **in M** that is in some point tangential to S is already contained in S.