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Minimal Surfaces

Sheet 1

For discussion on **25 October 2022**

Exercise 1: "Minimal" Submanifolds

A k-dimensional submanifold in an n-dimensional Riemannian manifold, $n \geq 2$ and $1 \leq k \leq n-1$, can have the following properties. Indicate which implications between them always hold and give counterexamples (for any combination of dimensions) if a property does not imply another.

- stationary minimal
- stably minimal
- locally area minimizing
- (globally) area minimizing

Exercise 2: Second Fundamental Form

The *second fundamental form* A of a submanifold S in a Riemannian manifold (M, g) is

$$A(X, Y) := (\nabla^g_X Y)^\perp$$

for vector fields X, Y tangential to S, where $^\perp$ denotes the component perpendicular to $TS \subset TM$ (i.e., the projection onto the normal bundle $\perp S \subset TM$). Prove:

(a) the *Gauß equation*

$$\langle R^M(X,Y)V, W\rangle = \langle R^S(X,Y)V, W\rangle + \langle A(X,V), A(Y,W)\rangle - \langle A(X,W), A(Y,V)\rangle$$

for vector fields tangential to S, and

(b) the *Codazzi–Mainardi equation*

$$(R^M(X,Y)Z)^\perp = (\nabla_X A)(Y, Z) - (\nabla_Y A)(X, Z)$$

for vector fields tangential to S.

Exercise 3: Finding totally geodesic submanifolds

(a) Let H^n be a submanifold of M^{n+k}. Show that the following are equivalent:

(i) Every geodesic in H^n with respect to the induced metric is also a geodesic for the ambient metric in M.

(ii) For any vector $V \in T_x H$ tangential to H, the geodesic with respect to the ambient metric with initial velocity V is locally contained in H.

(iii) The second fundamental form A of H vanishes everywhere.

(b) Assume that H^n is submanifold of a Riemannian manifold (M^{n+1}, g) and $\varphi : M \to M$ an isometry with H as set of fixed points. Show that H is totally geodesic.