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Scalar curvature and hammocks

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1 Introduction

Scalar curvature is the simplest generalization of Gaussian curvature to higher dimensions. However there are many questions open with regard to its relation to other geometric quantities and topology. Here we will prove and illustrate some features of scalar curvature in higher dimensions related to a general hammock effect for scalar curvature, namely the one-sided affinity for curvature decreasing deformations.

The first one is concerned with some prescribed decrease of the scalar curvature $\mathrm{Scal}(g)$ of some Riemannian metric g on a given manifold M^n of dimension ≥ 3. We denote the $\varepsilon-$neighborhood of some set U with respect to g by U_ε.

Theorem 1. *Let $U \subset M$ be an open subset and f any smooth function on M with $f < \mathrm{Scal}(g)$ on U and $f \equiv \mathrm{Scal}(g)$ on $M \setminus U$.*
Then, for each $\varepsilon > 0$, there is a smooth metric g_ε on M with

$$g \equiv g_\varepsilon \text{ on } M \setminus U_\varepsilon \text{ and } f - \varepsilon \leq \mathrm{Scal}(g_\varepsilon) \leq f \text{ on } U_\varepsilon.$$

Actually, the metric g_ε can also be chosen arbitrarily near to g in C^0-topology. As will become clear from proof when combined with results from [L2].

The corresponding statements for $f > \mathrm{Scal}(g)$ are false even without the C^0-condition: in general one may decrease but not increase curvatures locally. As we will see the most prominent example is provided by the following "positive energy theorem" originally proved by Schoen and Yau and somewhat later by Witten (cf. [S], [PT] and [LP]). We will present a different and short proof relying on the *non*-existence of certain curvature increasing deformations underlining its relation to Theorem 1.

Theorem 2. *Let (M, g) be an asymptotically flat spin manifold with $\mathrm{Scal}(g) \geq 0$. Then the energy $E(g)$ is non-negative and $E(g) = 0$ iff (M, g) is flat.*

(The precise definitions and some explanations are given below).

Now we will briefly discuss the meaning of these results. It should be quite obvious that Theorem 1 implies a rough version of the so-called Kazdan-Warner Trichotomy (cf. [KW1],[KW2]) which can be stated as follows: Every closed manifold $M^n, n \geq 3$ belongs to exactly one of the following three classes described by properties of scalar curvature functions $f \in C^\infty(M, \mathbb{R})$ on these spaces:

A. Every f can be realized as the scalar curvature of some metric $g, i.e. f = \mathrm{Scal}(g)$
B. f can be written as $f = \mathrm{Scal}(g)$ iff f is identically zero or negative somewhere
C. f can be realized as $f = \mathrm{Scal}(g)$ iff f is negative somewhere

However, our point of view differs from such a type of classification. First of all note that the metrics in Theorem 1 will have nearly the same geometry as the original one (as a first application this will be used in the proof of Theorem 3 below) while the Kazdan-Warner metrics are rather special.

Supported by results such as those in Theorem 2 we think of some sort of maximal amount of positive curvature which can be carried by a given manifold. Starting from such *maximal metrics* other curved metrics are basically obtained by decreasing curvatures (and scalings) without necessarily essential changes of the coarse "metric geometry". Thus there should be a more profound individual upper curvature bound whose sign is just either $>, =$ or < 0 (reflecting cases A-C). We refer to [L1] for a more detailed discussion.

Next we turn to a related more concrete problem. Recall that a manifold admits an almost flat structure (i.e. a family of metrics g_ε with diam$(M, g_\varepsilon) = 1$ and $|\mathrm{Sec}(g_\varepsilon)| < \varepsilon$, for each $\varepsilon > 0)$ iff it is an infranilmanifold (cf. [G] and [R]). The corresponding notions of almost Ricci or scalar flatness have not been considered yet, partly because of a lack of techniques. Furthermore, up to now, such structures have not appeared as frequently as almost flat manifolds. This might change with further progress made in understanding Ricci and scalar curvatures.

We will show that almost scalar flat structures exist on each closed (i.e. compact without boundary) manifold M^n in dimension $n > 3$. More precisely, we have

Theorem 3. *M^n admits a family of Riemannian metrics g_ε with*

$$\operatorname{diam}(M, g_\varepsilon) = 1 \;\; and \;\; |\operatorname{Scal}(g_\varepsilon)| < \varepsilon, \text{for any } \varepsilon > 0.$$

Note that there are many manifolds which do not admit any metric g with $\operatorname{Scal}(g) \equiv 0$. This is related to Theorem 2 (cf. Sect. 5 below) while the proof makes use of Theorem 1 (cf. Sect. 8).

The paper splits into three parts. In Sect. 2-4 we prove Theorem 1, in Sects. 5-7 we are concerned with the positive energy theorem and in Sects. 8-10 we derive Theorem 3.

2 Singular conformal deformations

The proof of Theorem 1 is subdivided in several stages (occupying the next three sections):

1. We will show how to obtain a metric G_ε with:

$$G_\varepsilon \equiv g \text{ on } M \setminus U_{\varepsilon/3} \text{ and } f - \varepsilon/3 \leq \operatorname{Scal}(G_\varepsilon) \leq f + \varepsilon/3 \text{ on } U_{\varepsilon/3}.$$

For this construction we start with a conformal deformation outside a small set and subsequently (cf. Sect. 3) these singular parts are smoothed changing the conformal class.

2. In order to get the desired metric with $g \equiv g_\varepsilon$ on $M \setminus U_\varepsilon$ and $f - \varepsilon \leq \operatorname{Scal}(g_\varepsilon) \leq f$ on U_ε. we must check that the freedoms in these constructions can be exploited in such a way that the curvature strictly decreases (cf. Sect. 4).

It suffices to prove the following specialized version of Theorem 1 where U is a cube in $M = \mathbb{R}^n$. The general case is a straightforward extension using suitable coverings by charts.

For the same reason it suffices (via scalings) to consider metrics g with $\|g - g_{\text{Eucl.}}\|_{C^3_{g_{\text{Eucl}}}} \ll 1$. This allows us to restrict the descriptions of some of the geometric configurations to the Euclidean case.

Proposition 2.1 *Let f be a smooth function on $\mathbb{R}^n$ with supp $(f - \operatorname{Scal}(g)) = [-1,1]^n \subset \mathbb{R}^n$, $f < \operatorname{Scal}(g)$ on $]-1,1[^n$ and let $\varepsilon > 0$ be given. Then we can find a metric g_ε on $\mathbb{R}^n$ with $g_\varepsilon \equiv g$ on $\mathbb{R}^n \setminus [-2,2]^n$ and $f - \varepsilon \leq \operatorname{Scal}(g_\varepsilon) \leq f$ on $[-2,2]^n$.*

Our strategy is to superpose a lot of small deformations. In order to add up such modifications of g it is convenient to use conformal changes of

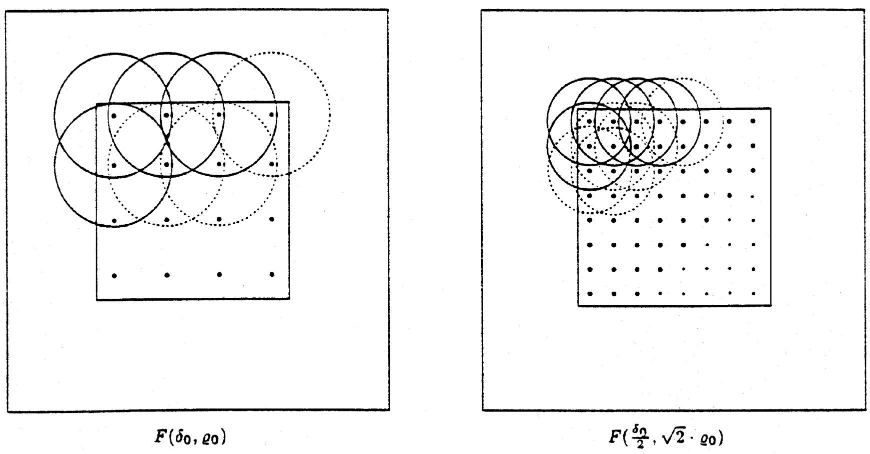

Fig. 1

g. The transformation law for Scal becomes particularly simple when the conformal factor is written as $e^{2f} \cdot g$, for some smooth f. (We always use the sign convention $\Delta f = +f''$, for f on $\mathbb{R}$).

Then one gets

$$\begin{aligned}&\mathrm{Scal}\ (e^{2f} \cdot g)\\ &\quad = e^{-2f} \cdot (\mathrm{Scal}(g) - 2(n-1) \cdot \Delta f - (n-1)(n-2) \cdot \|\nabla f\|^2)\ .\end{aligned}$$

In particular, for $\Psi = \sum_i \varphi_i$, with finitely many smooth $\varphi_i > 0$, we note:

$$\begin{aligned}\mathrm{Scal}(e^{2\Psi} \cdot g) = e^{-2\Psi} \cdot (\mathrm{Scal}(g) - 2(n-1)\\ \cdot \sum_i \Delta\varphi_i - (n-1)(n-2) \cdot \|\sum_i \nabla\varphi_i\|^2).\end{aligned} \tag{1}$$

Now we start as follows. Define lattices $L_\delta \subset \mathbb{R}^n$ with $L_\delta = \{\delta \cdot (z_1, \cdots z_n) | z_i \in \mathbb{Z}\}$,$\delta > 0$, take the following family $F(\delta, \varrho) = \{B_{\delta\cdot\varrho}(p) \mid p \in L_\delta \cap [-1,1]^n\}$, $\varrho \in [1, \delta^{-\frac{1}{2}}]$ of balls and also their union

$$\mathcal{F}(\delta, \varrho) = \bigcup_{p \in L_\delta \cap [-1,1]^n} , B_{\delta\cdot\varrho}(p),\ \varrho \in [1, \delta^{-\frac{1}{2}}]$$

(cf. Fig. 1) and construct certain metrics on each of them.

Thus we first consider a single ball and notice the following elementary

Lemma 2.2 *For* $\mu \in]0,1[, q \in \mathbb{R}^n$ *we can find a smooth function* $\varphi = \varphi_q(g, \delta, \varrho, \mu) \geq 0$ *on* $\mathbb{R}^n \setminus \{q\}$ *with* $\varphi \equiv 0$ *on* $\mathbb{R}^n \setminus B_{\delta\cdot\varrho}(q)$ *and*

(i) $\Delta_g \varphi \equiv 1$ *on* $B_{\delta\cdot\varrho\cdot(1-\mu)}(q) \setminus \{q\}$
(ii) $0 \leq \Delta_g\varphi \leq 1$ *on* $B_{\delta\cdot\varrho}(q) \setminus B_{\delta\cdot\varrho\cdot(1-\mu)}(q)$

(iii) Also we may assume that the φ are chosen uniformly in the sense that for $g \to g_{\text{Eucl.}}$ in C^3 and for every fixed triple $\delta, \varrho, \mu : \varphi_0(g) \to \varphi_0(g_{\text{Eucl.}})$ compactly on $\mathbb{R}^n \setminus \{0\}$ in C^2 and also:

$$\varphi_0(g_{\text{Eucl.}}, \delta, \varrho, \mu)\left(\left(\frac{\delta' \cdot \varrho'}{\delta \cdot \varrho}\right) \cdot x\right) = \left(\frac{\delta' \cdot \varrho'}{\delta \cdot \varrho}\right)^2 \cdot \varphi_0(g_{\text{Eucl.}}, \delta', \varrho', \mu)(x)$$

Remark. In each of the following steps we will specify further properties of φ_q which however can readily be adjusted.

In order to find such a family of functions one may choose a radially symmetric function $\varphi(x) = \overline{\varphi}(\|\, x \,\|)$ on the Euclidean $\mathbb{R}^n$ and choose geodesic coordinates in order to generalize this definition to the curved case choosing $\varphi_q(g, \delta, \varrho, \mu)(x) = \overline{\varphi}_q(g, \delta, \varrho, \mu)(dist(p, x)) = \overline{\varphi}(dist(p, x))$

We use these functions to build up a singular metric $G(\delta, \varrho, \mu)$ which will be further smoothed leading to the desired metric g_ε subsequently. We set $G(\delta, \varrho, \mu) := e^{2\Psi(\delta,\varrho,\mu)} \cdot g$ with $\Psi(\delta, \varrho, \mu) := \sum_{q \in L_\delta \cap [-\frac{3}{2}, \frac{3}{2}]^n} m(q) \cdot \varphi_q$, for some $m(q) \in \mathbb{R}^{>0}$.

Lemma 2.3 *For each $\varepsilon > 0$ we can find small $\delta_0 > 0, \mu_0 > 0$ and a $\varrho_0 > 0$ such that for each $\delta \in]0, \delta_0[$ there are multipliers $m(q)$ with*

$$(2) \qquad \mid \Delta\Psi(\delta, \varrho_0, \mu_0) - (\text{Scal}(g) - f) \mid \leq \frac{\varepsilon}{3} \text{ on } \mathbb{R}^n \setminus L_\delta.$$

Proof. We start by showing how to obtain these $m(q)$.

Define the following covering numbers $a(q) = a(q, \delta, \varrho, \mu) := \# \{p \in L_\delta \cap [-\frac{3}{2}, \frac{3}{2}]^n \mid q \in B_{\delta\cdot\varrho\cdot(1-\mu)}(p)\}$, $b(q) = b(q, \delta, \varrho, \mu) := \# \{p \in L_\delta \cap [-\frac{3}{2}, \frac{3}{2}]^n | q \in B_{\delta\cdot\varrho}(p) \setminus B_{\delta\cdot\varrho\cdot(1-\mu)}(p)\}$ and finally set $m(q) := (\text{Scal}(g) - f)(q)/\, a(q), (resp.\ m(q) := 0$ when $a(q) = 0)$.

Directly from the definitions, we have:

$$\begin{aligned} \sum_{z \in B_{\delta\cdot\varrho\cdot(1-\mu)}(p)} (\text{Scal}(g) - f)(p)/a(p) &\leq \Delta\Psi(\delta, \varrho, \mu) \\ \leq \sum_{z \in B_{\delta\cdot\varrho}(p)} (\text{Scal}(g) - f)(p)/a(p) \end{aligned}$$

(The sums are taken over all the balls (around points in $L_\delta \cap [-\frac{3}{2}, \frac{3}{2}]^n$) containing z.)

Now, in order to find δ_0, μ_0 and ϱ_0 we observe, setting $\varrho = \delta^{-\frac{1}{2}}, \mu = \delta$ in what follows:

$$\begin{aligned} a(\delta, \varrho, \mu) = a(\delta, \delta^{-\frac{1}{2}}, \delta) &\geq \text{ const. } \cdot \delta^{-n/2} && \text{on } [-1, 1]^n \\ b(\delta, \varrho, \mu) = b(\delta, \delta^{-\frac{1}{2}}, \delta) &\leq \text{ const. } \cdot \delta^{-(n-1)/2} && \text{on } [-2, 2]^n \end{aligned}$$

and $a \equiv b \equiv 0$ on $\mathbb{R}^n \setminus [-1.6, 1.6]^n$ (for any sufficiently small $\delta > 0$).

Then the difference between upper and lower bound can be estimated as follows:

$$0 \leq \sum_{z \in B_{\delta \cdot \varrho}(p) \setminus B_{\delta \cdot \varrho \cdot (1-\mu)}(p)} (\mathrm{Scal}(g) - f)(p)/a(p)$$
$$\leq \max_{B_{2 \cdot \delta \cdot \varrho}(z)} (\mathrm{Scal}(g) - f) \cdot \frac{b(p)}{a(p)} \leq \mathrm{const.} \cdot \sqrt{\delta}$$

Moreover

$$\mathcal{S}_\delta(g, f)(z) := \sum_{z \in B_{\delta \cdot \varrho \cdot (1-\mu)}(p)} (\mathrm{Scal}(g) - f)(p)/a(p)$$
$$\xrightarrow[\delta \to 0]{} (\mathrm{Scal}(g) - f)(z),$$

uniformly in z on $\mathbb{R}^n$ since $(\mathrm{Scal}(g) - f)$ is a continuous function on $\mathbb{R}^n$ with $\mathrm{Scal}(g) - f \equiv 0$ on $\mathbb{R}^n \setminus] - 1, 1[^n$ and

$$\max_{p \in L_\delta \cap B_{2 \cdot \delta \cdot \varrho}(z)} \frac{a(p)}{a(z)} \xrightarrow[\delta \to 0]{} 1 \xleftarrow[0 \leftarrow \delta]{} \min_{p \in L_\delta \cap B_{2 \cdot \delta \cdot \varrho}(z)} \frac{a(p)}{a(z)}$$

uniformly in *z* on compacta $\subset] - \frac{3}{2}, \frac{3}{2}[^n$.

Hence, for sufficiently small $\delta > 0$ we get
$| \Delta\Psi(\delta, \delta^{-\frac{1}{2}}, \delta) - (\mathrm{Scal}(g) - f) | \leq \frac{\varepsilon}{4}$ on $\mathbb{R}^n \setminus L_\delta$.

Now, we can *decouple* δ, ϱ, μ. Namely, we can choose a *fixed* sufficiently small $\delta_0 > 0$ as above and such that for $\varrho_0 = \delta_0^{-\frac{1}{2}}$ and $\mu_0 = \delta_0$ $| a(\delta, \varrho_0, \mu_0)/a(\delta, \delta^{-\frac{1}{2}}, \delta) - 1 | \ll 1$ and $| b(\delta, \varrho_0, \mu_0)/b(\delta, \delta^{-\frac{1}{2}}, \delta) - 1 | \ll 1$ for $\delta \in]0, \delta_0[$, because decreasing δ merely corresponds to a scaling of the whole setting by δ^{-2} which does not affect the covering numbers. Thus we may adjust ϱ_0 and μ_0 such that inequality (2) is satisfied for any sufficiently small $\delta > 0$. □

While this sounds rather like (2.1) we must take into account the fact that $\Delta\Psi$ describes the new scalar curvature only when Ψ and $\nabla\Psi$ can be assumed to be arbitrarily small. Unfortunately φ looks very much like the pole of a usual Green's function near q. However, outside "fixed" small balls around $p \in L_\delta \cap [-\frac{3}{2}, \frac{3}{2}]^n$ and in addition to 2.3 we have:

Lemma 2.4 *For any given $\eta > 0$, $\varrho_0 > 0$, $\mu_0 \in]0, \frac{1}{2}[$ and $\lambda \in]0, \frac{1}{2}[$, we can find a sufficiently small $\delta_0 \in]0, \varrho_0^{-2}]$, such that for every $\delta \in]0, \delta_0[$*

$$\|\Psi(\delta, \varrho_0, \mu_0)\|_{C^1(\mathbb{R}^n \setminus \mathcal{F}(\delta, \lambda))} < \eta$$

This is seen in the Euclidean case $(g \equiv g_{\text{Eucl.}})$ by considering a single φ on $B_\delta(0)$ (i.e. $q = 0$). Set $\varphi_m(x) = \varphi(m \cdot x)$ on $B_{\frac{1}{m} \cdot \delta}(0)$. Then $(\Delta\varphi_m)(x) = m^2 \cdot \Delta\varphi(m \cdot x)$, while $\|\nabla\varphi_m\|(x) = m \cdot \|\nabla\varphi\|(m \cdot x)$ and $\varphi_m(x) = \varphi(m \cdot x)$. Thus multiplying φ_m by m^{-2} reproduces the original value of $\Delta\varphi$ while $\|\nabla\varphi\|$ and φ decrease at the same rate as m^{-1} and m^{-2}, respectively. Thus noting that φ and $\nabla\varphi$ are bounded on $B_\delta(0) \setminus B_{\delta \cdot \lambda}(0)$ simply taking a large $m \gg 1$ (or choosing directly some small $\delta \ll 1$) will serve our purpose. Now consider $\Psi(\delta, \varrho_0, \mu_0)$. Here we have to add up the components $m(q) \cdot \varphi_q$. Applying the previous considerations to each component we get the result for Ψ, since (for $\delta \to 0$), the covering number $a(p)$ and $b(p)$ of (2.3) remain bounded (because ϱ_0 and μ_0 are fixed).

Obviously, the same argument carries over to the general case $(g \neq g_{\text{Eucl.}})$ $\square$

As a consequence we can find a $\delta_0(\varrho_0, \mu_0, \lambda) > 0$ with $\mid Scal(G(\delta, \varrho_0, \mu_0)) - f \mid \ll 1$ on $\mathbb{R}^n \setminus \mathcal{F}(\delta, \lambda)$ for every $\delta \in]0, \delta_0[$.

3 Hammocks

The effective conformal deformation above is defined outside a certain set $F(\delta, \varrho)$ of small balls. Here we will define the desired metrics on the remaining part.

This cannot be done while staying in the same conformal class (e.g. flat metrics on a torus cannot be conformally deformed into a $\mathrm{Scal} < 0$-metric). In other words scalar curvature "hammocks" correspond to a change of conformal classes.

The advantage resulting from those deformations described in the previous section is that the prescribed curvature function on each of those balls is nearly constant. This allows us to use the following device.

We start with a metric $g_{f(p)}$ on $S^n, n \geq 3$ (the dimensional restriction has not been used so far) with $\mid \mathrm{Scal}(g_{f(p)}) - f(p) \mid \ll 1$ and such that there is a ball $B \subset S^n$ with $(B, g_{f(p)})$ isometric to $B_r(0) \subset F^n_{f(p)}$ where the latter is the n-dimensional space of constant sectional curvature $\equiv f(p)$. We retain this identification in what follows. Such a metric can be obtained from another one with constant scalar curvature $f(p)$ by some local deformation (cf. (9.1) for some details).

Then, for some suitable $0 < r' < r \quad B_r(0) \setminus B_{r'}(0)$ can be deformed in such a way that the new metric $G_{f(p)}$ equals $g_{f(p)}$ near $\partial B_r(0)$, $\mid \mathrm{Scal}(G_{f(p)}) - f(p) \mid \ll 1$ and $\partial B_{r'}(0)$ becomes totally geodesic and we may even assume that $\partial B_{r'}(0)$ is isometric to the round sphere of radius r, for any $r < r_0$ for some r_0 depending only on $f(p)$.

This type of deformations has been considered in [GL] and [L2].

Next we will prepare $G(\delta, \varrho, \mu)$ to allow this prepared sphere to be glued onto $\mathbb{R}^n\setminus$ small balls around $p \in L_\delta \cap [-\frac{3}{2}, \frac{3}{2}]^n$. Here one chooses some $\lambda, 0 < \lambda \ll \varrho$ and deforms $B_{\delta\cdot\lambda}(p) \setminus \{p\}$ in such a way that for some $\theta \in]0, \lambda[, \partial B_{\delta\cdot\theta}(q)$ becomes nearly totally geodesic. Then one just cuts out $B_{\delta\cdot\theta}(q)$ and glues in $S^n \setminus B_{r'}(0)$, arriving at the desired metric g_ε.

Now we will put this sketch into more concrete form:

Lemma 3.1 *For any given $\varepsilon > 0$ we can find $\delta_0 > 0, \varrho_0 > 0, \mu_0 > 0$ such that for $\delta \in]0, \delta_0[, \lambda \in]0, \frac{\varrho_0}{2}[$ there is a metric $G_\theta(\delta, \varrho, \mu_0)$, for some $\theta \in]0, \lambda[$, defined on $\mathbb{R}^n \setminus \mathcal{F}(\delta, \theta)$ with:*

(i) $G_\theta(\delta, \varrho_0, \mu_0) \equiv G(\delta, \varrho_0, \mu_0)$ *on* $\mathbb{R}^n \setminus \mathcal{F}(\delta, \lambda)$
(ii) $|\operatorname{Scal}(G_\theta(\delta, \varrho_0, \mu_0)) - f| < \frac{\varepsilon}{3}$ *on* $\mathbb{R}^n \setminus \mathcal{F}(\delta, \theta)$
(iii) $\partial B_\theta(p)$ *is almost totally geodesic: that is* $G_\theta(\delta, \varrho, \mu_0)$ *can be extended smoothly near* $\partial B_\theta(p)$ *by the metric* $G_{f(p)}$ *on the prepared sphere.*

Proof. Choosing sufficiently small fixed $\lambda > 0$ and fixed $\varrho > 0$ we find that for $\delta \to 0 : \Psi \ll 1, \|\nabla\Psi\| \ll 1$ on $B_{\delta\cdot\lambda}(p) \setminus B_{\frac{\delta}{2}\cdot\lambda}(p)$ and by the same argument $\sum_{q\in L_\delta\setminus\{p\}} m(q) \cdot \varphi_q \ll 1, \ \|\sum_{q\in L_\delta\setminus\{p\}} m(q) \cdot \nabla\varphi_q\| \ll 1$, $p \in L_\delta$, (φ_p as in (2.2)) on $B_{\delta\cdot\lambda}(p)$.

Then we can substitute $B_{\delta\cdot\lambda}(p) \setminus \{p\}$ equipped with $e^{2\Psi} \cdot g$ by another conformal metric on $B_{\delta\cdot\varrho}(p) \setminus B_{\delta\cdot\theta}(p)$ for some suitable $\theta \in]0, \lambda[$ as follows.

We will choose a smooth function Ψ_1 on $B_{\delta\cdot\lambda}(p) \setminus B_{\delta\cdot\theta}(p)$ for some $\theta > 0$ with $\Psi_1 \equiv \Psi$ near $\partial B_{\delta\cdot\lambda}(p)$ and $|\operatorname{Scal}(e^{2\Psi_1} \cdot g) - f| < \frac{\varepsilon}{4}$. This is possible for sufficiently small δ (ϱ and μ fixed) as φ_p converges (in the sense of (2.2)) to $\varphi_p(g_{\text{Eucl.}})$ which is rotationally symmetric around p. Hence, as the above inequality (ii) can be taken for granted near $\partial B_{\delta\cdot\lambda}(p)$ one just increases the radial 2nd order derivative of φ_p in such a way (dictated by the Scal-transformation law (1)) that it leads to $\varphi_{1,p}$ (substituting φ_1) adding up to Ψ_1 with property (ii) on $B_{\delta\cdot\lambda}(p) \setminus B_{\delta\cdot\theta}(p)$. This is possible because for this step we can assume that $f, \operatorname{Scal}(g)$ and $\Delta \sum_{q\in L_\delta\setminus\{p\}} m(q) \cdot \varphi_q$ are constant ($\equiv c_1, c_2$ and c_3 respectively) on $B_{\delta\cdot\lambda}(p)$ and thus one only needs to find $\varphi_{1,p}$ with

$$\big|c_1 - e^{-2\varphi_{1,p}} \cdot (-2(n-1) \cdot \Delta\varphi_{1,p} - (n-1) \cdot (n-2) \cdot \|\nabla\varphi_{1,p}\|^2 \\ + c_2 - 2(n-1)c_3)\big| < \frac{\varepsilon}{4}.$$

Now, we notice that $\partial B_{\delta\cdot\lambda}(p)$ is far from being totally geodesic (having nearly the 2nd fundamental form of $\partial B_{\delta\cdot\lambda}(0) \subset \mathbb{R}^n$), whereas the radius of $\partial B_{\delta\cdot\lambda}(p)$ (measured with respect to $e^{2\Psi_1} \cdot g$) becomes arbitrarily *large* for small r because $\varphi_0(g_{\text{Eucl.}}) = \text{const.} \cdot \|x\|^{-n+2}$ and $r \cdot exp(r^{-n+2}) \xrightarrow[r\to 0]{} +\infty$.

Hence, summing up, for sufficiently small $\delta > 0$, there is a $\theta > 0$ with $\theta < \lambda$ such that $\partial B_{\delta\cdot\theta}(p)$ becomes (nearly) totally geodesic for $e^{2\Psi_1} \cdot g$ and this can be improved on arbitrarily by choosing some smaller $\delta > 0$ in such a way that the prepared sphere can be glued smoothly for suitably small $r' > 0$ with $\delta \cdot \theta < r^1 < \delta \cdot \lambda$. $\square$

Summarizing we get a metric satisfying the condition

$$G_\varepsilon \equiv g \text{ on } M \setminus U_{\varepsilon/3} \text{ and } f - \alpha \leq \mathrm{Scal}(G_\varepsilon) \leq f + \alpha \text{ on } U_{\varepsilon/3}.$$

(We will use this for very small $\alpha \ll \varepsilon$.)

4 Curvature decreasing effect

Finally we will show how to refine the definition of G_ε in order to get g_ε with the following additional property

Lemma 4.1 $\mathrm{Scal}(g_\varepsilon) \leq f$ *on* U_ε

Proof. We can use the procedure of the previous sections applied to a function F with

$F \equiv f - \beta$ on $U_{\varepsilon/3}$, $f - \beta \leq F \leq f$ on $U_{2\cdot\varepsilon/3} \setminus U_{\varepsilon/3}$ and $F \equiv f$ on $M \setminus U_{2\cdot\varepsilon/3}$

for some $\beta > 0$ with $\alpha \ll \beta \ll \varepsilon$.

Therefore it is enough to show that we may assume that $\mathrm{Scal}(G_\varepsilon) \leq \mathrm{Scal}(g)$ on U_ε.

As becomes clear from the previous step in the construction it suffices to prove that the function Ψ can be chosen such that in addition to the properties above:

$$\mathrm{Scal}(e^{2\Psi} \cdot g) \leq \mathrm{Scal}(g) \text{ on } U_\varepsilon \setminus \mathcal{F}(\delta, \varrho) \tag{3}$$

From equality (1) we immediately get:

$$\mathrm{Scal}(e^{2\Psi} \cdot g) \;\leq\; e^{-2\Psi} \cdot (\mathrm{Scal}(g) - 2(n-1) \cdot \sum_i \Delta\varphi_i) \tag{4}$$

and hence for $\mathrm{Scal}(g) \geq 0$ this claim becomes obvious for any choice of Ψ with $\Delta\Psi \geq 0$.

Now, when $\mathrm{Scal}(g) < 0$ we set the *right-hand side* of (4) $\leq \mathrm{Scal}(g)$ and get the condition $(1 - e^{-2\Psi}) \cdot |\,\mathrm{Scal}(g)\,| \leq e^{-2\Psi} \cdot 2(n-1) \cdot |\sum_i \Delta\varphi_i\,|$.

Moreover, since we may assume that $e^{-2\Psi} > 1/2$ and (via scaling) that $|\,\mathrm{Scal}(g)\,| < 1$ it is enough to make sure that $|\,1 - e^{-2\Psi}\,| < C \cdot |\sum_i \Delta\varphi_i\,|$, for a certain $C = C(n) > 0$.

Finally (since $1 \geq e^{-2\Psi} > 1/2$) the mean value theorem shows that this can still be reduced ultimately to the

Claim. Ψ can be chosen such that $\mid \Psi \mid \leq c \cdot \mid \Delta\Psi \mid$, for some prescribed $c > 0$.

For this it suffices to consider the components $m(q_i) \cdot \varphi_{q_i}$. For notational convenience, we consider $\varphi_i := \varphi_{q_i}$ as being defined on $B_{\delta \cdot \varrho}(0) \subset \mathbb{R}^n$ centered at $q_i = 0$. Now $\mid \Delta\varphi_i \mid$ can be estimated as described in [L2](2.1): Once again we further adjust the definition of φ_i. Recall that when $f \in C^\infty(\mathbb{R}^n, \mathbb{R})$ is of the form $f(x) = F(\| x \|)$ for some smooth function F on $\mathbb{R}^{>0}$. Then one can write for $r = \| x \|$: $\Delta f(x) = F''(r) + \frac{(n-1)}{r} \cdot F'(r)$.

We may assume that φ_i can be written as $\varphi_i(x) = \phi_i(r)$ (outside $\mathcal{F}(\delta, \varrho)$) for some smooth functions ϕ_i. Moreover one can choose this function $\phi_i(r)$ such that for a given $k > 0 \quad \phi_i(r)'' \geq k \cdot \phi_i(r)$ and $\geq k \cdot \mid \phi_i'(r) \mid$ on $\mathbb{R}^{>0}$, which is the key point here.

Finally, since we are on $U_\varepsilon \setminus \mathcal{F}(\delta, \varrho)$ we have the following estimate $\Delta\varphi_i \geq a(n) \cdot \phi_i' + b(n) \cdot \phi_i''$, with $b(n, \varrho) > 0$. Hence, for a suitably chosen $k > 0$ we get $\Delta\varphi_i \geq a(n) \cdot \phi_i' + b(n) \cdot \phi_i'' \geq c \cdot \phi_i$ □

5 Positive energy theorem

Roughly speaking the positive energy theorem (cf. [LP] and [PT]) asserts that the total energy of any “sufficiently reasonable” universe is nonnegative and zero exactly when the system is in the vacuum state. This can be cut down to a problem concerning “maximal” spacelike asymptotically flat hypersurfaces (in Lorentz geometry). This maximality and a further reasonable assumption (the so-called “dominant energy condition”) imply that this hypersurface H(which will be the manifold under consideration) has $\mathrm{Scal} \geq 0$ (via Gauss-equations). Now the energy E becomes a numerical invariant of the geometry of H near infinity and the problem is to show $E \geq 0$.

The first known proof is due to Schoen and Yau (cf. [SY 1], [SY 2]). One assumes $E < 0$ and produces a contradiction by showing the existence of a certain minimal hypersurface $\subset H$ which could not exist assuming that the positive energy theorem holds in lower dimensions. This proof does not need the spin assumption. However the argument works perfectly only in dimension ≤ 7. In higher dimensions there might be hypersurface singularities which destroy the possibility of applying the induction hypothesis.

The second proof by Witten (cf. [PT]) can be extended to spin manifolds of arbitrary dimensions. It proves and uses the existence of a certain spinor on the manifold allowing the energy to be expressed in terms of scalar curvature, (squares of) the norm of this spinor and its covariant derivative. This leads to quite obvious estimates for the energy as $E \geq 0$ for $\mathrm{Scal} \geq 0$.

In the following we describe a more geometric argument which works for spin manifolds in arbitrary dimensions via contradiction assuming $E < 0$. We use the negativity of E to deform the manifold in such a way that the resulting one is Euclidean outside a compact set K with $\mathrm{Scal} > 0$ in *interior*K. Using this manifold we can construct various *compact* manifolds with $\mathrm{Scal} > 0$. On the other hand, one can use spin geometry to show that some of these manifolds cannot admit any metrics with $\mathrm{Scal} > 0$. Hence $E \geq 0$.

Now we describe the actual mathematical set-up:

Let (M^n, g) be a Riemannian manifold containing a compact subset $K \subset M$ such that each component ("end") C_k of $M \setminus K$ is diffeomorphic to $\mathbb{R}^n \setminus \overline{B_1(0)}$. With respect to this chart we assume that the metric g on $M \setminus K$ can be written as

$$g = g_{\mathrm{Eucl.}} + h, \text{ with } h_{ij} = O\left(||x||^{-p}\right) \text{ and}$$

$$\begin{aligned} ||x|| \cdot \left|\frac{\partial g_{ij}}{\partial x_k}\right|(x) + ||x||^2 \cdot \left|\frac{\partial^2 g_{ij}}{\partial x_k \partial x_l}\right|(x) \\ = O\left(||x||^{-p}\right) \text{ with } p > (n-2)/2 \end{aligned}$$

In other words, outside a compact set M looks more and more (i.e. for $||x|| \to +\infty$) like the Euclidean $\mathbb{R}^n$ with a certain deviation motivated by physics (one may think of a "distribution of matter" in a *compact* domain). If (M, g) satisfies these conditions we will say M is *asymptotically flat*.

Moreover, one is also interested in measuring the "total energy" $E(M, g)$ of M. This can be done (i.e. can be defined properly) if $\mathrm{Scal}(g) = O(||x||^{-p})$, for $p > n$ and we will also assume that this condition is satisfied.

The actual "physical" definition of E would be rather too much of a digression here (it can be found in [LP], [PT] or [S]). The important point, which we make use of, is that E can be recovered as a coefficient in a suitable expansion of g near infinity and we will use it as a definition of E.

More precisely, we first deform the ends making them conformally Euclidean as follows (cf. [S](4.1)): For some suitably chosen cut-off function φ on $\mathbb{R}^n \setminus K$ with $\varphi \equiv 1$ near ∂K and $\varphi \equiv 0$ near infinity one may first define $g_\varphi := \varphi \cdot g + (1 - \varphi) \cdot g_{\mathrm{Eucl.}}$. Then, for suitable φ (here the Scal-condition enters) one can, by applying a (by now) standard argument, find a conformal deformation (namely taking a non-trivial element h of the kernel of the conformal Laplacian) $h^{4/n-2} \cdot g_\varphi$ with $\mathrm{Scal}(h^{4/n-2} \cdot g_\varphi) \equiv 0, h > 0, h \to 1$ near infinity such that $h^{4/n-2} \cdot g_\varphi$ is again asymptotically flat.

Moreover it is not hard to choose φ in such a way that $E(M, h^{4/n-2} \cdot g_\varphi)$ becomes arbitrarily close to $E(M, g)$. In other words it suffices to prove the positive energy theorem for this type of asymptotically flat manifold. In this particular case we may define E as follows:

Noting that h can be written as

$$h = 1 + \frac{A_k}{\|x\|^{n-2}} + O(\|x\|^{1-n}) \text{ on } \mathbb{R}^n \setminus B_1(0) \text{ with } A_k \in \mathbb{R}$$

being *uniquely* determined, we set $E(M, h^{4/n-2} \cdot g_\varphi) := 4(n-1) \cdot A_k$. (Note that this yields an energy for each end C_k.) In general, one could approximate (M, g) by certain $(M, h^{4/n-2}\varphi_n)$ and define $E(M, g)$ as the limit of $E(M, h^{4/n-2} \cdot g_{\varphi_n})$.

Now, finally we can formulate

Proposition 5.1 $E(M, h^{4/n-2} \cdot g_\varphi) \geq 0$ *for each end* C_k *of* M.

Up to this point we have described the reductions used by Schoen and Yau on their way to the proof. Having reached this point the real geometric work begins. In their minimal hypersurface proof Schoen and Yau use the modified metric mainly to simplify geometric constructions and calculations made in order to gain control to handle such a hypersurface.

We will also utilize it but in a different way. In particular, we use the (trivial) observation that h is harmonic on $\mathbb{R}^n \setminus B_R(0)$:
$-\gamma \cdot \Delta h + \mathrm{Scal}(g_{\mathrm{Eucl.}}) \cdot h = \mathrm{Scal}(h^{4/n-2} \cdot g_{\mathrm{Eucl}}) \cdot h^\alpha$ with $\gamma, \alpha > 0$ and $\mathrm{Scal}(g_{\mathrm{Eucl.}}) = \mathrm{Scal}(h^{4/n-2} \cdot g_{\mathrm{Eucl}}) \equiv 0$.

6 Negative energy yields positive curvature

Now we assume that the energy E of at least one end of our asymptotically flat manifold M^n (equipped with $g_0 = h^{4/n-2} \cdot g$) is *negative*. We will prove that this assumption can be "transformed" into a purely geometric condition.

We can (and will henceforth) assume that M has only this one end. For instance, one could close the other ends by bending them together and compactifiying them by adding points "at infinity". This can be done in such a way that $\mathrm{Scal} \geq 0$-metrics result (cf. the argument in (6.2) below).

Proposition 6.1 *If* $E < 0$*, then we can find another complete metric* g_1 *on* M^n *with* $\mathrm{Scal} \geq 0$ *(and* $\mathrm{Scal} > 0$ *in some points) and such that there is a compact set* $K \subset M^n$ *with* $(M^n \setminus K, g_1)$ *being isometric to* $(\mathbb{R}^n \setminus B_R(0), g_{\mathrm{Eucl.}})$ *for some* $R > 0$.

We will reduce the proof of (6.1) to the following

Lemma 6.2 *Let* $h = \frac{1}{||x||^{n-2}} + f$ *be harmonic on* $S = \overline{B_6(0)} \setminus B_1(0) \subset \mathbb{R}^n$ *(equipped with* $g_{\mathrm{Eucl.}}$*). Then there is a* $\delta = \delta(n) > 0$ *such that, provided* $|f| < \delta$*, there is a smooth function* $H > 0$ *on* S *with*

(i) $\Delta H \geq 0$ on S and $\Delta H > 0$ in some interior points of S

(ii) $H \equiv h$ near $\partial B_1(0)$
(iii) $H \equiv$ const. > 0 near $\partial B_6(0)$.

Proof. As already mentioned above for a function F on $\mathbb{R}^n \backslash (0)$ with $F(x) = G(||x||)$ for some smooth G one finds for $r = \|x\| : \Delta F(x) = G''(r) + \frac{n-1}{r} \cdot G'(r)$.

Our first aim is to obtain a smooth function $h_1 > 0$ on $\mathbb{R}^{>0}$ with $h_1(r) \equiv \frac{1}{r^{n-2}}$ on $]0,2], h_1 \equiv$ const. > 0 on $\mathbb{R}^{\geq 5}$ with $h_1''(r) + \frac{n-1}{r} \cdot h_1'(r) \geq 0$ for $r \in]2,5[$ and > 0 on an interval $I \subset [3,4]$.

For this purpose we define $f = f(d,s) \in C^\infty(\mathbb{R}, \mathbb{R}^{\geq 0})$ as follows:

$$f(d,s)(r) = s \cdot exp(-d/5 - r) \text{ on } \mathbb{R}^{\leq 5}, f(d,s) \equiv 0 \text{ on } \mathbb{R}^{\geq 5}.$$

Claim. For suitably chosen $s, d > 0$ we have

(i) $f''(r) + (n-1) \cdot f'(r) > 0$ on $]1,5[$
(ii) $f''(4) > \left(\frac{1}{r^{n-2}}\right)'' |_{r=4}$, $f'(4) > \left(\frac{1}{r^{n-2}}\right)' |_{r=4}$ and $f < \frac{1}{2} \cdot \frac{1}{r^{n-2}}$ on $]1,5[$

Proof of this claim.

$$f'(r) = -s \cdot \frac{d}{(5-r)^2} \cdot \exp(-d/5 - r), f''(r)$$
$$= s \cdot \left(\frac{-2d}{(5-r)^3} + \frac{d^2}{(5-r)^4} \right) \cdot \exp(-d/5 - r)$$

Hence, for each $k > 0$ we can find $d_0 = d_0(k) > 0$ such that for any $d \geq d_0, s > 0 : f'' + k \cdot f' > 0$ on $]1,5[$. Assuming $k \geq n-1$ we can be sure that (i) is satisfied. For (ii) we choose some $k_0 \geq 3 \cdot \mid \left(\frac{1}{r^{n-2}}\right)'' |_{r=4}$ / $\left(\frac{1}{r^{n-2}}\right)' |_{r=4}$. Then, for any $d \geq d_0(k)$ we can find some $s > 0$ such that $f''(4) = 2 \cdot \left(\frac{1}{r^{n-2}}\right)' |_{r=4}$ and (hence) $f'(4) > \left(\frac{1}{r^{n-2}}\right)' |_{r=4}$. By choosing a sufficiently large d we can be sure that $f < \varepsilon$ on $]1,5[$ (for every given $\varepsilon > 0$).

Thus taking $k = k_0 + n$ we can find some f as claimed.

Next we use $\frac{1}{r^{n-2}}$ and f to get h_1 by double integration: Using properties (i) and (ii) of f we can find a smooth $k_1 > 0$ with:

$k_1 \equiv \left(\frac{1}{r^{n-2}}\right)''$ on $[1,3], k_1 \equiv f''$ on $\mathbb{R}^{\geq 4}$ and satisfying $k_1 \geq \left(\frac{1}{r^{n-2}}\right)''$ on $[3,4]$ and in order to fix the integration constant:

$$\int_3^4 k_1(r)dr = f'(4) - \left(\frac{1}{r^{n-2}}\right)' |_{r=3}$$

Define h_1 as $h_1(z) := \int_1^z \left((\int_1^y k_1(x)dx) + (\frac{1}{r^{n-2}})' |_{r=1} \right) dy + \frac{1}{r^{n-2}} |_{r=1}$

Then $h_1 > 0$ (since $f < \frac{1}{2} \cdot \frac{1}{r^{n-2}}$), smooth and $h_1 \equiv \mathrm{const.} > 0$ on $\mathbb{R}^{\geq 5}$; moreover $h_1 = \frac{1}{r^{n-2}}$ on $[1, 2]$ and $h_1'' + \frac{n-1}{r} h_1' \geq 0$ on $\mathbb{R}^{\geq 1}$.

This latter inequality is obvious outside]3, 4[. On]3, 4[we know that $h_1'' = k_1 \geq \left(\frac{1}{r^{n-2}}\right)''$ and (therefore) $h_1' \geq \left(\frac{1}{r^{n-1}}\right)'$. Thus $\left(\frac{1}{r^{n-2}}\right)'' + \frac{n-1}{r} \cdot \left(\frac{1}{r^{n-2}}\right)' = 0$ leads to the desired inequality.

We may assume $\Delta(h_1(\|x\|)) = h_1''(r) + \frac{n-1}{r} \cdot h_1'(r) > c > 0$ on [3.1, 3.9].

Now we may attack the general (non-rotationally) symmetric case $h = \frac{1}{\|x\|^{n-2}} + f$, $\Delta h = 0$.

Choose a fixed cut-off function $\psi \in C^\infty(\mathbb{R}^n, [0, 1])$ with $\psi \equiv 1$ on $B_{3.4}(0)$ and $\psi \equiv 0$ on $\mathbb{R}^n \setminus B_{3.6}(0)$ and define $H := h_1 + \psi \cdot f$

Claim. There is a $\delta > 0$ depending only on the dimension n such that, provided $\sup_S |f| < \delta$, $\Delta H \geq 0$ (and > 0 in some interior points of S).

The proof of this claim starts by noting that $\Delta f = 0$. Thus for each $\varepsilon > 0$ there is a $\delta > 0$ such that $\sup_S |f| < \delta$ implies $\|f\|_{C^3(B_4(0) \setminus \overline{B_3(0)})} < \varepsilon$ via elliptic theory.

In particular, we may assume $|\Delta(\psi \cdot f)| < \varepsilon$. This obviously implies the claim for sufficiently small δ resp ε and since h_1 was fixed the general case of the Lemma is proved. □

Now we derive (6.1) from this Lemma.

Proof of (6.1). As explained in the previous section we may assume that $g_0 = h^{4/n-2} \cdot g_{\text{Eucl.}}$ on $M \setminus K \cong \mathbb{R}^n \setminus B_1(0)$ for some harmonic function $h(x) = 1 + \frac{E}{4(n-1) \cdot \|x\|^{n-2}} + f$, with $sup\left(|f| \cdot \|x\|^{n-1}\right) < +\infty$. In order to apply (6.2) we first note that $\frac{1}{\|x\|^{n-2}}$, which is the fundamental solution of $\Delta G = 0$, reproduces under scalings.

That is, if we substitute $g_1 = g_{\text{Eucl.}}$ for $g_\lambda = \lambda^2 \cdot g_{\text{Eucl.}}$ then $\frac{1}{\|x\|_{g_1}^{n-2}} = \frac{\lambda^{n-2}}{\|x\|_{g_\lambda}^{n-2}}$ and therefore for our purpose we can identify (via rescaling) for any positive integer m:

$$\frac{6^{m(n-2)}}{\|x\|^{n-2}_{g_{\text{Eucl.}}}} \text{living on } B_{6^{m+1}}(0) \setminus \overline{B_{6^m}(0)} \text{ with}$$
$$\frac{1}{\|x\|^{n-2}_{g_{\text{Eucl.}}}} \text{ living on } B_6(0) \setminus \overline{B_1(0)}$$

Now the point is that under this identification the C^0-Norm of the perturbation term $f = O(\|x\|^{1-n})''$ converges to zero for $m \to +\infty$ (note that) $| \, 6^{m(n-2)} \cdot f(6^m) \, | \leq const. \cdot 6^{-m}$.

In particular, for sufficiently large m we may restrict our considerations to the case that $|f| < \frac{|E|}{4(n-1)} \cdot \delta$ on $B_6(0) \setminus \overline{B_1(0)}$ and we may apply (6.2) to get a function $\bar{h} = 1 + \frac{E}{4(n-1)} \cdot H$. The geometry of $\bar{h}^{4/n-2} \cdot g_{\text{Eucl.}}$ is as follows:

Near $\partial B_1(0)$ it is just $h^{4/n-2} \cdot g_{\text{Eucl.}}$, near $\partial B_6(0)$ it is Euclidean and from the transformation rule for Scal under conformal changes (starting from the Euclidean one) we get:

$$-\gamma \cdot \Delta\bar{h} = \text{Scal}(\bar{h}^{4/n-2} \cdot g_{\text{Eucl.}}) \cdot \bar{h}^{\alpha},$$
$$\text{for some } \gamma, \alpha > 0 \text{ depending only on } n.$$

Thus we find that $\text{Scal} \geq 0$ and > 0 in some points since $E < 0$. □

7 Detecting flatness

Now we are able to reduce the problem to the question of whether certain *compact* manifolds admit a $\text{Scal} > 0$-metric. It is here that we employ spin geometric arguments. Henceforth we shall assume that M admits a spin structure and note that the connected sum of spin manifolds is still spin.Also we show how to prove that $E = 0$ iff (M, g) is isometric to $(\mathbb{R}^n g_{\text{Eucl}})$.

We will outline two proofs for (6.1), both using versions of the Lichnerowicz obstruction to $\text{Scal} > 0$-metrics: non-vanishing $\hat{A}$-genus and generalizations thereof.

Proof 1 (using just the "standard" Lichnerowicz argument). Take (M^n, g_1) described in (6.1) and use it to construct (by induction) complete manifolds $(N^k, g(k))$ of dimension $k \geq m$ with $(N^k \setminus K(k), g(k))$ isometric to $(\mathbb{R}^k \setminus B_1(0), g_{\text{Eucl}})$ and $\text{Scal}(g(k)) > 0$ on *interior*$(K(k))$ for some compact set $K(k) \subset N^k$. This can be done in the same way as in ([L1], sect. 6). Choose some $k \geq m$ with $k = 4d$ for some integer $d > 0$ and take a spin manifold A^k with $\hat{A}([A^k]) \neq 0$. Now "cover" A^k (as in [L1], sect.4) by sufficiently many N^k. That is, take any metric on A^k and scale it by a huge constant (making it nearly flat) and choose a well-ordered "Besicovitch" covering of A^k by nearly isometric balls B_i. Cut out some smaller (disjoint) balls $B'_i \subset B_i$ (i.e. $B'_i \cap B'_j = \emptyset$ for $i \neq j$) and replace them by suitably scaled $(K(k), g(k))$. Then one can distribute Scal all over this new manifold B^k and one gets a $\text{Scal} > 0$-metric on B^k while it can obviously be arranged that $\hat{A}([B^k]) \neq 0$.

However these two conditions are not compatible as the Lichnerowicz formula (cf. [LM], p.161) $D^*D = \nabla^*\nabla + \frac{1}{4}\text{Scal}$ (where D and ∇ denote the Dirac operator and the covariant derivative respectively) implies that $kerD = CokerD = 0$, while *index* $D = \hat{A}([B^k]) \neq 0$. Hence $E \geq 0$. □

Proof 2 (using Gromov-Lawson's Twisted Dirac bundles): The Riemannian manifold (M^n, g_1) obtained in (6.1) can be "imbedded" in a flat torus T^n.

More precisely, take a cube containing $B_{10}(0) \subset \mathbb{R}^n$ and identify opposite sides producing a flat torus. Now substitute $M \setminus (\mathbb{R}^n \setminus B_6(0))$ for $B_6(0) \subset T^n$ equipped with the natural metric. This is a manifold diffeomorphic to $\widehat{M} \# T^n$ (where $\widehat{M}$ is a one point compactification of M) equipped with some metric with $\mathrm{Scal} \geq 0$ (and $\mathrm{Scal} > 0$ in some points which implies that the whole manifold admits a $\mathrm{Scal} > 0$-curved metric).

However if $\widehat{M} \# T^n$ is covered by some spin manifold (this is certainly the case when M is spin) then we can use a generalized version of the Lichnerowicz formula by Gromov and Lawson to show that $\widehat{M} \# T^n$ cannot admit any $\mathrm{Scal} > 0$-metric. Once again this means $E \geq 0$. □

Remarks. 1. The classical three-dimensional case is *completely* covered by these arguments, as every *orientable* 3-manifold is already spin. Thus, for the general non-spin manifolds one just takes the orientation covering.

2. It turned out recently that the second proof can be shown to be "equivalent" to the positive energy theorem, i.e. the non-existence of $\mathrm{Scal} > 0$-metrics on enlargeable manifolds can be deduced from the validity of the positive energy theorem (cf. [L3]).

Finally, for completeness, we will discuss the case $E = 0$, whose derivation from the above is pretty standard (cf. [S]).

If $E(M, g) = 0, \mathrm{Scal}(g) \geq 0$, then (M, g) is isometric to $(\mathbb{R}^n, g_{\mathrm{Eucl}})$

This can be seen as follows:

If $\mathrm{Scal}(g) > 0$ in some point, then one can easily deform (M, g) to (M, g_1) with (M, g_1) still asymptotically flat with $\mathrm{Scal}(g_1) > 0$, but with $E(M, g_1) < 0$. This contradicts (6.1) and also proves that g cannot be deformed to such a metric with $\mathrm{Scal} > 0$ in some point. This in turn implies that g is Ricci flat. Now one again uses $E(M, g) = 0$ and applies some Bochner type argument to show the extension of n linear independent parallel vector fields on M, implying that M is isometric to the flat $\mathbb{R}^n$. □

8 Shortening geodesics

Here we will start our proof of the existence of scalar flat structures. Three types of deformations are used in the proof. We begin by proving that any closed geodesic (without selfintersection) can be shortened without essential (decreasing) effects on Scal. This is the place where dimension $n > 3$ is used substantially and it is therefore rather unclear whether almost scalar flat structures generally exist in dimension 3. Secondly, we show how to find a closed geodesic lying quite densely spread all over the manifold (after some changes of the given metric but again without negative influences on Scal). Finally we use the methods of Theorem 1 to adjust the upper bound of Scal.

Let $g_L = L^2 \cdot g_{S^1} + g_{\mathrm{Eucl.}}$ be the standard product metric on $S^1 \times S^{n-2} \times]0,1[\subset S^1 \times B_1(0) \subset S^1 \times \mathbb{R}^{n-1},\ n \geq 4$ for some (large) $L > 0$.

Proposition 8.1 *For each $\delta > 0$ there is a metric g_δ on $S^1 \times B_1(0)$ with $g_\delta \equiv g_L$ near the boundary,* $\mathrm{Scal}(g_\delta) \geq -\delta$ *and* $\mathrm{diam}(S^1 \times B_1(0)) \leq 3$.

First of all a direct computation leads to the following formula

$$\begin{aligned} &\mathrm{Scal}(f^2 \cdot g_{S^1} + g^2 \cdot g_{S^{n-2}} + g_{\mathbb{R}}) \\ &\quad = -\Bigg((n-2) \cdot \frac{g''}{g} + \frac{f''}{f} + (n-2) \cdot \frac{f'}{f} \cdot \frac{g'}{g} \\ &\quad\quad + \frac{(n-3)\cdot(n-2)}{2} \cdot \frac{(g')^2 - 1}{g^2} \Bigg) \end{aligned} \tag{5}$$

with $f(r), g(r)$ being smooth functions on $\mathbb{R}$.

We claim that the standard metric (corresponding to $f \equiv L,\ g(r) = r$) can be substituted for another (nearly) scalar flat metric, but with f being very small near 0.

Proof of (8.1). Step 1: Choose some smooth $f = f_{\varepsilon,\delta,L}$, for constants $\varepsilon, \delta > 0$ given, as follows (cf. Fig. 2)

$$f = \begin{cases} L & \text{on } \mathbb{R}^{\geq -\varepsilon} \\ \text{having } f'' < 0 & \text{on }]-2\varepsilon, -\varepsilon[\\ \text{linear} & \text{on } [-3\varepsilon, -2\varepsilon] \\ \text{having } f'' > 0 & \text{on }]-4\varepsilon, -3\varepsilon[\\ \delta & \text{on } \mathbb{R}^{\leq -4\varepsilon} \end{cases}$$

Step 2. Choose some smooth positive $g = g_{\varepsilon,\delta,L,\eta_0}$, for constants $\varepsilon, \delta, \mu, \eta_0 > 0$ given, with

$$g(r) = \begin{cases} r & \text{on } \mathbb{R}^{\geq 1} \\ \text{having } \mu > g'' \geq 0 & \text{on } [\frac{3}{4}, 1] \\ \text{linear} & \text{on } [0, \frac{3}{4}] \\ F & \text{on } [-\varepsilon, 0] \quad (F \text{ defined below}) \\ \eta & \text{on } [-4\varepsilon, -\varepsilon] \text{ for some } \eta > 0, \eta \leq \eta_0 \\ \text{having } g'' < 0 & \text{on }]-5\varepsilon, -4\varepsilon[\text{ and being equal to} \\ & \quad \sin(r + 5\varepsilon) \text{ near } -5\varepsilon \end{cases}$$

We construct a positive smooth F on $\mathbb{R}$ as follows (this is basically the same thing as in [GL], chap.1):

Start with $f_0 = \alpha_0 \cdot r + \beta_0$ for some $\alpha_0, \beta_0 > 0, \alpha_0 < 1$ on $[-\frac{\beta}{\alpha}, 1]$. This fulfills the inequality $f_0'' \cdot f_0 + f_0' \cdot f_0' < 1$. Next we bend f_0 "a little bit", i.e. we define $f_1 \equiv f_0$ on $[\frac{1}{2} + \frac{1}{8}, 1]$, linear on $\mathbb{R}^{\leq \frac{1}{2} - \frac{1}{8}}$ with

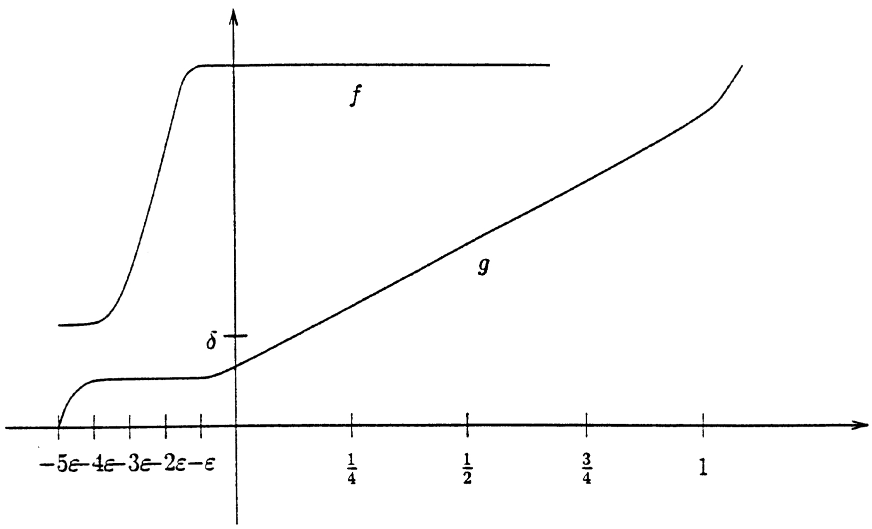

Fig. 2

$0 \leq f_1' = \alpha_1 < \alpha_0 = f_0', f_1'' \geq 0$ on $\mathbb{R}$ and such that $f_1'' \cdot f_1 + f_1' \cdot f_1' < 1$ is still satisfied. Now this "same" bending can be iterated until $f_n' = 0$. More precisely, after some rescaling we may repeat the previous step with f_0 replaced by f_1 etc.

The point is that $\alpha_{i+1} - \alpha_i$ can be chosen arbitrarily within $[0, \alpha_i - \alpha_{i-1}]$ as in each step $f_i' \cdot f_i'$ decreases, while $f_0 \cdot f_0''$ remains upper bounded (as can be seen by rescaling).

Thus after finitely many steps we have $F = f_n$ with $F = \alpha \cdot r + \beta$ on $\mathbb{R}^{\geq 1}, F, F', F'' \geq 0$,

$$F \equiv c > 0 \text{ on } \mathbb{R}^{\leq \gamma}, \text{for some } \gamma \in \mathbb{R}.$$

Now, after some obvious rescaling and translating F fits into the definitions of g.

Step 3. We claim we can choose δ_0 in such a way that $\mathrm{Scal}\,(f^2 \cdot g_{S^1} + g^2 \cdot g_{S^{n-2}} + g_{\mathbb{R}}) \geq 0$ on $S^1 \times S^{n-2} \times] - 5\varepsilon, \frac{1}{2}]$

From Step 2 and formula (*) we know that $\mathrm{Scal} \geq 0$ on $S^1 \times S^{n-2} \times] - \varepsilon, \frac{1}{2}] \cup [-5\varepsilon, -4\varepsilon]$. Also from step 1 we see that $\mathrm{Scal} \geq 0$ on $S^1 \times S^{n-2} \times [-3\varepsilon, -\varepsilon]$

The only domain we have to devote care to is $S^1 \times S^{n-2} \times [-4\varepsilon, -3\varepsilon]$. Here we have $\mathrm{Scal} = -\frac{f''}{f} + \frac{1}{\eta^2}$, which is obviously positive for some small $\eta_0 > 0$.

Now we can deduce (8.1): Choose $\delta = \frac{1}{8}, \varepsilon = \frac{1}{100}$. Then we can find some small $\mu, \eta_0 > 0$ such that f and g as constructed in steps 1-3 satisfy: $\mathrm{Scal} \geq -\delta$.

Also the distance of any two points in $S^1 \times B_1(0)$ (and hence the diameter) is < 3:

For any two points $p, q \in S^1 \times B_1(0), p = (p_1, p_2), q = (q_1, q_2)$ and join these points along the radius with $(p_1, 0)$ and $(q_1, 0)$ respectively and then join these two points along $S^1 \times \{0\}$. The length of this part is $\leq \frac{1}{8} + 2 \cdot \frac{5}{100} + 2 < 3$. $\square$

Actually, we will use the "hyperbolic version" of (8.1), which we are going to deduce now.

Corollary 8.2 *For any $\delta > 0$ we can find a metric g_δ on $S^1 \times B_1(0)$ satisfying $g_\delta = L^2 \cdot \cosh^2 r \cdot g_{S^1} + \sinh^2 r \cdot g_{S^{n-2}} + g_{\mathbb{R}}$ (i.e. g_δ is hyperbolic) near the boundary,*

$$\mathrm{Scal}(g_\delta) \geq -1 - \delta \textit{ and diam } (S^1 \times B_1(0)) \leq 3$$

Proof. We reduce this case to (8.1) as follows:

Substitute $\sinh r$ (defined on [0,1]) for f_ε on $[r_0, 1]$ for some $r_0 < 0$ and $\varepsilon > 0$ with

$$f_\varepsilon = \begin{cases} h \cdot \sinh(r + \varepsilon) + (1 - h) \cdot \sinh r \text{ on } \mathbb{R}^{\geq 0} \\ \text{having } \frac{f''_\varepsilon}{f_\varepsilon} \leq 1, f_\varepsilon \geq 1 \text{ and } f'_\varepsilon = 1 \text{ near } r_0 (\text{with } f_\varepsilon(r_0) = 0) \end{cases}$$

for some fixed $h \in C^\infty(\mathbb{R}, [0, 1])$, with $h = 1$ on $\mathbb{R}^{\leq \frac{1}{4}}, h \equiv 0$ on $\mathbb{R}^{\geq \frac{3}{4}}$.

Substitute $\cosh r$ on $[0, 1]$ for g_ε on $[r_0, 1]$ defined as follows:

$$g_\varepsilon = \begin{cases} L \cdot \cosh \qquad \text{on} \qquad \mathbb{R}^{\geq 0} \\ \text{satisfying } g'' \leq \cosh, \ g' \leq 0 \text{ and } g \equiv \text{const.} \ \geq L \text{ near } r_0. \end{cases}$$

It is easy to check that, for sufficiently small $\varepsilon > 0$,

$$\mathrm{Scal}\left(g_\varepsilon^2 \cdot g_{S^1} + f_\varepsilon^2 \cdot g_{S^{n-2}} + g_{\mathbb{R}}\right) \geq -1 - \frac{\delta}{2}$$

Moreover, near $S^1 \times S^{n-2} \times \{r_0\}$ the metric is precisely the Euclidean one as assumed in (8.1). Thus applying (8.1) for $\frac{\delta}{2}$ yields the claim. $\square$

Remark. The reader might be tempted to expect that some more care in choosing the warping function could even produce scalar flat curve shortenings. However there is no way of accomplishing this: just imbed the resulting tube in a (flat) torus . The metric will become scalar flat. But in the torus case we have already mentioned that any scalar flat metric has to be flat (cf. Sect. 7), while the tube containing the shortened geodesic cannot be flat.

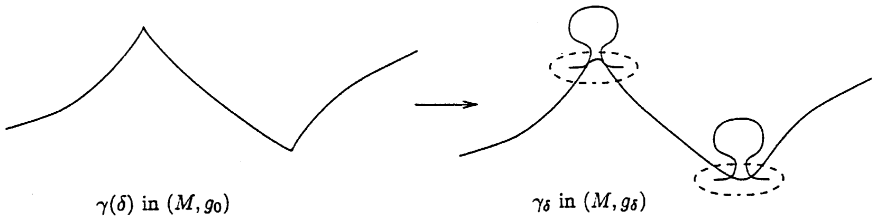

Fig. 3

9 Closed geodesics

We want to show the existence of closed geodesics lying arbitrarily "dense" in M after carrying out some well-controlled (stepwise) metrical deformations.

As $|\mathrm{Scal}|$ decreases under scaling by large constants, every closed manifold $M^n,\ n \geq 3$ admits a metric g_0 with $\mathrm{Scal}(g_0) \geq -1$. (Of course, M even admits a metric with $\mathrm{Scal} \equiv -1$ but we do not need it here.) We start with such a metric and prove:

Proposition 9.1 *For each $\delta > 0$ there is a metric g_δ on M and a closed geodesic γ_δ (without selfintersections) such that*

(i) $\mathrm{dist}(x, \gamma_\delta) \leq \delta$ *for each point* $x \in M$
(ii) $-1 - \delta \leq \mathrm{Scal}(g_\delta)$

Proof. First of all choose a piecewise smooth geodesic curve $\gamma(\delta)$ in (M, g_0) without selfintersections and with $\mathrm{dist}(x, \gamma(\delta)) \leq \frac{\delta}{2}$, for each $x \in M$. This can easily be done by taking some suitable $\frac{\delta}{2}$-dense set of points and joining pairs of points step by step by geodesic arcs.

Now we deform our metric g_0 near the edges of $\gamma(\delta)$ in such a way that the new metric g_δ will have $\mathrm{Scal} \geq -1 - \delta$ and $\gamma(\delta)$ can be substituted for an entirely smooth (closed and selfintersection-free) geodesic γ_δ.

First of all we use geodesic coordinates around each edge p and choose disjoint balls $B_r(p)$ around them (lying in the coordinate neighbourhood).

Now we can choose C^∞ cut-off functions h_p on $B_r(p)$ with $h_p \in C^\infty(M, [0,1]),\ h_p \equiv 1$ near $\partial B_r(p)$, $h_p \equiv 0$ near p such that $-1 - \mu < \mathrm{Scal}(h_p \cdot g_0 + (1 - h_p) \cdot g_{\mathrm{hyp.}})$ for any given $\mu > 0$. Here $g_{\mathrm{hyp.}}$ denotes the hyperbolic metric pushed forward from the tangent space at p by the exponential map. This is a simple consequence of the fact that both $1-$jets of g_0 and $g_{\mathrm{hyp.}}$ coincide in p, $\mathrm{Scal}(g_0) \geq \mathrm{Scal}(g_{\mathrm{hyp.}}) \equiv -1$ and Scal is computed from the 2-jet of the metric and the 2nd order term enters linearly. We call the new metric g_μ.

We may assume $\gamma(\delta)$ is still (piecewise) geodesic with respect to g_μ (otherwise one can easily redefine $\gamma(\delta)$).

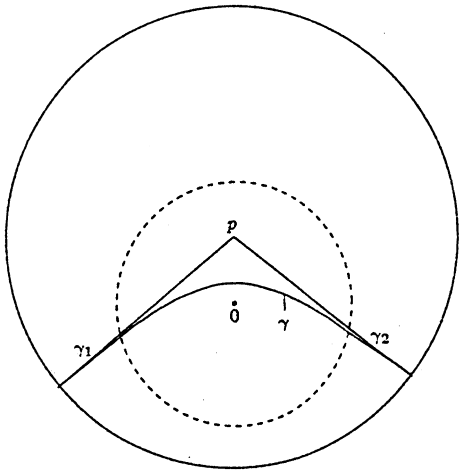

Fig. 4

Thus we have reduced the problem to the following

Lemma 9.2 *Let $B \subset \mathbb{H}^n$ be a ball around the origin p, $\gamma_1,\ \gamma_2$ two distinct geodesic arcs starting from p. Then we can find a metric g_δ on B with $g_\delta \equiv g_{\mathrm{hyp.}}$ near ∂B, a geodesic $\gamma_\delta \subset B$ without selfintersection with $\gamma \equiv \gamma_1$ and $\gamma \equiv \gamma_2$ respectively near ∂B and such that $-1-\delta \leq \mathrm{Scal}(g_\delta)$, for $\delta > 0$ given.*

Proof. We consider B as a small hyperbolic ball $B \equiv B_r(p) \subset \mathbb{H}^n$. In polar coordinates hyperbolic metrics may be described as $[0, r[\times S^{n-1}$ equipped with $g_{\mathbb{R}} + \sinh^2 r \cdot g_{S^{n-1}}$, with the usual identification in 0.

Now we choose a distinguished point “0” outside p and consider a small ball $B_{r''}(0) \subset B_r(p)$, $r' \ll r$. Next, we delete 0 and deform the remaining part of $B_{r'}(0)$. We replace $\sinh$ with some smooth $f > 0$ with $f \equiv$ *sinh* near r' and extend f to the negative axis (as in the proof of (8.1)) in such a way that for some $r'' < 0, |r''| \ll 1,'' \partial B_{r''}(0)''$ becomes totally geodesic and such that for small given $\delta > 0 : -1-\delta < \mathrm{Scal}(g_{\mathbb{R}} + f^2 \cdot g_{S^{n-1}})$ on $B_{r'}(0) \setminus B_{r''}(0)$ and $\mathrm{Scal} \geq 0$ near $\partial B_{r''}$.

All this can be done quite easily using equation (5) as in Sect. 8 and is left to the reader. The positive curvature of S^{n-1} plays the role of the antagonist of the first and second derivative of f.

Next we start with the standard spherical metric on $S^n, n \geq 3$ and deform this metric on a ball $B \subset S^n$ in such a way that we get (after cutting out a small ball $\subset B$) a totally geodesic S^{n-1} boundary sphere (looking the same as in the hyperbolic case), without changes outside B and such that Scal ≥ 0 in the transition domain. Once again this can be done as in (8.1).

Now we may form the "connected sum" $B_r(0) \# S^n = B_r(0)$ to get our metric on $B_r(0)$: we may adjust these deformations to be able to glue these pieces along their totally geodesic boundary spheres reconstructing $B_r(0)$ with a smooth metric with $-1-\delta < \mathrm{Scal}(g)$. Note that the sphere can be chosen arbitrarily small and moreover the construction (from (3.1)) to glue $B_r(0)$ and S^n leads to arbitrarily short "necks". In other words the increase of distances can be made as small as ever needed.

Now the main point is that this configuration containing a totally geodesic sphere serves to straighten our geodesic:

We perform the procedure in a point near but outside p and place the center "0" in the plane spanned by γ_1 and γ_2 (cf. Fig. 4). As our construction has led to a totally geodesic slicing sphere one can easily check that the germs of γ_1 and γ_2 near $\partial B_r(0)$ can be "joined" by a smooth geodesic γ given some suitable choice of the centre. As already mentioned the auxiliary sphere attached can be assumed being arbitrarily small. Hence we can be sure that $\mathrm{dist}(x, \gamma(\delta)) < 2\delta$ for each $x \in M$. $\square$

10 Small diameters

The results of the previous section allow us to start with a metric g_δ on M with diam$(M, g_\delta) = D > 0$ and $-1-\delta < \mathrm{Scal}(g_\delta)$ for any given $\delta > 0$ such that there is a closed geodesic γ_δ without selfintersections with dist $(x, \gamma_\delta) \leq \delta$ for any point $x \in M$ and D independent of δ. The idea (originating from [BG]) is to use this geodesic after shortening it as a general shortcut for joining any two points in M.

We can find some small μ with $\delta/2 > \mu > 0$ such that $U_\mu(\gamma) := \{x \in M | \mathrm{dist}(x, \gamma_\delta) \leq \mu\}$ is a tube (parametrized by Fermi-coordinates) and we want to apply the curve shortening results (8.2) to such a tube. We first deform the metric near γ_δ. Once again we paste two metrics together: g_δ on $U_\mu(\gamma)$ and $g_{\mathrm{hyp.}}$ on $S^1 \times B_\mu(0) \subset S^1 \times \mathbb{H}^{n-1}$, with *length*$S^1 = L$ measured along γ and $S^1 \times \{0\}$ being equal (under Fermi-coordinates) and $\mathrm{Scal}(g_\delta) \geq -1-\delta$. Also as both curves are smooth geodesics we observe that the metrics g_δ and $g_{\mathrm{hyp.}}$ have the same 1-jet when restricted to the curves. (The construction of g_δ is easily accomplished in such a way that the holonomy of γ_δ is trivial.) Once again, this ensures the existence of some cut-off function $h \in C^\infty(M, [0,1])$ such that $h \equiv 0$ near $\gamma_\delta, h \equiv 1$ near $M \setminus U_\mu(\gamma_\delta)$ and

$$1 - 2\delta < \mathrm{Scal}(h \cdot g_\delta + (1-h) \cdot g_{\mathrm{hyp.}})$$

This time $g_{\mathrm{hyp.}}$ denotes the push forward metric via Fermi-coordinates of a hyperbolic tube of the same length L (i.e. $S^1 \times B_\mu(0)$ equipped with $L^2 \cdot \cosh^2 r \cdot g_{S^1} + g_{\mathbb{R}} + \sinh^2 r \cdot g_{S^{n-2}}$)

Applying (8.2) we can get a metric g with $g \equiv g_\delta$ on $M \setminus U_\mu(\gamma_\delta)$ and $-1-3\delta < \mathrm{Scal}(g)$, such that $\mathrm{diam}_g U_\mu(\gamma_\delta) \leq 2\mu$ with respect to g_δ. Now apply (the construction used for) Theorem 1 to some $f \equiv -1-3\delta$ and $U = M$ for $\varepsilon \ll 1$: resuming that argument we first notice that the ratio θ/δ can be made arbitrarily small. Moreover the deformations near the "neck" needed in order to glue the spheres to the manifold contribute w.l.o.g. at most a total factor $\frac{11}{10}$ to the diameter. Thus it remains to explain why the sphere attached to $\mathbb{R}^n$ in Sect. 3 can be chosen to have arbitrarily small diameter. Namely one starts with a scalar flat metric g_0 on S^n such that there are arbitrarily close (in C^∞-topology) metrics $g_\pm$ with $\mathrm{Scal} \equiv \pm c$, for any $c \in]-\alpha, \alpha[$ for some small $\alpha > 0$ and with diameter $\equiv D > 0$. Now one considers the scaled metric $\theta^{-2} \cdot g$ and notices that for $\theta \ll 1$, $|\mathrm{Scal}| \ll 1$. Thus we may carry out the glueing process uniformly increasing the scaled diameter by $\frac{1}{10}\mathrm{diam} + 2D$ and therefore by increasing diam(M) by at most $\frac{1}{10}\mathrm{diam}(M) + \delta$. Moreover $\mu < \delta$ and for each $x \in M$ we have $\mathrm{dist}_g(x, \gamma_\delta) < \mu + \delta < 2\delta$ thus $\mathrm{diam}(M) < 6 \cdot \delta$ and, since $-1-3\delta < \mathrm{Scal}(g) < -1+3\delta$, Theorem 3 is proved. □

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