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Algebraic Geometry
Exercise Sheet 10

To be hand in on 9.1.2019

Exercise 1: Let $f : X \to Y$ be a morphism of schemes. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module and $\mathcal{E}$ be an $\mathcal{O}_Y$-module. Assume that $\mathcal{E}$ is locally free, i.e. there exist an open covering Y_i of Y such that $\mathcal{E}|_{Y_i} \cong \mathcal{O}_{Y_i}^{\oplus n_i}$ for each i. Show that there exists a natural isomorphism:

$$(f_*\mathcal{F}) \otimes_{\mathcal{O}_Y} \mathcal{E} \cong f_*(\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{E}).$$

(*Hint: Use the adjointness of f_* and f^* to construct a canonical and functorial map. Then localize on Y to show that this map is an isomorphism.*)

Exercise 2:

Let $f : X \to Y$ be a morphism of schemes.

(i) Assume

$$\begin{aligned} &f \text{ is quasi–compact and for any } W \subseteq Y \text{ quasi–compact open} \\ &U, V \subseteq f^{-1}(W) \text{ quasi–compact open} \Rightarrow U \cap V \text{quasi–compact open} \end{aligned} \tag{1}$$

Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Show that $f_*\mathcal{F}$ is a quasi-coherent $\mathcal{O}_Y$-module.

(*Hint: Reduce to the case $Y = \operatorname{Spec} B$ and consider a finite affine cover $X = \bigcup U_i$.*)

(ii) Assume X is noetherian (as a topological space). Show that f satisfies the property (1).

Exercise 3:

Let k be an (algebraically closed) field and fix a k-rational point $\infty \in \mathbb{P}^1_k = X$. Define sheaves of $\mathcal{O}_X$-modules $\mathcal{O}(1)$ and $\mathcal{O}(-1)$ by

$$\mathcal{O}(-1) : U \mapsto \{f \in \Gamma(U, \mathcal{O}_X) | f(\infty) = 0\}$$

and $\mathcal{O}(1) := \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}(-1), \mathcal{O}_X)$. Further for an integer $n \geq 0$ we define $\mathcal{O}(n) := \mathcal{O}(1)^{\otimes n}$ and $\mathcal{O}(-n) := \mathcal{O}(-1)^{\otimes n}$. Let $n \in \mathbb{Z}$.

(i) Show that $\mathcal{O}(n)$ is a quasi-coherent $\mathcal{O}_X$-module.

(ii) Show that for $m, n \in \mathbb{Z}$ one has $\mathcal{O}(m) \otimes_{\mathcal{O}_X} \mathcal{O}(n) \cong \mathcal{O}(m+n)$.

(iii) Show that $\Gamma(X, \mathcal{O}(n))$ is a finite dimensional k-vector space of dimension

$$\dim_k \Gamma(X, \mathcal{O}(n)) = \begin{cases} 0 & n < 0 \\ n+1 & n \geq 0 \end{cases}$$

(iv) Show that the functor $\Gamma(X, -)$ on the category of quasi-coherent $\mathcal{O}_X$-module does not commute with tensor products, i.e. find quasi-coherent $\mathcal{O}_X$-modules $\mathcal{F}$ and $\mathcal{G}$ such that $\Gamma(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}) \ncong \Gamma(X, \mathcal{F}) \otimes_{\Gamma(X, \mathcal{O}_X)} \Gamma(X, \mathcal{G})$.

Exercise 4: Show that the following schemes are not affine by either giving an example of a quasi-coherent sheaf $\mathcal{F}$ that is not generated by its global sections (i.e. the canonical morphism $\Gamma(X, \mathcal{F}) \otimes_{\Gamma(X, \mathcal{O}_X)} \mathcal{O}_X \to \mathcal{F}$ is not surjective) or giving an example of a short exact sequence of quasi-coherent sheaves that is not exact on global sections.

(i) $\mathbb{A}^n_k \backslash \{0\}$ for $n \geq 2$ and a field k.

(ii) $\mathbb{P}^1_k$ for a field k.