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Algebriac Geometry
Exercise Sheet 1

Exercise 1:

Let R be a commutative ring and M be a R-module. For a mutiplicative subset $S \subset R$, there is a natural partial order on S: for any $f_1, f_2 \in S$, set $f_1 \leq f_2$ if $f_1|f_2$ in R.

(i) Show that for any $f_1|f_2$, there are natural homomorphisms $M_{f_1} \to M_{f_2}$, that make $\{M_f\}_{f\in S}$ into a direct system.

(ii) Show that $\varinjlim_{f\in S} M_f = S^{-1}M$.

Exercise 2:

(i) Consider the partial order on $\mathbb{N}$ given by divisibility (i.e. for any m and $n \in \mathbb{N}$, $m \leq n$ if $m|n$). Then for any $m|n$, the natural map $\mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$ make $\{\mathbb{Z}/n\mathbb{Z}\}_{n\in\mathbb{Z}}$ into a inverse system, show that

$$(\hat{\mathbb{Z}} :=) \varprojlim_{n\in\mathbb{N}} \mathbb{Z}/n\mathbb{Z} = \prod_{p \text{ prime}} \mathbb{Z}_p$$

(ii) Consider the usual order "$\leq$" on $\mathbb{N}$. Then for any $n \in \mathbb{N}$, the natural maps between abelian groups $\mathbb{Z}/p^n\mathbb{Z} \xrightarrow{\times p} \mathbb{Z}/p^{n+1}\mathbb{Z}$ make $\{\mathbb{Z}/p^n\mathbb{Z}\}_{n\in\mathbb{N}}$ into a direct system. Show that

$$\varinjlim_{n\in\mathbb{N}} \mathbb{Z}/p^n\mathbb{Z} = \mathbb{Q}_p/\mathbb{Z}_p$$

(iii) Recall that a sequence $\{a_n\}_{n\in\mathbb{N}}$ in $\mathbb{Z}_p$ converges to 0 if for any $n \in \mathbb{N}$, the set $\{k \in \mathbb{N} | a_k \notin p^n\mathbb{Z}_p\}$ is finite. Consider the usual order of $\mathbb{N}$ and for any $n \in \mathbb{N}$, the natural maps $\mathbb{Z}[T]/p^{n+1}\mathbb{Z}[T] \to \mathbb{Z}[T]/p^n\mathbb{Z}[T]$. This makes $\{\mathbb{Z}[T]/p^n\mathbb{Z}[T]\}_{n\in\mathbb{N}}$ into a inverse system. Show that $\mathbb{Z}_p\langle T\rangle = \varprojlim_{n\in\mathbb{N}} \mathbb{Z}[T]/p^n\mathbb{Z}[T]$, here

$$\mathbb{Z}_p\langle T\rangle := \left\{ \sum_{k\in\mathbb{N}} a_k T^k \in \mathbb{Z}_p[\![T]\!] | a_k \to 0 \text{ when } k \to \infty \right\}$$

Exercise 3:

(i) Let $\mathcal{C}$ be the category of the category of R-modules for a ring R or commutative rings. Construct a right adjoint functor of the forgetful functor $i : \mathcal{C} \to \text{Sets}$.

(ii) Let $S \to R$ be a homomorphsim between commutative rings. Construct a left adjoint functor of the forgetful functor $i : \text{Mod}(R) \to \text{Mod}(S)$.

(iii) Construct a left adjoint functor of the forgetful functor $i : \text{Ab} \to \text{Grp}$ from the category of abelian groups to the category of groups.

Exercise 4:

Let $\mathcal{C}$ and $\mathcal{D}$ be categories. Assume that $\mathcal{C}$ admits limits. Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be a pair of adjoint functors such that F is right ajoint functor of G. It means that for any $M \in \mathcal{C}$ and $N \in \mathcal{D}$, there is a natural bijection:

$$\text{Hom}_{\mathcal{D}}(N, F(M)) \cong \text{Hom}_{\mathcal{C}}(G(N), M)$$

Show that F commutes with limits.

Remark: Symmetrically, if $\mathcal{D}$ admits colimits, then the functor G commutes with colimits.