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WS 2018/2019

Algebraic Geometry
Exercise Sheet 9

To be hand in on 20.12.2018

Exercise 1:

(i) Let X be a scheme of finite type over $\operatorname{Spec} \mathbb{C}$ and let $Y := \operatorname{Res}_{\mathbb{C}/\mathbb{R}}(X)$. Show that there is an isomorphism of $\mathbb{C}$-schemes: $Y \times_{\operatorname{Spec} \mathbb{R}} \operatorname{Spec} \mathbb{C} \cong X \times_{\operatorname{Spec} \mathbb{C}} X$.

(ii) Let U_n be the unitary group in n-variables over $\mathbb{R}$. Show that there is an isomorphism of $\mathbb{C}$-schemes: $\mathrm{U}_n \times_{\operatorname{Spec} \mathbb{R}} \operatorname{Spec} \mathbb{C} \cong \mathrm{GL}_{n,\mathbb{C}}$.
(*Hint: Consider the isomorphism :* $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C} \times \mathbb{C}$, $z_1 \otimes z_2 \mapsto (z_1 z_2, z_1 \overline{z_2})$)

Exercise 2:

Let X be a topological space and $\{X_i\}_{i \in I}$ be a family of subspaces of X such that

$$X = \bigcup_{i \in I} X_i$$

(i) Show that $\dim X_i \leq \dim X$ for any $i \in I$.

(ii) If each X_i is open subspace of X, show that $\dim X = \sup_{i \in I} \{\dim X_i\}$

(iii) If each X_i is closed subspace of X and I is finite, show that $\dim X = \max_{i \in I} \{\dim X_i\}$

Exercise 3:

Let k be a field.

(i) Let $X = \operatorname{Spec} A$ for a k-algebra A of finite type that is a domain. Let $0 \neq f \in A$. Show that the closed subscheme $\operatorname{Spec} A/(f)$ of X has dimension $\dim X - 1$.
(*Hint: Krull's principle ideal theorem*)

(ii) Consider the homogeneous ideal $I := (y^2 - xz, z^2 - yw, xw - yz)$ in $k[x, y, z, w]$.

(a) Show that $V_+(I)$ is isomorphic to $\mathbb{P}^1_{\mathbb{Z}}$ which induces a closed embedding:

$$\mathbb{P}^1_{\mathbb{Z}} \hookrightarrow \mathbb{P}^3_{\mathbb{Z}}.$$

(b) Show that I can not be generated by 2 elements.

Exercise 4:

Let X be a scheme and let $Z \subseteq X$ be a closed subset of X .

(i) Show that the following presheaf:

$$\mathcal{I}_Z(U) := \{f \in \mathcal{O}_X(U) | \forall x \in Z \cap U, f(x) = 0 \in k(x)\}$$

is a sheaf of ideals.

(ii) Consider the natural $\mathcal{O}_X$-algebra structure of $\mathcal{O}_X/\mathcal{I}_Z$. Show that $(Z, i^{-1}(\mathcal{O}_X/\mathcal{I}_Z)$ is a reduced scheme, here i is the natural map: $i : Z \to X$.

(iii) Show that the composition of natural morphisms of sheaves

$$\mathcal{O}_X \to \mathcal{O}_X/\mathcal{I}_Z \to i_* i^{-1}(\mathcal{O}_X/\mathcal{I}_Z)$$

induces a closed embedding $(Z, i^{-1}(\mathcal{O}_X/\mathcal{I}_Z)) \to (X, \mathcal{O}_X)$.

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