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Algebraic Geometry
Exercise Sheet 7

To be hand in on 06.12.2018

Exercise 1:

Let k be a field with $\mathrm{char}(k) \neq 2$ and let

$$i : C := V_+(X^2 + Y^2 - Z^2) \to \mathbb{P}^2_k \backslash V_+(Y, Z) \subseteq \mathbb{P}^2_k$$

be the natural closed immersion.

(i) Consider the affine covering $\mathbb{P}^2_k \backslash V_+(Y, Z) = X_1 \cup X_{-1}$, where $X_i = \mathbb{P}^2_k \backslash V_+(Y + iZ)$. Show that $X_i \cap C$ is isomorphic to $\mathbb{A}^1_k$ for $i \in \{1, -1\}$.

(ii) Show that C is isomorphic to $\mathbb{P}^1_k$.

(iii) Let $k = \mathbb{Q}$, show that $C' := V_+(X^2 + Y^2 + Z^2 = 0)$ is not isomorphic to $\mathbb{P}^1_{\mathbb{Q}}$ but $C' \times_{\operatorname{Spec} \mathbb{Q}} \operatorname{Spec} \mathbb{Q}(i)$ is isomorphic to $\mathbb{P}^1_{\mathbb{Q}(i)}$.

Exercise 2:

Let k be a field and n be a positive integer. For any $I \in \{1, \dots, n\}$, let

$$X_I := \operatorname{Spec} k[X_1, \dots, X_n] \backslash V(X_i)_{i \in I}.$$

Show that X_I is not affine if $|I| \geq 2$.
(*Hint: Compute the global section of* X_I)
(*Remark: This means that the natural open immersion* $X_I \to \mathbb{A}^n_k$ *is not affine.*)

Exercise 3:

Let k be a field.

(i) Let

$$X := \operatorname{Spec} k[X_1, X_2, \dots] \backslash (X_i)_{i \in \mathbb{N}}.$$

Show that X is not quasi-compact, which implies $\operatorname{Spec} k[X_1, X_2, \dots]$ is not noetherian.

(ii) Let $(a_i)_{i \in \mathbb{N}}$ be a family of pairwise distinct elements of k. Set

$$A = k[U, T_1, T_2, \dots]/((U - a_i)T_{i+1} - T_i, T_i^2).$$

Describe $\operatorname{Spec} A$ as a topological space and show that $A_{\mathfrak{p}}$ is noetherian for every prime ideal $\mathfrak{p} \in \operatorname{Spec} A$. Show that A is not noetherian.
(*Hint: Show that the nilradical of* A *is not finitely generated.*)

Exercise 4:

Let $f : Y \to X$ be a morphism of schemes.

(i) Show that X is quasi-compact if and only if X be can be covered by finitely many affine open subschemes.

(ii) Show that f is quasi-compact if and only if there is an open affine covering U_i of X such that $f^{-1}(U_i)$ is quasi-compact.

(iii) Assume f is quasi-compact. Let $g : X' \to X$ the a morphism of schemes. Show that the natural morphism $X' \times_X Y \to X'$ is quasi-compact.

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