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# Algebraic Geometry
# Exercise Sheet 6

To be hand in on 29.11.2018

## Exercise 1:

A scheme $X$ is called reduced if for any open subset $U \subset X$, the ring $\mathcal{O}_X(U)$ is reduced (i.e. $\mathcal{O}_X(U)$ has no nilpotent elements).

(i) Show that $X$ is reduced if an only if for every $x \in X$, the local ring $\mathcal{O}_{X,x}$ is reduced.

(ii) Let $Z$ be a closed subset of $X$, show that there exist an unique closed subscheme $(Z, \mathcal{O}_Z)$ on $Z$ such that $(Z, \mathcal{O}_Z)$ is reduced.

## Exercise 2:

Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of schemes. Assume $i : (U, \mathcal{O}_U) \to (Y, \mathcal{O}_Y)$ is an immersion and $f(X) \subseteq U$. Consider about the following assertion: there exists an unique morphism $g : (X, \mathcal{O}_X) \to (U, \mathcal{O}_U)$ such that $f = i \circ g$.

(i) If $i$ is an open immersion, show that the assertion holds.

(ii) If $i$ is a closed immersion, find a conterexample for the assertion.

(iii) If $i$ is a closed immersion and $(X, \mathcal{O}_X)$ is reduced, show that the assertion holds.

## Exercise 3:

Let $k$ be a field, and denote $X_l := \mathbb{A}^l_k \backslash \{0\}$ for any $l$. Let $m, n \in \mathbb{N}$. Consider the following $k$-algebra homomorphism:

$$f : k[T_{i,j}]_{\substack{0 \leq i \leq m \\ 0 \leq j \leq n}} \to k[X_i, Y_j]_{\substack{0 \leq i \leq m \\ 0 \leq j \leq n}}$$
$$T_{i,j} \mapsto X_i Y_j.$$

(1) Show that $f$ induces a morphism:

$$g : X_{m+1} \times X_{n+1} \to X_{mn+m+n+1}.$$

(2) Denote $p_l : X_{l+1} \to \mathbb{P}^l_k$ as the natural projection. Show that there exists an unique morphism:

$$\tilde{g} : \mathbb{P}^m_k \times \mathbb{P}^n_k \to \mathbb{P}^{mn+m+n}_k$$

such that $\tilde{g} \circ (p_m \times p_n) = p_{mn+m+n} \circ g$.

(3) Show that $\tilde{g}$ is a closed immersion.

## Exercise 4:

Let $k$ be a algebraically closed field and let $A := k[x, y, u, v]/(xy + ux^2 + vy^2)$. Let $X = \operatorname{Spec} A$ be the corresponding affine scheme. Let $I := (x, y)$ and $U := X \backslash V(I)$.

(i) Show that $(U, \mathcal{O}_X|_U)$ is affine.
(*Hint: Show that* $fy + gx = 1$ *for* $f := \frac{-v}{x}$ *and* $g := \frac{-u}{y}$, *and use the affine criterion.*)

(ii) *Bonus*: show that there does not exist an element $f \in A$ such that $(U, O_X|_U) = D(f)$.
(*Hint: Assume such* $f$ *exists, consider the natural projection* $p : X \to \operatorname{Spec} k[x, y]$. *1.Show that for a closed point* $z = (x - a, y - b) \neq (x, y)$ *the fiber* $p^{-1}(z)$ *is isomorphic to* $\mathbb{A}^1_k$. *2.Show that for* $z \neq (x, y)$, *the restriction of* $f$ *to this fiber is constant. 3.Show that this constant glue to give a function on* $\operatorname{Spec} k[x, y] \backslash \{(0, 0)\}$. *4.Deduce that* $f$ *is a constant and nonzero function on* $X \backslash V(I)$.)

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