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Algebraic Geometry
Exercise Sheet 4

To be hand in on 15.11.2018

Exercise 1:

Let k be an algebraically closed field,

(i) describe the following schemes:
   (a) $\operatorname{Spec} k[x, y]$,
   (b) $\operatorname{Spec} k[x, y]/(xy)$.

(ii) describe the scheme $\operatorname{Spec} \mathbb{Z}[i]$ and the fibers of the morphism :

$$\operatorname{Spec} \mathbb{Z}[i] \to \operatorname{Spec} \mathbb{Z}.$$

Exercise 2:

Let A be a commutative ring, show that

(i) $V(\mathfrak{p}) = \overline{\{\mathfrak{p}\}}$ for any prime ideal $\mathfrak{p}$,

(ii) $\mathfrak{p} \mapsto V(\mathfrak{p})$ defines a bijection between prime ideals and irreducible closed subsets,

(iii) each irreducible closed subset has an unique generic point.

Exercise 3:

Let A and B be commutative rings.

(i) Construct a natural map:

$$\alpha : \operatorname{Hom}_{\mathrm{Sch}}(\operatorname{Spec} A, \operatorname{Spec} B) \to \operatorname{Hom}_{\mathrm{Rings}}(B, A).$$

(ii) Show that α is bijective.

Exercise 4:

Let X be a scheme. Assume X has a finite cover by open affine subschemes U_i such that each intersection $U_i \cap U_j$ is quasi-compact. Let $f \in \mathcal{O}_X(X)$ and define

$$X_f := \{x \in X | f(x) = 0 \text{ in } k(x)\},$$

here $k(x)$ is the residue field of $\mathcal{O}_{X,x}$.

(i) Let $U = \operatorname{Spec} B$ be an open affine subscheme of X, and denotes $f_B \in B = \mathcal{O}_X(U)$ the restriction of f. Show that $U \cap X_f = D(f_B)$, which implies that X_f is open.

(ii) Show that for any $a \in \ker(\mathcal{O}_X(X) \to \mathcal{O}_X(X_f))$, there exists some $n > 0$, such that $f^n a = 0$.

(iii) Show that for any $b \in \mathcal{O}_X(X_f))$, there exists some $n > 0$, such that $f^n b$ is the restriction of an element of $\mathcal{O}_X(X)$.

(iv) Show that $\mathcal{O}_X(X_f) \cong \mathcal{O}_X(X)_f$.

Homepage: http://www.uni-muenster.de/Arithm/hellmann/veranstaltungen