Prof. Dr. E. Hellmann
Shanxiao Huang WS 2018/2019

Algebraic Geometry
Exercise Sheet 3

To be hand in on 08.11.2018

Exercise 1:

A sheaf $\mathcal{F}$ on a topological space X is *flasque* if for every inclusion $V \subset U$ of open sets, the restriction map $\mathcal{F}(U) \to \mathcal{F}(V)$ is surjective.

(i) Show that a constant sheaf on an irreducible topological space is flasque.

(ii) Let $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ be an exact sequence of sheaves with $\mathcal{F}'$ flasque. Show that for any open set U, the sequence $0 \to \mathcal{F}'(U) \to \mathcal{F}(U) \to \mathcal{F}''(U) \to 0$ is also exact.

(iii) Let $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ be an exact sequence of sheaves with $\mathcal{F}$ and $\mathcal{F}'$ flasque. Show that $\mathcal{F}''$ is flasque.

Exercise 2:

Let $\{*\}$ be a one-point topological space, and x be a closed point in X. Let A be an abelian group.

(i) Consider the map $i : \{*\} \to X$, $* \mapsto x$. Show that $i_*(\underline{A}_{\{*\}}) = Sky^{(A,x)}$.

(ii) Let $\mathcal{F}$ be a sheaf on X and let i be as in (i). Show that $i^{-1}(\mathcal{F}) = \mathcal{F}_x$.

(iii) Consider the unique map $f : X \to \{*\}$. Show that $f^{-1}(\underline{A}_{\{*\}}) = \underline{A}_X$.

Exercise 3:

Let $f : X \to Y$ be a continuous map of topological spaces.

(i) Show that for any sheaf $\mathcal{F}$ on X, there is natural map: $f^{-1}f_*(\mathcal{F}) \to \mathcal{F}$.

(ii) Show that for any sheaf $\mathcal{G}$ on Y, there is natural map: $\mathcal{G} \to f_*f^{-1}\mathcal{G}$.

(iii) Use the maps in (i) and (ii) to show that there is a natural bijection of sets, for any sheaves $\mathcal{F}$ on X and $\mathcal{G}$ on Y,

$$\operatorname{Hom}_X(f^{-1}\mathcal{G}, \mathcal{F}) = \operatorname{Hom}_Y(\mathcal{G}, f_*(\mathcal{F})).$$

Hence we say that (f^{-1}, f_*) is a pair of adjoint functor.

Exercise 4:

Let $\mathfrak{U} = \{U_i\}$ be an open covering of X, and suppose we are given for each i a sheaf $\mathcal{F}_i$ on U_i, and for each i, j, an isomorphism $\varphi_{i,j} : \mathcal{F}_i|_{U_{i,j}} \xrightarrow{\sim} \mathcal{F}_j|_{U_{i,j}}$ (here $U_{i,j} := U_i \cap U_j$), such that

(i) $\varphi_{i,i} = \mathrm{id}_{\mathcal{F}_i}$ for each i,

(ii) $\varphi_{i,k} = \varphi_{j,k} \circ \varphi_{i,j}$ on $U_i \cap U_j \cap U_k$ for each i, j, k.

Show that there exists a sheaf $\mathcal{F}$ on X, together with isomorphisms $\psi_i : \mathcal{F}|_{U_i} \xrightarrow{\sim} \mathcal{F}_i$ such that for each i, j, $\psi_j = \varphi_{i,j} \circ \psi_i$ on $U_{i,j}$. Moreover, the tuple $(\mathcal{F}, \{\psi_i\})$ is unique up to unique isomorphism.

Exercise 5:

Let $i : Z \to X$ be a closed embedding of topological spaces, and let $j : U(:= X \backslash Z) \to X$ be the inclusion of the open complement.

(i) Let $\mathcal{F}$ be a sheaf of abelian groups on Z. Show that for any $x \in X$

$$(i_*\mathcal{F})_x = \begin{cases} \mathcal{F}_x, & \textit{if } x \in Z \\ 0, & \textit{if } x \notin Z \end{cases}.$$

(ii) Let $\mathcal{F}$ be a sheaf of abelian groups on U. Show that there is an unique sheaf $j_!(\mathcal{F})$ on X such that $j_!(\mathcal{F})|_U = \mathcal{F}$ and for any $x \notin U$, $j_!(\mathcal{F})_x = 0$.

(iii) Let $\mathcal{F}$ be a sheaf of abelian groups on X. Show that there is an exact sequence of sheaves on X,

$$0 \to j_!(\mathcal{F}|_U) \to \mathcal{F} \to i_*(i^{-1}(\mathcal{F})) \to 0.$$

Homepage: http://www.uni-muenster.de/Arithm/hellmann/veranstaltungen