Homogenisation of variational problems: an overview

Caterina Zeppieri

Applied Mathematics - University of Münster

Oberseminar Differentialgeometrie - July 5, 2021

Starting point: scale-dependent mechanical system

Multi-scale variational modeling of materials

Starting point: scale-dependent mechanical system

Aim: study the equilibrium configurations of the system

Multi-scale variational modeling of materials

Starting point: scale-dependent mechanical system

Aim: study the equilibrium configurations of the system

Approach: identify the equilibrium configurations of the system with the ground states of an associated energy

Multi-scale variational modeling of materials

Starting point: scale-dependent mechanical system

Aim: study the equilibrium configurations of the system

Approach: identify the equilibrium configurations of the system with the ground states of an associated energy

$$\rightsquigarrow \quad \min\{E_\varepsilon(u) \colon u \in X\}$$

with

- $E_\varepsilon : X \to \overline{\mathbb{R}}$ scale-dependent energy
- $\varepsilon > 0$ and “small” is a microscopic/mesoscopic scale (of geometrical, constitutive, or physical nature)

Micro to macro

Idea: "Let $\varepsilon \to 0$" to replace a complex, scale-dependent problem

$$\min\{E_\varepsilon(u)\colon u \in X\}$$

with a (simpler) scale-free problem

$$\min\{E_0(u)\colon u \in X\}$$

which captures the relevant features of the original problem

Idea: "Let $\varepsilon \to 0$" to replace a complex, scale-dependent problem

$$\min\{E_\varepsilon(u) \colon u \in X\}$$

with a (simpler) scale-free problem

$$\min\{E_0(u) \colon u \in X\}$$

which captures the relevant features of the original problem

- If v_ε minimises E_ε, then $v_\varepsilon \to v_0$ with v_0 minimiser of E_0;
- $E_\varepsilon(v_\varepsilon) \to E_0(v_0)$.

Micro to macro

Idea: “Let $\varepsilon \to 0$” to replace a complex, scale-dependent problem

$$\min\{E_\varepsilon(u) \colon u \in X\}$$

with a (simpler) scale-free problem

$$\min\{E_0(u) \colon u \in X\}$$

which captures the relevant features of the original problem

- If v_ε minimises E_ε, then $v_\varepsilon \to v_0$ with v_0 minimiser of E_0;
- $E_\varepsilon(v_\varepsilon) \to E_0(v_0)$.

The pointwise limit of E_ε (if it exists) in general *does not* fulfil these requirements

Prototipical example: homogenisation in $1D$

$$E_{\varepsilon}(u) = \int_0^L a\left(\frac{x}{\varepsilon}\right)(u')^2 dx - 2\int_0^L g\, u\, dx, \quad u \in W_0^{1,2}(0,L)$$

with

- $a \in L^{\infty}(\mathbb{R})$, 1-periodic, $0 < \alpha \leq a(x) \leq \beta$ a.e. in $\mathbb{R}$
- $g \in L^2(0,L)$

Prototipical example: homogenisation in $1D$

$$E_\varepsilon(u) = \int_0^L a\left(\frac{x}{\varepsilon}\right)(u')^2 dx - 2\int_0^L g\,u\,dx, \quad u \in W_0^{1,2}(0,L)$$

with

- $a \in L^\infty(\mathbb{R})$, 1-periodic, $0 < \alpha \le a(x) \le \beta$ a.e. in $\mathbb{R}$
- $g \in L^2(0,L)$

Question:

$$\min\{E_\varepsilon(u) \colon u \in W_0^{1,2}(0,L)\} \quad \xrightarrow{\varepsilon\to 0} \quad ?$$

Prototipical example: homogenisation in $1D$

$$E_\varepsilon(u) = \int_0^L a\left(\frac{x}{\varepsilon}\right)(u')^2 dx - 2\int_0^L g\,u\,dx, \quad u \in W_0^{1,2}(0,L)$$

with

- $a \in L^\infty(\mathbb{R})$, 1-periodic, $0 < \alpha \le a(x) \le \beta$ a.e. in $\mathbb{R}$
- $g \in L^2(0,L)$

Question:

$$\min\{E_\varepsilon(u) \colon u \in W_0^{1,2}(0,L)\} \quad \xrightarrow{\varepsilon\to 0} \quad ?$$

The pointwise limit of E_ε exists and is given by

$$E(u) = \langle a \rangle \int_0^L (u')^2 dx - 2\int_0^L g\,u\,dx$$

with

$$\langle a \rangle := \int_0^1 a(t)\,dt$$

Prototipical example: homogenisation in $1D$

$$E_\varepsilon(u) = \int_0^L a\left(\frac{x}{\varepsilon}\right)(u')^2 dx - 2\int_0^L g\,u\,dx, \quad u \in W_0^{1,2}(0,L)$$

v_ε minimizes E_ε in $W_0^{1,2}(0,L)$ $\iff$ v_ε solves the Euler-Lagrange equation

$$\begin{cases} -\dfrac{d}{dx}\left(a\left(\dfrac{x}{\varepsilon}\right)v_\varepsilon'(x)\right) = g \quad \text{in } (0,L) \\ v_\varepsilon(0) = v_\varepsilon(L) = 0 \end{cases}$$

Prototipical example: homogenisation in $1D$

$$E_\varepsilon(u) = \int_0^L a\left(\frac{x}{\varepsilon}\right)(u')^2 dx - 2\int_0^L g\, u\, dx, \quad u \in W_0^{1,2}(0,L)$$

v_ε minimizes E_ε in $W_0^{1,2}(0,L)$ $\iff$ v_ε solves the Euler-Lagrange equation

$$\begin{cases} -\dfrac{d}{dx}\left(a\left(\dfrac{x}{\varepsilon}\right)v_\varepsilon'(x)\right) = g & \text{in } (0,L) \\ v_\varepsilon(0) = v_\varepsilon(L) = 0 \end{cases}$$

$$\rightsquigarrow\ v_\varepsilon(x) = -\int_0^x \frac{G(t)}{a_\varepsilon(t)}\,dt + \left(\frac{\int_0^L \frac{G(t)}{a_\varepsilon(t)}\,dt}{\int_0^L \frac{1}{a_\varepsilon(t)}\,dt}\right)\int_0^x \frac{1}{a_\varepsilon(t)}\,dt \quad (G' = g)$$

where $a_\varepsilon(t) := a(t/\varepsilon)$ and

$$\frac{1}{a_\varepsilon} \rightharpoonup \left\langle \frac{1}{a} \right\rangle$$

Prototipical example: homogenisation in $1D$

$v_\varepsilon \rightharpoonup v_0$ in $W^{1,2}(0,L)$ satisfying

$$\begin{cases} -\dfrac{d}{dx}\left(\left\langle \dfrac{1}{a}\right\rangle^{-1} v_0'(x)\right) = g \quad \text{in } (0,L) \\ v_0(0) = v_0(L) = 0 \end{cases}$$

$$\Updownarrow$$

v_0 minimises the functional E_0

$$E_0(u) = \left\langle \frac{1}{a}\right\rangle^{-1} \int_0^L (u')^2 dx - 2\int_0^L g\, u\, dx$$

$\langle \frac{1}{a}\rangle^{-1} \leq \langle a \rangle \quad \rightsquigarrow \quad$ in general $E_0 \leq E$

Prototipical example: homogenisation in $1D$

$$v_\varepsilon \rightharpoonup v_0 \quad \text{in} \quad W^{1,2}(0,L) \quad \text{satisfying}$$

$$\begin{cases} -\dfrac{d}{dx}\left(\left\langle \dfrac{1}{a}\right\rangle^{-1} v_0'(x)\right) = g \quad \text{in } (0,L) \\ v_0(0) = v_0(L) = 0 \end{cases}$$

$$\Updownarrow$$

v_0 minimises the functional E_0

$$E_0(u) = \left\langle \frac{1}{a}\right\rangle^{-1} \int_0^L (u')^2 dx - 2\int_0^L g\, u\, dx$$

$\langle \frac{1}{a}\rangle^{-1} \leq \langle a\rangle \quad \rightsquigarrow \quad$ in general $E_0 \leq E$

E_0 is the limit of E_ε in a *variational* sense

(De Giorgi 1975)

$$E_{\varepsilon} \xrightarrow{\Gamma} E_0$$

(De Giorgi 1975)

$$E_\varepsilon \xrightarrow{\Gamma} E_0 \quad \Longleftrightarrow$$

- (Ansatz-free lower bound) $\forall\, u_\varepsilon, u \in X$ such that $u_\varepsilon \to u$ it holds
$$E_0(u) \leq E_\varepsilon(u_\varepsilon) + o(1) \quad \text{as } \varepsilon \to 0$$

- (Existence of a “recovery sequence”) $\forall\, u \in X \;\; \exists\, \bar{u}_\varepsilon \in X$ such that $\bar{u}_\varepsilon \to u$ and
$$E_\varepsilon(\bar{u}_\varepsilon) \to E_0(u) \quad \text{as } \varepsilon \to 0$$

Γ-convergence

(De Giorgi 1975)

$$E_\varepsilon \xrightarrow{\Gamma} E_0 \quad \Longleftrightarrow$$

- (Ansatz-free lower bound) $\forall\, u_\varepsilon, u \in X$ such that $u_\varepsilon \to u$ it holds

$$E_0(u) \le E_\varepsilon(u_\varepsilon) + o(1) \quad \text{as } \varepsilon \to 0$$

- (Existence of a “recovery sequence”) $\forall\, u \in X \;\; \exists\, \bar{u}_\varepsilon \in X$ such that $\bar{u}_\varepsilon \to u$ and

$$E_\varepsilon(\bar{u}_\varepsilon) \to E_0(u) \quad \text{as } \varepsilon \to 0$$

Fundamental property of Γ-convergence

$$E_\varepsilon \xrightarrow{\Gamma} E_0 \quad + \quad \text{“compactness”}$$

$$\Downarrow$$

$$\inf\{E_\varepsilon(u) \colon u \in X\} \quad \rightsquigarrow \quad \min\{E_0(u) \colon u \in X\}$$

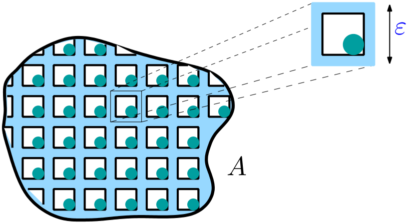
Properties of the Γ-limit

- the Γ-limit is always lower semicontinuous
- if $E_\varepsilon = E$ is the constant sequence, in general $\Gamma\text{-}\lim E \neq E$
- if $E_\varepsilon \stackrel{\Gamma}{\longrightarrow} E_0$ and G is continuos then

$$E_\varepsilon + G \stackrel{\Gamma}{\longrightarrow} E_0 + G$$

- if $E_\varepsilon \stackrel{\Gamma}{\longrightarrow} E_0$ and $E_\varepsilon \longrightarrow E$ pointwise then $E_0 \leq E$

Nonlinear homogenisation



A = reference configuration of a multi-phase periodic composite

ε = period of the composite

$u\colon A \subset \mathbb{R}^n \to \mathbb{R}^m$ deformation

$$F_\varepsilon(u) = \int_A f\left(\frac{x}{\varepsilon}, \nabla u\right)\, dx, \quad u \in W^{1,p}(A, \mathbb{R}^m) \quad \text{(elastic energy)}$$

with $f\colon \mathbb{R}^n \times \mathbb{R}^{m\times n} \to [0, +\infty)$ Borel function satisfying

- $x \to f(x, \xi)$ $(0,1)^n$-periodic
- $\xi \to f(x, \xi)$ continuous
- $c_1|\xi|^p \leq f(x, \xi) \leq c_2(1 + |\xi|^p) \quad p > 1$

The nonlinear homogenisation Theorem

Theorem (Braides 85, Müller 87)

The functionals

$$F_\varepsilon(u) = \int_A f\left(\frac{x}{\varepsilon}, \nabla u\right)\, dx, \quad u \in W^{1,p}(A, \mathbb{R}^m)$$

Γ-converge, with respect to the $L^p(A, \mathbb{R}^m)$-convergence, to the *homogenised* functional

$$F_0(u) = \int_A f_{\mathrm{hom}}(\nabla u)\, dx, \quad u \in W^{1,p}(A, \mathbb{R}^m)$$

where

$$f_{\mathrm{hom}}(\xi) = \lim_{t \to +\infty} \frac{1}{t^n} \inf\left\{ \int_{tQ} f(y, \nabla u)\, dy \colon u \in W^{1,p}(tQ, \mathbb{R}^n),\ u = \xi x \text{ on } \partial tQ \right\}$$

with $Q := \left(-\frac{1}{2}, \frac{1}{2}\right)^n$.

(Idea of the) Proof of the lower bound by blow-up

(Fonseca-Müller 92)

$$F_{\varepsilon_j}(u) = \int_A f\left(\frac{x}{\varepsilon_j}, \nabla u\right)\, dx, \quad u \in W^{1,p}(A, \mathbb{R}^m)$$

Claim: $u \in W^{1,p}(A, \mathbb{R}^m)$, $u_j \to u$ in $L^p \implies \liminf_{j\to+\infty} F_{\varepsilon_j}(u_j) \geq F_0(u)$

(Idea of the) Proof of the lower bound by blow-up

(Fonseca-Müller 92)

$$F_{\varepsilon_j}(u) = \int_A f\left(\frac{x}{\varepsilon_j}, \nabla u\right)\, dx, \quad u \in W^{1,p}(A, \mathbb{R}^m)$$

Claim: $u \in W^{1,p}(A, \mathbb{R}^m)$, $u_j \to u$ in $L^p \implies \liminf_{j \to +\infty} F_{\varepsilon_j}(u_j) \geq F_0(u)$

Step 0: localise the functionals: For $B \in \mathcal{B}(A)$ set

$$F_{\varepsilon_j}(u, B) = \int_B f\left(\frac{x}{\varepsilon_j}, \nabla u\right)\, dx$$

and

$$\mu_j := f\left(\frac{x}{\varepsilon_j}, \nabla u_j\right) \mathcal{L}^n$$

so that

$$\mu_j(B) = F_{\varepsilon_j}(u_j, B)$$

(Idea of the) Proof of the lower bound by blow-up

(Fonseca-Müller 92)

$$F_{\varepsilon_j}(u) = \int_A f\left(\frac{x}{\varepsilon_j}, \nabla u\right) dx, \quad u \in W^{1,p}(A, \mathbb{R}^m)$$

Claim: $u \in W^{1,p}(A, \mathbb{R}^m)$, $u_j \to u$ in L^p $\Longrightarrow$ $\liminf_{j\to+\infty} F_{\varepsilon_j}(u_j) \geq F_0(u)$

Step 0: localise the functionals: For $B \in \mathcal{B}(A)$ set

$$F_{\varepsilon_j}(u, B) = \int_B f\left(\frac{x}{\varepsilon_j}, \nabla u\right) dx$$

and

$$\mu_j := f\left(\frac{x}{\varepsilon_j}, \nabla u_j\right) \mathcal{L}^n$$

so that

$$\mu_j(B) = F_{\varepsilon_j}(u_j, B)$$

Notice that

$$\sup_j |\mu_j|(A) < +\infty$$

(Idea of the) Proof of the lower bound by blow-up

Step 1: definition of a limit measure: up to subsequences

$$\mu_j \rightharpoonup \mu$$

consider the Radon-Nikodym decomposition

$$\mu = \frac{d\mu}{dx}\,\mathcal{L}^n + \mu^s, \quad \text{with} \quad \mu^s \perp \mathcal{L}^n$$

(Idea of the) Proof of the lower bound by blow-up

Step 1: definition of a limit measure: up to subsequences

$$\mu_j \rightharpoonup \mu$$

consider the Radon-Nikodym decomposition

$$\mu = \frac{d\mu}{dx}\,\mathcal{L}^n + \mu^s, \quad \text{with} \quad \mu^s \perp \mathcal{L}^n$$

Step 2: local analysis: let $x_0 \in A$ be s.t.

$$\frac{d\mu}{dx}(x_0) = \lim_{\rho\to 0+} \frac{\mu(Q_\rho(x_0))}{\mathcal{L}^n(Q_\rho(x_0))} = \lim_{\rho\to 0+} \frac{\mu(Q_\rho(x_0))}{\rho^n}$$

We also have

$$\mu(Q_\rho(x_0)) = \lim_j \mu_j(Q_\rho(x_0))$$

(for all $\rho > 0$ but a countable set)

$$\rightsquigarrow \quad \frac{d\mu}{dx}(x_0) = \lim_j \frac{\mu_j(Q_{\rho_j}(x_0))}{\rho_j^n} = \lim_j \frac{1}{\rho_j^n} \int_{Q_{\rho_j}(x_0)} f\left(\frac{x}{\varepsilon_j}, \nabla u_j\right) dx$$

(Idea of the) Proof of the lower bound by blow-up

$$\rightsquigarrow \quad \frac{d\mu}{dx}(x_0) = \lim_j \frac{1}{\rho_j^n} \int_{Q_{\rho_j}(x_0)} f\left(\frac{x}{\varepsilon_j}, \nabla u_j\right) dx$$

Step 3: blow-up: modify (u_j) to get (v_j) s.t. $v_j \rightharpoonup \nabla u(x_0)x$ in $W^{1,p}$

(Idea of the) Proof of the lower bound by blow-up

$$\rightsquigarrow \quad \frac{d\mu}{dx}(x_0) = \lim_j \frac{1}{\rho_j^n} \int_{Q_{\rho_j}(x_0)} f\left(\frac{x}{\varepsilon_j}, \nabla u_j\right) dx$$

Step 3: blow-up: modify (u_j) to get (v_j) s.t. $v_j \rightharpoonup \nabla u(x_0)x$ in $W^{1,p}$

Set

$$w_j^\rho(x) := \frac{u_j(x_0 + \rho x) - u(x_0)}{\rho} \qquad \text{(blow-up sequence)}$$

since $u_j \rightharpoonup u$ in $W^{1,p}$, choosing x_0 s.t.

$$\lim_{\rho \to 0^+} \frac{1}{\rho^n} \int_{Q_\rho(x_0)} \frac{|u(x) - u(x_0) - \nabla u(x_0)(x - x_0)|^p}{\rho^p} dx = 0$$

we find $\rho_j \to 0^+$ such that

$$v_j := {w_j}^{\rho_j} \rightharpoonup \nabla u(x_0)x$$

(Idea of the) Proof of the lower bound by blow-up

$$\rightsquigarrow \quad \frac{d\mu}{dx}(x_0) = \lim_j \frac{1}{\rho_j^n} \int_{Q_{\rho_j}(x_0)} f\left(\frac{x}{\varepsilon_j}, \nabla u_j\right) dx$$

Step 3: blow-up: modify (u_j) to get (v_j) s.t. $v_j \rightharpoonup \nabla u(x_0)x$ in $W^{1,p}$

Set

$$w_j^\rho(x) := \frac{u_j(x_0 + \rho x) - u(x_0)}{\rho} \qquad \text{(blow-up sequence)}$$

since $u_j \rightharpoonup u$ in $W^{1,p}$, choosing x_0 s.t.

$$\lim_{\rho \to 0^+} \frac{1}{\rho^n} \int_{Q_\rho(x_0)} \frac{|u(x) - u(x_0) - \nabla u(x_0)(x - x_0)|^p}{\rho^p} dx = 0$$

we find $\rho_j \to 0^+$ such that

$$v_j := {w_j}^{\rho_j} \rightharpoonup \nabla u(x_0)x$$

$$\rightsquigarrow \quad \frac{d\mu}{dx}(x_0) = \lim_j \int_{Q_1(0)} f\left(\frac{\rho_j x + x_0}{\varepsilon_j}, \nabla v_j\right) dx$$

(Idea of the) Proof of the lower bound by blow-up

A further modification of v_j is needed to obtain a new sequence

$$\hat{v}_j = \nabla u(x_0)x \quad \text{on} \quad \partial Q_1$$

without essentially increasing the energy (via the energy bounds)

$$\frac{d\mu}{dx}(x_0) = \lim_j \int_{Q_1(0)} f\left(\frac{\rho_j x + x_0}{\varepsilon_j}, \nabla \hat{v}_j\right) dx$$

(Idea of the) Proof of the lower bound by blow-up

A further modification of v_j is needed to obtain a new sequence

$$\hat{v}_j = \nabla u(x_0)x \quad \text{on} \quad \partial Q_1$$

without essentially increasing the energy (via the energy bounds)

$$\begin{aligned} \frac{d\mu}{dx}(x_0) &= \lim_j \int_{Q_1(0)} f\left(\frac{\rho_j x + x_0}{\varepsilon_j}, \nabla \hat{v}_j\right) dx \\ &\geq \liminf_j \left\{ \int_{Q_1(0)} f\left(\frac{\rho_j x + x_0}{\varepsilon_j}, \nabla w\right) dx \colon w = \nabla u(x_0)\, x \text{ on } \partial Q_1 \right\} \end{aligned}$$

(Idea of the) Proof of the lower bound by blow-up

A further modification of v_j is needed to obtain a new sequence

$$\hat{v}_j = \nabla u(x_0)x \quad \text{on} \quad \partial Q_1$$

without essentially increasing the energy (via the energy bounds)

$$\begin{aligned}
\frac{d\mu}{dx}(x_0) &= \lim_j \int_{Q_1(0)} f\left(\frac{\rho_j x + x_0}{\varepsilon_j}, \nabla \hat{v}_j\right) dx \\
&\geq \lim_j \inf \left\{ \int_{Q_1(0)} f\left(\frac{\rho_j x + x_0}{\varepsilon_j}, \nabla w\right) dx \colon w = \nabla u(x_0)\, x \text{ on } \partial Q_1 \right\} \\
&\geq \lim_j \frac{\varepsilon_j^n}{\rho_j^n} \inf \left\{ \int_{Q_{\frac{\rho_j}{\varepsilon_j}}\left(\frac{x_0}{\varepsilon_j}\right)} f(y, \nabla w)\, dy \colon w = \nabla u(x_0)\, x \text{ on } \partial \Big(Q_{\frac{\rho_j}{\varepsilon_j}}\Big(\frac{x_0}{\varepsilon_j}\Big)\Big) \right\}
\end{aligned}$$

with $t := \frac{\rho_j}{\varepsilon_j} \to +\infty$ as $j \to +\infty$

(Idea of the) Proof of the lower bound by blow-up

Step 4: local estimate: using the periodicity of f we get the existence and homogeneity of the limit

$$\lim_{j} \frac{\varepsilon_j^n}{\rho_j^n} \inf \left\{ \int_{Q_{\frac{\rho_j}{\varepsilon_j}}\left(\frac{x_0}{\varepsilon_j}\right)} f(y, \nabla w)\, dy \colon w = \nabla u(x_0)\, x \text{ on } \partial\Big(Q_{\frac{\rho_j}{\varepsilon_j}}\Big(\frac{x_0}{\varepsilon_j}\Big)\Big) \right\}$$

(Idea of the) Proof of the lower bound by blow-up

Step 4: local estimate: using the periodicity of f we get the existence and homogeneity of the limit

$$\lim_j \frac{\varepsilon_j^n}{\rho_j^n} \inf\left\{ \int_{Q_{\frac{\rho_j}{\varepsilon_j}}\left(\frac{x_0}{\varepsilon_j}\right)} f(y, \nabla w)\, dy \colon w = \nabla u(x_0)\, x \text{ on } \partial\Big(Q_{\frac{\rho_j}{\varepsilon_j}}\Big(\frac{x_0}{\varepsilon_j}\Big)\Big) \right\}$$

$$= \lim_{t \to +\infty} \frac{1}{t^n} \inf\left\{ \int_{Q_t(0)} f(y, \nabla w)\, dy \colon w = \underbrace{\nabla u(x_0)}_{\xi}\, x \text{ on } \partial Q_t(0) \right\}$$

(Idea of the) Proof of the lower bound by blow-up

Step 4: local estimate: using the periodicity of f we get the existence and homogeneity of the limit

$$
\lim_j \frac{\varepsilon_j^n}{\rho_j^n} \inf \left\{ \int_{Q_{\frac{\rho_j}{\varepsilon_j}}\left(\frac{x_0}{\varepsilon_j}\right)} f(y, \nabla w)\, dy \colon w = \nabla u(x_0)\, x \text{ on } \partial\Big(Q_{\frac{\rho_j}{\varepsilon_j}}\Big(\frac{x_0}{\varepsilon_j}\Big)\Big) \right\}
$$

$$
= \lim_{t \to +\infty} \frac{1}{t^n} \inf \left\{ \int_{Q_t(0)} f(y, \nabla w)\, dy \colon w = \underbrace{\nabla u(x_0)}_{\xi}\, x \text{ on } \partial Q_t(0) \right\}
$$

$$
= f_{\hom}(\nabla u(x_0))
$$

$$
\rightsquigarrow \quad \frac{d\mu}{dx}(x_0) \geq f_{\hom}(\nabla u(x_0)) \quad \text{a.e. in } A
$$

(Idea of the) Proof of the lower bound by blow-up

Step 4: local estimate: using the periodicity of f we get the existence and homogeneity of the limit

$$\lim_j \frac{\varepsilon_j^n}{\rho_j^n} \inf \left\{ \int_{Q_{\frac{\rho_j}{\varepsilon_j}}\left(\frac{x_0}{\varepsilon_j}\right)} f(y, \nabla w)\, dy \colon w = \nabla u(x_0)\, x \text{ on } \partial\left(Q_{\frac{\rho_j}{\varepsilon_j}}\left(\frac{x_0}{\varepsilon_j}\right)\right) \right\}$$

$$= \lim_{t\to+\infty} \frac{1}{t^n} \inf \left\{ \int_{Q_t(0)} f(y, \nabla w)\, dy \colon w = \underbrace{\nabla u(x_0)}_{\xi}\, x \text{ on } \partial Q_t(0) \right\}$$

$$= f_{\rm hom}(\nabla u(x_0))$$

$$\rightsquigarrow \quad \frac{d\mu}{dx}(x_0) \geq f_{\rm hom}(\nabla u(x_0)) \quad \text{a.e. in } A$$

Step 5: global estimate: integrating the local estimate gives

$$\liminf_j F_{\varepsilon_j}(u_j) = \liminf_j \mu_j(A) \geq \mu(A)$$

$$\geq \int_A \frac{d\mu}{dx}\, dx \geq \int_A f_{\rm hom}(\nabla u)\, dx = F_0(u)$$

More homogenisation problems: the BV-setting

Homogenisation of perimeters

$$G_\varepsilon(E) := \int_{\partial^* E \cap \Omega} g\left(\frac{x}{\varepsilon}, \nu_E\right) d\mathcal{H}^{n-1}$$

E set of finite perimeter, ν_E inner normal to E (defined at all points of $\partial^* E$).

More homogenisation problems: the BV-setting

Homogenisation of perimeters

$$G_\varepsilon(u) := \int_{S_u \cap \Omega} g\left(\frac{x}{\varepsilon}, \nu_u\right) d\mathcal{H}^{n-1}$$

$u = \chi_E$, $u \in BV(\Omega)$, S_u discontinuity set of u.

More homogenisation problems: the BV-setting

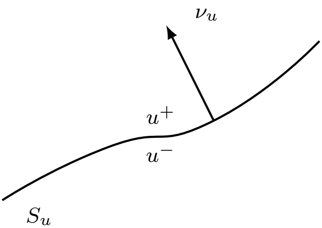
Homogenisation of perimeters

$$G_\varepsilon(u) := \int_{S_u \cap \Omega} g\Big(\frac{x}{\varepsilon}, \nu_u\Big)\, d\mathcal{H}^{n-1}$$

$u = \chi_E$, $u \in BV(\Omega)$, S_u discontinuity set of u.

Homogenisation of free-discontinuity problems

$$E_\varepsilon(u) := \int_\Omega f\Big(\frac{x}{\varepsilon}, \nabla u\Big)\, dx + \int_{S_u \cap \Omega} g\Big(\frac{x}{\varepsilon}, \nu_u\Big)\, d\mathcal{H}^{n-1}, \quad u \in SBV(\Omega, \mathbb{R}^m)$$

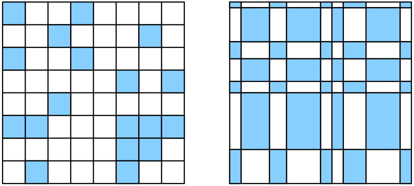


$$Du = \nabla u\, \mathcal{L}^n + (u^+ - u^-) \otimes \nu_u\, \mathcal{H}^{n-1} \llcorner S_u$$

- S_u discontinuity set of u
- $u^+ - u^-$ jump of u across S_u
- ν_u normal to S_u (pointing towards u^+)

More homogenisation problems: the random-setting

- $(\Omega, \mathcal{T}, P)$ probability space
- $\omega \in \Omega$ random parameter



Examples of random checkerboards

$$E_\varepsilon(\omega)(u) = \int_A f\Big(\omega, \frac{x}{\varepsilon}, \nabla u\Big)\, dx + \int_{A\cap S_u} g\Big(\omega, \frac{x}{\varepsilon}, u^+ - u^-, \nu_u\Big) d\mathcal{H}^{n-1} \quad u \in SBV(A;\mathbb{R}^m)$$

f and g are stationary random variables

$\rightsquigarrow$ periodicity in law replaces periodicity

Assumptions on f and g (ω fixed)

$f\colon \mathbb{R}^n \times \mathbb{R}^{m\times n} \to [0, +\infty)$, $f = f(x, \xi)$ volume energy density

- f is Borel-measurable
- $c_1|\xi|^p \le f(x,\xi) \le c_2(1+|\xi|^p) \quad (p > 1)$
- $\xi \mapsto f(x,\xi)$ is continuous for every $x \in \mathbb{R}^n$

$\mathcal{F}$

Assumptions on f and g (ω fixed)

$f\colon \mathbb{R}^n \times \mathbb{R}^{m\times n} \to [0,+\infty)$, $f = f(x,\xi)$ volume energy density

- f is Borel-measurable
- $c_1|\xi|^p \le f(x,\xi) \le c_2(1+|\xi|^p) \quad (p>1)$
- $\xi \mapsto f(x,\xi)$ is continuous for every $x \in \mathbb{R}^n$

$\mathcal{F}$

$g\colon \mathbb{R}^n \times (\mathbb{R}^m \setminus \{0\}) \times \mathbb{S}^{n-1} \to [0,+\infty)$, $g = g(x,\zeta,\nu)$ surface energy density

- g is Borel-measurable
- $c_3(1+|\zeta|) \le g(x,\zeta,\nu) \le c_4(1+|\zeta|)$
- $\zeta \mapsto g(x,\zeta,\nu)$ is continuous for every $x \in \mathbb{R}^n$, $\nu \in \mathbb{S}^{n-1}$
- $g(x,\zeta,\nu) = g(x,-\zeta,-\nu)$

$\mathcal{G}$

Homogenisation Theorem

Theorem (Cagnetti, Dal Maso, Scardia, Z. - *Arch. Ration. Mech. Anal.* 2019)

Let $f \in \mathcal{F}$ and $g \in \mathcal{G}$ be stationary. Then there exist $\Omega' \subset \Omega$ with $P(\Omega') = 1$ and homogenoeus random integrands $f_0 \in \mathcal{F}$, $g_0 \in \mathcal{G}$ such that

$$E_\varepsilon(\omega)(u) = \int_A f\Big(\omega, \frac{x}{\varepsilon}, \nabla u\Big)\, dx + \int_{A\cap S_u} g\Big(\omega, \frac{x}{\varepsilon}, u^+ - u^-, \nu_u\Big) d\mathcal{H}^{n-1} \quad u \in SBV(A)$$

$\Gamma(L^1)$-converges to

$$E_0(\omega)(u) = \int_A f_0(\omega, \nabla u)\, dx + \int_{S_u\cap A} g_0(\omega, u^+ - u^-, \nu_u) d\mathcal{H}^{n-1} \quad u \in SBV(A)$$

for every $\omega \in \Omega'$.

Homogenisation Theorem

Theorem (Cagnetti, Dal Maso, Scardia, Z. - *Arch. Ration. Mech. Anal.* 2019)

Let $f \in \mathcal{F}$ and $g \in \mathcal{G}$ be stationary. Then there exist $\Omega' \subset \Omega$ with $P(\Omega') = 1$ and homogenoeus random integrands $f_0 \in \mathcal{F}$, $g_0 \in \mathcal{G}$ such that

$$E_\varepsilon(\omega)(u) = \int_A f\Big(\omega, \frac{x}{\varepsilon}, \nabla u\Big)\, dx + \int_{A \cap S_u} g\Big(\omega, \frac{x}{\varepsilon}, u^+ - u^-, \nu_u\Big) d\mathcal{H}^{n-1} \quad u \in SBV(A)$$

$\Gamma(L^1)$-converges to

$$E_0(\omega)(u) = \int_A f_0(\omega, \nabla u)\, dx + \int_{S_u \cap A} g_0(\omega, u^+ - u^-, \nu_u) d\mathcal{H}^{n-1} \quad u \in SBV(A)$$

for every $\omega \in \Omega'$.

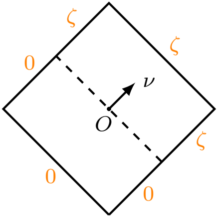
Moreover

$$f_0(\omega, \xi) := \lim_{t \to +\infty} \frac{1}{t^n} \inf \Big\{ \int_{tQ} f(\omega, x, \nabla u)\, dx \colon u \in W^{1,p}(tQ; \mathbb{R}^m),\ u = \xi x \text{ near } \partial(tQ) \Big\}$$

Homogenisation Theorem (continues...)

$$g_0(\omega,\zeta,\nu) := \lim_{t\to+\infty} \frac{1}{t^{n-1}} \inf \left\{ \int_{tQ^\nu \cap S_u} g(\omega, x, u^+ - u^-, \nu_u)\, d\mathcal{H}^{n-1} : \right.$$
$$\left. u \in SBV(tQ^\nu;\mathbb{R}^m),\ \nabla u = 0 \text{ a.e.},\ u = u_{0,\zeta,\nu} \text{ near } \partial(tQ^\nu) \right\}$$

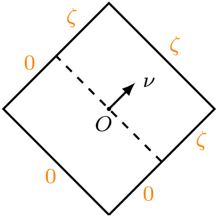
$$u_{0,\zeta,\nu}(x) := \begin{cases} \zeta & \text{if } x\cdot\nu \geq 0 \\ 0 & \text{if } x\cdot\nu < 0 \end{cases}$$



Homogenisation Theorem (continues...)

$$g_0(\omega,\zeta,\nu) := \lim_{t\to+\infty} \frac{1}{t^{n-1}} \inf\left\{ \int_{tQ^\nu\cap S_u} g(\omega,x,u^+-u^-,\nu_u)d\mathcal{H}^{n-1} : u \in SBV(tQ^\nu;\mathbb{R}^m),\ \nabla u = 0 \text{ a.e.}, u = u_{0,\zeta,\nu} \text{ near } \partial(tQ^\nu)\right\}$$

$$u_{0,\zeta,\nu}(x) := \begin{cases} \zeta & \text{if } x\cdot\nu \geq 0 \\ 0 & \text{if } x\cdot\nu < 0 \end{cases}$$



Further, if f and g are ergodic f_0 and g_0 are deterministic

Remarks

- The Γ-limit

$$E_0(\omega)(u) = \int_A f_0(\omega, \nabla u)\, dx + \int_{A\cap S_u} g_0(\omega, u^+ - u^-, \nu_u) d\mathcal{H}^{n-1}$$

is a random homogeneous functional

Remarks

- The Γ-limit

$$E_0(\omega)(u) = \int_A f_0(\omega, \nabla u)\, dx + \int_{A \cap S_u} g_0(\omega, u^+ - u^-, \nu_u) d\mathcal{H}^{n-1}$$

  is a random homogeneous functional

- When f and g are ergodic, the Γ-limit E_0 is deterministic

Remarks

- The Γ-limit

$$E_0(\omega)(u) = \int_A f_0(\omega, \nabla u)\, dx + \int_{A \cap S_u} g_0(\omega, u^+ - u^-, \nu_u) d\mathcal{H}^{n-1}$$

  is a random homogeneous functional

- When f and g are ergodic, the Γ-limit E_0 is deterministic

- the class of functionals

$$E_\varepsilon(\omega)(u) = \int_A f\Big(\omega, \frac{x}{\varepsilon}, \nabla u\Big)\, dx + \int_{A \cap S_u} g\Big(\omega, \frac{x}{\varepsilon}, u^+ - u^-, \nu_u\Big) d\mathcal{H}^{n-1}$$

  is compact by Γ-convergence; i.e. $f_0(\omega, \cdot) \in \mathcal{F}$, $g_0(\omega, \cdot, \cdot) \in \mathcal{G}$

Remarks

- The Γ-limit

$$E_0(\omega)(u) = \int_A f_0(\omega, \nabla u)\, dx + \int_{A\cap S_u} g_0(\omega, u^+ - u^-, \nu_u) d\mathcal{H}^{n-1}$$

  is a random homogeneous functional

- When f and g are ergodic, the Γ-limit E_0 is deterministic

- the class of functionals

$$E_\varepsilon(\omega)(u) = \int_A f\Big(\omega, \frac{x}{\varepsilon}, \nabla u\Big)\, dx + \int_{A\cap S_u} g\Big(\omega, \frac{x}{\varepsilon}, u^+ - u^-, \nu_u\Big) d\mathcal{H}^{n-1}$$

  is compact by Γ-convergence; i.e. $f_0(\omega, \cdot) \in \mathcal{F}$, $g_0(\omega, \cdot, \cdot) \in \mathcal{G}$

- in the limit there is no interaction between volume and surface energy

Blow up: main difference with the $W^{1,p}$-case

$$u_j \to u \text{ in } L^1(A;\mathbb{R}^m) \text{ with } u \in SBV(A;\mathbb{R}^m)$$

- $\mu_j := f\left(\frac{x}{\varepsilon_j}, \nabla u_j\right)\mathcal{L}^n + g\left(\frac{x}{\varepsilon_j}, {u_j}^+ - {u_j}^-, \nu_{u_j}\right)\mathcal{H}^{n-1} \llcorner S_{u_j}$

Blow up: main difference with the $W^{1,p}$-case

$$u_j \to u \text{ in } L^1(A;\mathbb{R}^m) \text{ with } u \in SBV(A;\mathbb{R}^m)$$

- $\mu_j := f\left(\frac{x}{\varepsilon_j}, \nabla u_j\right)\mathcal{L}^n + g\left(\frac{x}{\varepsilon_j}, u_j^{+} - u_j^{-}, \nu_{u_j}\right)\mathcal{H}^{n-1}\llcorner S_{u_j}$

- $\mu = \frac{d\mu}{dx}\mathcal{L}^n + \frac{d\mu}{d\mathcal{H}^{n-1}\llcorner S_u}\mathcal{H}^{n-1}\llcorner S_u + \mu^s$

Blow up: main difference with the $W^{1,p}$-case

$$u_j \to u \text{ in } L^1(A;\mathbb{R}^m) \text{ with } u \in SBV(A;\mathbb{R}^m)$$

- $\mu_j := f\left(\frac{x}{\varepsilon_j}, \nabla u_j\right)\mathcal{L}^n + g\left(\frac{x}{\varepsilon_j}, {u_j}^+ - {u_j}^-, \nu_{uj}\right)\mathcal{H}^{n-1} \llcorner S_{u_j}$
- $\mu = \frac{d\mu}{dx}\mathcal{L}^n + \frac{d\mu}{d\mathcal{H}^{n-1}\llcorner S_u}\mathcal{H}^{n-1}\llcorner S_u + \mu^s$
- blow-up at a differentiability point for u gives

$$\frac{d\mu}{dx}(x_0) \geq f_{\text{hom}}(\nabla u(x_0))$$

Blow up: main difference with the $W^{1,p}$-case

$$u_j \to u \text{ in } L^1(A;\mathbb{R}^m) \text{ with } u \in SBV(A;\mathbb{R}^m)$$

- $\mu_j := f\left(\frac{x}{\varepsilon_j}, \nabla u_j\right)\mathcal{L}^n + g\left(\frac{x}{\varepsilon_j}, {u_j}^+ - {u_j}^-, \nu_{u_j}\right)\mathcal{H}^{n-1} \llcorner S_{u_j}$
- $\mu = \frac{d\mu}{dx}\mathcal{L}^n + \frac{d\mu}{d\mathcal{H}^{n-1}\llcorner S_u}\mathcal{H}^{n-1}\llcorner S_u + \mu^s$
- blow-up at a differentiability point for u gives

$$\frac{d\mu}{dx}(x_0) \geq f_{\text{hom}}(\nabla u(x_0))$$

- blow-up at a jump point for u gives

$$\frac{d\mu}{d\mathcal{H}^{n-1}\llcorner S_u}(x_0) \geq g_{\text{hom}}(u^+(x_0) - u^-(x_0), \nu_u(x_0))$$

Blow up: main difference with the $W^{1,p}$-case

$$u_j \to u \text{ in } L^1(A;\mathbb{R}^m) \text{ with } u \in SBV(A;\mathbb{R}^m)$$

- $\mu_j := f\left(\frac{x}{\varepsilon_j}, \nabla u_j\right)\mathcal{L}^n + g\left(\frac{x}{\varepsilon_j}, {u_j}^+ - {u_j}^-, \nu_{uj}\right)\mathcal{H}^{n-1} \llcorner S_{u_j}$
- $\mu = \frac{d\mu}{dx}\mathcal{L}^n + \frac{d\mu}{d\mathcal{H}^{n-1}\llcorner S_u}\mathcal{H}^{n-1}\llcorner S_u + \mu^s$
- blow-up at a differentiability point for u gives

$$\frac{d\mu}{dx}(x_0) \geq f_{\text{hom}}(\nabla u(x_0))$$

- blow-up at a jump point for u gives

$$\frac{d\mu}{d\mathcal{H}^{n-1}\llcorner S_u}(x_0) \geq g_{\text{hom}}(u^+(x_0) - u^-(x_0), \nu_u(x_0))$$

- ergodic theory is needed to prove the existence of the homogenisation formulas