

SPHERICAL RANK RIGIDITY AND BLASCHKE MANIFOLDS

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INTRODUCTION

In this paper we define a notion of rank for closed manifolds with positive upper curvature bound and prove a rigidity result for the same. More precisely, consider the following definition. If not explicitly stated otherwise all geodesics are assumed to be parameterized by arc length.

Definition. Let M be a complete Riemannian manifold with sectional curvature bounded above by 1. We say that M has *positive spherical rank* if every geodesic $\gamma: [0, \pi] \rightarrow M$ has a conjugate point at $t = \pi$.

By the Rauch comparison theorem we know that along any geodesic there cannot be a conjugate point before π . The well known equality discussion implies that for any normal geodesic $c: [0, \pi] \rightarrow M$ there exists a spherical Jacobi field i.e., a Jacobi field of the form $J(t) = \sin(t)E(t)$ where E is a parallel vector field (see for instance [Chav93, Theorem 2.15]). This latter characterization is analogous to the notions of (upper) Euclidean rank and (upper) hyperbolic rank studied by several people; see below for a more detailed description. In this paper curvature refers to sectional curvature and is denoted by \mathbf{sec} . The following is the main result of the paper.

Theorem 1. *Let M^n be a complete, simply connected Riemannian manifold with $\mathbf{sec} \leq 1$ and positive spherical rank. Then M is isometric to a compact, rank one symmetric space i.e., M is isometric to \mathbf{S}^n , $\mathbf{CP}^{\frac{n}{2}}$, $\mathbf{HP}^{\frac{n}{4}}$ or \mathbf{CaP}^2 .*

Note that the condition of $\mathbf{sec} \leq 1$ is not really an obstruction; any manifold, and in particular any compact manifold, admits a metric with upper curvature bound 1. So any theorem in this class must necessarily include an additional assumption on the geometry of the manifold. There are few general theorems about manifolds with $\mathbf{sec} \leq 1$; the main theorem and Toponogov's theorem mentioned in Section 2 are two theorems for such manifolds. We do not know of any others.

Several notions of 'rank' have been studied for manifolds under suitable curvature assumptions. Historically rank was defined for symmetric spaces and referred to the dimension of

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an embedded flat torus. This evidently descends from the definition of rank for Lie groups. In our paper we study a more recent notion of rank (also called the *geometric rank*) first defined in [BBE85] for non-positively curved manifolds. According to [BBE85] a complete Riemannian manifold with $\mathbf{sec} \leq 0$ has *higher (Euclidean) rank* if along every geodesic γ there exists at least one parallel Jacobi field orthogonal to γ' . It follows that the 2-plane spanned by this Jacobi field and γ' is extremal. The following theorem was proved by W. Ballmann ([Bal85]), and using completely different methods by K. Burns and R. Spatzier ([BS87]), building on previous work in [BBE85] and [BBS85]: Let M^n be a non-positively curved complete manifold of finite volume. Suppose along every geodesic there exists at least one parallel Jacobi field. Then the universal cover of M is either a symmetric space or isometric to a Riemannian product.

The next result was for compact manifolds with $\mathbf{sec} \leq -1$ due to U. Hamenstädt; she used a weaker notion of hyperbolic rank by only assuming that along every geodesic there exists a Jacobi field J such that $\mathbf{sec}(J, \gamma') = -1$ i.e., J and γ' span an extremal curvature 2-plane. She proved the following theorem (cf. [Ham90]): Let M^n be a compact manifold with upper curvature bound -1 and hyperbolic rank at least 1. Then M is isometric to a locally symmetric space. We will refer to this notion of rank as (*upper*) *hyperbolic rank*.

In order to exhibit the analogy of these results to the main theorem we restate it in a slightly weaker form.

Corollary 2. *Let M^n be a complete, simply connected Riemannian manifold with $\mathbf{sec} \leq 1$. Suppose that along every geodesic γ , there exists a normal parallel vector field E such that $\mathbf{sec}(E, \gamma') = 1$. Then M^n is isometric to a compact, rank one symmetric space.*

Indeed the corollary is an immediate consequence of Theorem 1 as $\sin(t)E(t)$ is then a Jacobi field along γ and consequently M has positive spherical rank.

Several questions remain open. For instance, one could turn the above definitions around for (closed) manifolds with suitable lower curvature bounds and ask whether any rigidity is possible. This is known to be false if the lower bound is zero but analogous questions for $\mathbf{sec} \geq -1$ and $\mathbf{sec} \geq 1$ remain untouched. We refer the reader to Table 1 where some of the known results are presented; the table is not meant to be a survey rather a point of departure for further investigations. In this paper we only deal with the case of spherical rank for upper curvature bound 1.

The paper is organized into three sections. In Section 1 we show that positive spherical rank implies that the manifold is a so-called *Blaschke manifold*. A Blaschke manifold is a Riemannian manifold with the property that its injectivity radius equals its diameter. Note that the definition has no curvature assumptions. For an excellent and rather complete treatment of Blaschke manifolds see [Bes78]. The study of these manifolds has a rich history motivated by the following open problem.

Blaschke Conjecture. *Let M be a Riemannian manifold such that $\text{inj } M = \text{diam } M$. Then M is isometric to a compact, rank one symmetric space (CROSS).*

Once we have established that M is Blaschke the remaining step may be regarded as a special case of the Blaschke conjecture. In Section 2 we show that a Blaschke manifold with upper curvature bound 1 and injectivity radius π must be isometric to a CROSS. This latter result has already been proved by V. Rovenskii and V. Toponogov in [RT96]; they prove this using comparison arguments. For the sake of completeness we present a shorter proof whose arguments may be useful in other contexts. In the final section of the paper we give an example, the Berger spheres, which shows that the conclusion of the main theorem fails for a weaker notion of spherical rank, namely the analogue of Hamenstädt's notion of rank. More precisely, we show that there are non-symmetric, simply connected, compact Riemannian manifolds with upper (respectively lower) curvature bound 1, such that along every geodesic γ there exists a normal Jacobi field J with $\mathbf{sec}(\gamma'(t), J(t)) = 1$ for all t .

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1. POSITIVE SPHERICAL RANK IMPLIES BLASCHKE

Let M be a complete, simply connected Riemannian manifold with $\mathbf{sec} \leq 1$ and positive spherical rank. By assumption, every geodesic hits its first conjugate point at π and therefore the diameter of M is bounded above by π . In order to show that M is a Blaschke manifold we only have to verify that the injectivity radius is at least π since we always have $\text{inj } M \leq \text{diam } M$. Since the conjugate radius of the manifold is π , it suffices to show $\text{inj } M = \text{conj } M$.

Consider the special case where M is even dimensional and positively curved. Then by Klingenberg's injectivity radius estimate $\text{inj } M = \pi$, and hence M is Blaschke. This observation, in fact, was the beginning for our investigations.

We now outline the argument for the general case. The starting point is the well-known generalization [CE75, Lemma 5.6] of an injectivity radius estimate of Klingenberg [Klin61, Lemma 4] that for M compact, $\text{inj } M$ is the smaller of $\text{conj } M$ and half the length of a shortest closed geodesic. We will argue by contradiction, and suppose that the length of some closed geodesic is less than 2π . Under these assumptions, it follows from Morse theoretic arguments that there exists a closed geodesic γ of length 2π and index 1.¹ For the actual argument it is important that γ satisfies a slightly stronger condition; see Lemma 1.4.

Moreover, we will show that γ is contained in a totally geodesic, isometrically immersed 2-sphere of constant curvature 1. The next step is to show that the same is true for all geodesics in the manifold and hence all geodesics are closed. Then in Section 1.4 we will show, by applying the index parity theorem [Wil01] that if all geodesics are closed, then they all have length at least 2π . This contradiction finishes the proof that M is indeed a Blaschke manifold.

1.1. Preliminaries.

First we state a useful generalization of Klingenberg's long homotopy lemma due to U. Abresch and W. Meyer [AM97].

¹Throughout this paper we always mean index in the free loop space.

Lemma 1.1 (Long Homotopy Lemma). *Let M be a compact Riemannian manifold and c a closed curve in M which is the union of at most two geodesic segments such that $l(c) < 2 \operatorname{conj} M$. Suppose $c = c_0$ is homotopic to a point via a continuous family of rectifiable closed curves c_t , $0 \leq t \leq 1$. Then some c_s has length $l(c_s) \geq 2 \operatorname{conj} M$.*

Now we adapt the second comparison theorem of Rauch to get the next proposition. We will need the following lemma in the proof.

Lemma 1.2. *Let M be a complete manifold with $\sec \leq 1$, and let \mathbf{S}^2 denote the 2-sphere of constant curvature 1. Suppose X is a normal Jacobi field along a geodesic b of length at most π such that $\|X(0)\| = 1$ and $\langle X', X \rangle = 0$. Let Y be a normal Jacobi field along a geodesic in \mathbf{S}^2 such that $\|Y(0)\| = 1$ and $Y'(0) = 0$.*

Then $\|X(s)\| \geq \|Y(s)\|$ for $0 \leq s \leq \pi/2$. Moreover, if $\|X(s_1)\| = \|Y(s_1)\|$ for some $0 < s_1 \leq \pi/2$, then $\sec(X(s), b'(s)) = 1$ and $\|X(s)\| = \|Y(s)\|$ for all $0 \leq s \leq s_1$.

Proof. Since the upper curvature bound is 1, we see from Cauchy-Schwarz and straightforward differentiation that

$$\|X\|'' + \|X\| \geq 0.$$

Let $a(s)$ be the function such that $\|X\|'' + a(s)\|X\| = 0$. Note that $a(s) \leq \sec(X(s), b'(s)) \leq 1$. By the Sturm comparison theorem, it follows that $\|X(s)\| \geq \|Y(s)\|$ on the closed interval $[0, \pi]$. We refer to [doCar92, p. 238] for a Sturm comparison theorem with different initial conditions. The same proof however applies equally well in our situation. Moreover if $\|X(s_1)\| = \|Y(s_1)\|$, then $a(s) = 1$ for all $s \leq s_1$. In particular,

$$\sec(X(s), b'(s)) = 1 \text{ for } 0 \leq s \leq s_1. \quad \square$$

An isometrically immersed surface (with piecewise smooth geodesic boundary) of constant curvature 1 will be called a *spherical slice*.

Let M be a complete, simply connected manifold with $\sec \leq 1$. Suppose c is a geodesic of length π between two points p

and q on M and suppose c_s is a smooth variation of c by curves connecting p and q such that $l(c_s) \leq \pi$ for all s . We pick a curve c_σ close to c and assume that c_σ is not a reparameterization of c .

Proposition 1.3. *Given c and c_σ as above, they span a totally geodesic spherical slice.*

Proof. If we choose σ sufficiently small we can find a normal vector field $Z(t)$ along c such that $c_\sigma(t) = \exp(Z(t))$ after possibly reparameterizing c_σ . We may assume $\|Z(t)\| < \pi/2$. This yields a proper variation $f(s, t) = \exp(s \cdot Z(t))$ of c . By construction, the curves $f_t(s)$ (t fixed) are geodesics which in turn implies that the vector field $X_t(s) = \frac{\partial f}{\partial t}$ is a Jacobi field along $f_t(s)$. It should be clear that f_t is not necessarily parameterized by arc length. However, according to our conventions c is and thus $\|X_t(0)\| = \|c'(t)\| = 1$.

Let \tilde{c} be a geodesic of length π on \mathbf{S}^2 , the 2-sphere with constant curvature 1. Consider the following variation of \tilde{c} ,

$$g(s, t) = \exp(s \cdot \|Z(t)\| \cdot \tilde{E}(t)),$$

where \tilde{E} is a unit parallel field along \tilde{c} orthogonal to \tilde{c}' . Notice that $g_s(t) = g(s, t)$ is a proper variation as well. Let $Y_t(s) = \frac{\partial g}{\partial t}$ be the vector field along the geodesics $g_t(s)$. Then

$$\|Y_t(0)\| = \left\| \frac{\partial g}{\partial t}(0) \right\| = \|\tilde{c}'(t)\| = 1.$$

We would like to apply the Sturm comparison theorem to the vector fields $X_t(s)$ and $Y_t(s)$. To do this we need to estimate the derivatives of these vector fields. A straightforward calculation yields

$$\begin{aligned} \|Y_t(0)\|' &= \frac{\partial}{\partial s}(\|Y_t(s)\|)_{s=0} = 0, \\ \|X_t(0)\|' &= \frac{\partial}{\partial s}(\|X_t(s)\|)_{s=0} = 0. \end{aligned}$$

As always each of the Jacobi fields Y_t and X_t can be decomposed into the sum of a normal and a tangential Jacobi field along f_t resp. g_t . The tangential parts of the Jacobi fields are given by $m_t s f_t'(s)$ respectively by $m_t s g_t'(s)$ with $m_t = \frac{\partial}{\partial t} \log(\|Z(t)\|)$. Since the geodesics g_t and f_t have the same speed it follows from the previous lemma that the norm of the normal part of X_t is bounded above by the norm of the normal part of Y_t . Combining the two statements we get $\|X_t(s)\| \geq \|Y_t(s)\|$ and so

$$\pi \leq \int \|Y_t(s)\| dt \leq \int \|X_t(s)\| dt$$

for all s . By construction equality holds at $s = 1$. Notice that this implies in particular that c_σ is a geodesic up to parameterization, as otherwise one could have replaced c_σ by

a nearby curve of length $< \pi$. The equality discussion implies $\|X_t(s)\| = \|Y_t(s)\|$ for $s \in [0, 1]$. This shows that the strip parameterized by f is intrinsically isometric to the strip parameterized by g . Furthermore the equality discussion also shows that the ambient curvature of the slice defined by f is 1 as well. By the Gauss Lemma we have the basic relation

$$0 = \mathbf{sec}_{\text{intrinsic}} - \mathbf{sec}_{\text{ambient}} = \langle B(X_1, X_1), B(X_2, X_2) \rangle - \|B(X_1, X_2)\|^2,$$

where $X_1 = \frac{\partial f}{\partial t}$ and $X_2 = \frac{\partial f}{\partial s}$ are linear independent vector fields on the slice. By construction, the curves $f_t(s)$ (t fixed) are geodesics so $B(X_2, X_2) = 0$. Therefore, $B(X_1, X_2)$ vanishes as well. It remains to show that $B(X_1, X_1) = 0$. Notice that in the 'model' slice parameterized by g the geodesics of length π connecting the end points of \tilde{c} pass through every point of the slice. Since the two slices are intrinsically isometric the same holds for the slice parameterized by f . Notice that these intrinsic geodesic have to be geodesics of the ambient manifold as well because otherwise we could find in M nearby curves which are strictly shorter. \square

1.2. Existence of a closed geodesic of length at least 2π and index 1.

The next lemma is a consequence of the long homotopy lemma; the first part is well known.

Lemma 1.4. *Let M be a complete, simply connected compact manifold with $\text{sec} \leq 1$ and injectivity radius less than π . Then there is a closed geodesic $c: [0, \ell] \rightarrow M$ of length $\ell \geq 2 \text{conj } M \geq 2\pi$ whose index in the free loop space of M is at most 1.*

Furthermore there is no free homotopy $c_s(t)$ with $s \in [0, 1]$ and $t \in [0, \ell]$ such that each of the following statements is true.

- (i) c_s is a closed geodesic of length $2 \text{conj } M$ and $c_0 = c$.
- (ii) The index of the closed geodesic c_s in the free loop space of M is at least 1, $s \in [0, 1]$.
- (iii) The index of the closed geodesic c_1 is at least 2.

For the proof of the above lemma we will apply the standard degenerate Morse Lemma, see for example [GM69].

Lemma 1.5. *Let B be a manifold of dimension b , $E: B \rightarrow \mathbf{R}$ a smooth proper function and let $p \in B$ be a critical point of E . Then we can find a neighborhood U of p and a map $x: U \rightarrow V \subset \mathbf{R}^b$ with $x(p) = 0$ such that*

$$E = E(p) - x_1^2 - \dots - x_\lambda^2 + x_{\lambda+1}^2 + \dots + x_{b-d}^2 + h(x_{b-d+1}, \dots, x_b)$$

where λ denotes the index of p , d the nullity of p and h is a smooth function.

Notice that any critical point of E in U is necessarily contained in $L := x^{-1}(0 \times \mathbf{R}^d)$. After replacing U by a smaller neighborhood we may assume that V is a bounded convex set. For the proof of Lemma 1.4 we make the following observation: Suppose that $\lambda > 0$. Let p^i be a sequence of points converging to p with $E(p^i) < E(p)$, $h(t) \in U$ a path of critical points with $E(h(t)) = E(p)$ and $h(0) = p$, $t \in [0, 1]$. Then there is a path $h_i(t)$ with $p^i = h_i(0)$, $E(h_i(t)) < E(p)$ such that $h_i(1)$ converges to $h(1)$. In order to construct h_i we will identify U with V via x . Consequently we write p_j^i instead of $x_j(p^i)$. First take a path given by the straight segment from p^i to another point q^i with $q_j^i = p_j^i$ for $j \geq 2$ and $|q_1^i| > \varepsilon$, where $\varepsilon > 0$ is a number which we can choose independent of i . Next consider the path from q^i to $\tilde{q}^i := (q_1^i, 0, \dots, 0)$ given by a straight segment. Since p_i converges to 0 it is easy to see that the energy E along this path stays strictly below $E(p)$ for almost all i . Next consider the path $\tilde{h}_i(t) = \tilde{q}^i + h(t)$ from \tilde{q}^i to $\tilde{q}^i + h(1)$ along which the energy E is constant. Finally, along the straight line from $h(1) + \tilde{q}^i$ to $h(1) + \frac{\tilde{q}^i}{i}$ the energy stays strictly below $E(p)$. Thus we may choose h_i as the composition of these paths for almost all i . Finally we can define h_i as the point curve for the finitely many remaining i .

Proof of Lemma 1.4. In this proof all curves are parameterized on $[0, 1]$. As usual we consider the energy functional on the free loop space of ΩM of M i.e., we define the energy of a piecewise smooth loop $c: [0, 1] \rightarrow M$ as

$$E(c) := \frac{1}{2} \int_0^1 \|\dot{c}(t)\|^2 dt.$$

For any value of $e \in (0, \infty)$ we let $\Omega M^{<e}$ (respectively $\Omega M^{\leq e}$) denote the loops in ΩM of energy $< e$ (respectively $\leq e$).

By Klingenberg's general injectivity radius estimate there is a closed geodesic of length $2 \operatorname{inj} M < 2\pi$. Furthermore the long homotopy lemma tells us that there is no free null homotopy of this geodesic contained in $\Omega M^{<e_0}$ with $e_0 = 2(\operatorname{conj} M)^2$. Consequently $\Omega M^{<e_0}$ has at least two connected components.

We now assume, on the contrary, that the statement of the Lemma is false. The first step is to verify that $\Omega M^{\leq e_0}$ is connected. In fact if $\Omega M^{\leq e_0}$ were not connected, then we could find an $\varepsilon > 0$ such that $\Omega M^{<e_0+\varepsilon}$ is not connected either. Since the statement of the lemma is assumed to be false, any closed geodesic of energy $> e_0$ has index at least 2. Thus it follows by the usual degenerate Morse theory argument (namely approximating E by Morse functions) that the free loop space itself is not connected either. This is a contradiction as M is simply connected. Hence, $\Omega M^{\leq e_0}$ is connected.

As usual given an e_1 one can find partition $0 < t_1 < \dots < t_k < 1$ of the unit interval such that for all $e \leq e_1$ the sub level $\Omega M^{\leq e}$ is homotopically equivalent to the subset of broken geodesics $B^{\leq e}$ contained in $\Omega M^{\leq e}$, whose points of non differentiability are points in the partition. We put $e_1 = e_0 + 1$ and fix a sufficiently fine partition. Then $B^{<e_1}$ is a finite dimensional submanifold and if we restrict the energy function to $B^{<e_1}$, then the critical point as well as the indices do not change. We have shown that $B^{\leq e_0}$ is connected while $B^{<e_0}$ is not.

Let C denote the set of closed geodesics of length $2(\operatorname{conj} M)$ and put $S = B^{<e_0} \cup C$. In other words S is obtained from $B^{\leq e_0}$ by removing all non-critical points from the boundary. Since $B^{\leq e_0}$ is connected it is easy to see that S is connected as well.

Let S_1 be an open and closed subset of $B^{<e_0}$ and suppose that neither S_1 nor its complement $S_2 := B^{<e_0} \setminus S_1$ is void. Let \bar{S}_i denote the closure of S_i in S . By construction $\bar{S}_1 \cup \bar{S}_2 = S$ and $\bar{S}_1 \cap \bar{S}_2$ is a nonempty subset of C . We claim that if $c \in \bar{S}_1 \cap \bar{S}_2$ then $C_0 \subset \bar{S}_1 \cap \bar{S}_2$ where C_0 is the path connected component of c in C . Since the index of any critical point in C is at least 1 this is an immediate consequence of the observation that we made after Lemma 1.5.

Let $C' \subset C$ denote the set of closed geodesics of index ≥ 2 . By assumption each path connected component can be represented by a geodesic of index ≥ 2 . By the previous argument $\bar{S}_1 \cap \bar{S}_2$ has a nontrivial intersection with C' . But this shows that $S' := B^{<e_0} \cup C'$ is connected as well. Since all points in C' have index at least 2 this implies as before that $B^{<e_0}$ is connected which is a contradiction. \square

1.3. All geodesics in M are closed.

We will assume from now on that M is a complete, simply connected manifold with $\operatorname{sec} \leq 1$ and positive spherical rank. In this subsection we want to prove that all geodesics of M are closed. There is nothing to prove if M is Blaschke. Thus we may assume $\operatorname{inj} M < \pi$. Then there is a geodesic γ satisfying the conclusion of Lemma 1.4. Using the existence of γ as a starting point we will show that all geodesics are contained in a totally geodesic immersed 2-sphere of constant curvature one.

Consider the set \mathcal{R} of geodesic segments $c : [0, \pi] \rightarrow M$ for which π is a conjugate point with multiplicity 1. Note that \mathcal{R} is an open set in the set of all geodesic segments of length π . Indeed, the multiplicity of a limit of geodesics of length π can only go up. Moreover, since the spherical rank is positive, every such geodesic has multiplicity at least 1. Finally \mathcal{R} is not empty as it contains $\gamma|_{[0, \pi]}$.

Lemma 1.6. *Suppose $c \in \mathcal{R}$ and J is a Jacobi field along c that vanishes at 0 and π with $\|J'(0)\| = 1$. Put $v := \dot{c}(0)$ and $w = J'(0)$.*

Then there is a unique maximal number $s_m \in (0, \pi]$ such that

$$h(s, t) = \exp(t(\cos(s)v + \sin(s)w)), \text{ with } t \in [0, \pi] \text{ and } s \in [-s_m, s_m]$$

parameterizes a totally geodesic immersed spherical slice of constant curvature 1. If $s_m < \pi$ then one of the boundary geodesics $h(\pm s_m, t)$ is not contained in \mathcal{R} . If $s_m = \pi$ then the image of h is a totally geodesic immersed 2-sphere.

Proof. We first want to show that we can indeed choose $s_m > 0$. Let $v \in T_p M$ be the initial vector of c . Denote by $T_p^1 M$ the unit sphere in $T_p M$. Consider the map $\phi : T_p^1 M \rightarrow M$ given by $w \mapsto \exp(\pi w)$, and set $q = \phi(v)$. Since the spherical rank is positive, ϕ has a singular differential everywhere. At v the kernel of the differential is precisely one dimensional since $c \in \mathcal{R}$. It follows that ϕ is of constant rank in a neighborhood of v . By the implicit function theorem, the fibers $\phi^{-1}(x)$ are 1-dimensional submanifolds for x in a neighborhood of q . The curve $\phi^{-1}(q)$ defines a variation of geodesics $f(s, t)$ of $c(t)$ of length π with constant starting and ending point. From Proposition 1.3 we deduce that c is contained in a spherical slice. Since there is up to constant factor only one Jacobi field along c that vanishes at 0 and π this spherical slice necessarily contains $h(s, t)$ for all $(t, s) \in [0, \pi] \times [-s_m, s_m]$ provided that $s_m > 0$ is chosen sufficiently small. This proves $s_m > 0$ and clearly we may choose $s_m \in (0, \pi]$ maximal.

Consider next the case of $s_m < \pi$. Notice that $J_{\pm}(t) = \frac{\partial h}{\partial s}(\pm s_m, t)$ is a Jacobi field along the boundary geodesic $c_{\pm}(t) = h(\pm s_m, t)$. If $c_{\pm} \in \mathcal{R}$, then the previous argument shows that we can increase s_m contradicting our choice of s_m . \square

Consider again the geodesic γ from Lemma 1.4. Since M has positive spherical rank and γ has index at most 1, it follows that $l(\gamma) = 2\pi$. We parameterize γ on the interval $[-\pi, \pi]$.

Proposition 1.7. *The closed geodesic γ of length 2π and index 1 is contained in a totally geodesic, isometrically immersed \mathbf{S}^2 of constant curvature 1.*

Proof. We construct two continuous vector fields X_+ and X_- along γ that are defined as follows:

- (i) $X_+(t) = 0$ for all $t < 0$ and $X_-(t) = 0$ for all $t > 0$.
- (ii) $X_+(t)$ is a non-vanishing Jacobi field on $[0, \pi]$ and $X_-(t)$ is a non-vanishing Jacobi field on $[-\pi, 0]$.

Note that the index form of γ restricted to the two dimensional subspace spanned by X_+ and X_- is 0. Since the index of γ is at most 1, it follows that the two dimensional

space spanned by X_+ and X_- must contain a Jacobi field $J = aX_+ + bX_-$. By the equality discussion of the Rauch comparison theorem, X_\pm looks like $\sin(t)E_\pm$ on the intervals where they are non-zero; here E_\pm are parallel vector fields. Since X is a smooth Jacobi field on $[-\pi, \pi]$ it follows by computing X' that $J(t) = \sin(t)X(t)$, where $X = aE_+ = bE_-$ i.e., J is a periodic Jacobi field. Moreover, $X(t)$ is a closed, parallel vector field along γ such that $\sec(\gamma'(t), X(t)) = 1$ for all t .

The vector fields $\sin(t)X(t)$ and $\cos(t)X(t)$ are periodic Jacobi fields along γ . From the previous lemma it follows that there is an $\epsilon > 0$ such that

$$\exp(sX(t)), \quad t \in [-\pi, \pi] \text{ and } s \in [-\epsilon, \epsilon]$$

parameterizes a totally geodesic spherical tube. Notice that there are lots of closed geodesics in this tube. Every one of the closed geodesics in the tube is homotopic to γ via a homotopy γ_s satisfying the first two conditions of Lemma 1.4. By the same lemma it follows that the third condition must be violated i.e., each of the closed geodesics in the tube must have index one in the free loop space. Therefore, there is no obstruction to increase ϵ . In other words, we may choose $\epsilon = \pi/2$ and thus γ is contained in a totally geodesic immersed \mathbf{S}^2 of constant curvature 1. \square

It is important to notice that each of the closed geodesics in the constant curvature \mathbf{S}^2 constructed above has index 1 in the free loop space. This implies that along every geodesic $t = \pi$ is a conjugate point with multiplicity 1.

Proposition 1.8. *Suppose M has $\sec \leq 1$ and positive spherical rank. Then all geodesics in M are closed.*

Proof. Consider the following subsets of T^1M , the unit tangent bundle of M .

$$S_1 = \left\{ v \in T^1M \mid \begin{array}{l} v \text{ tangent to a totally geodesic immersed } \mathbf{S}_v^2 \text{ and all geodesics} \\ c: [0, \pi] \rightarrow \mathbf{S}_v^2 \text{ have a conjugate point of multiplicity 1 at } \pi. \end{array} \right\}$$

$$S_2 = \left\{ v \in T^1M \mid \begin{array}{l} v \text{ tangent to a totally geodesic immersed } \mathbf{S}_v^2 \text{ and all geodesics} \\ c: [0, 2\pi] \rightarrow \mathbf{S}_v^2 \text{ have index 1 in the free loop space } \Omega M. \end{array} \right\}$$

Let $v_0 \in T^1M$ denote the initial velocity vector of the closed geodesic γ of length 2π . Then one can see that $S_2 \subset S_1 \subset T^1M$. Furthermore S_2 is non-empty since it contains v_0 and S_2 is closed.

Next we claim that S_1 is open. Let $w \in S_1$ and let \mathbf{S}_w^2 be as in the definition of S_1 . Suppose a sequence $w_i \in T^1M$ converges to w . For i sufficiently large there is a unique spherical Jacobi field $\sin(t)X_i(t)$ along the geodesic $c_i(t) = \exp(tw_i)$, $t \in [0, \pi]$. Suppose for a moment that w_i

is not tangent to a totally geodesic immersed 2 sphere. By Lemma 1.6 w_i is tangent to a spherical slice such that one of the boundary geodesics is not contained in \mathcal{R} . Since a subsequence of the boundary geodesics converges to a geodesic in \mathbf{S}_w^2 and \mathcal{R} is open this is impossible. In other words w_i is tangent to a totally geodesic immersed sphere $\mathbf{S}_{w_i}^2$. Since the geodesics in $\mathbf{S}_{w_i}^2$ converge to geodesics in \mathbf{S}_w^2 we deduce that $w_i \in S_1$ for almost all i .

Next we finish up the proof of the proposition by showing $S_2 = M$. Suppose, on the contrary that $S_2 \neq M$. Choose a path $h(s) \in T^1M$ with $h(0) = v_0$ and $h(1) \in T^1M \setminus S_2$. Since S_1 is an open neighborhood of the closed set S_2 we may assume that $h(s) \in S_1$ for all $s \in [0, 1]$. Furthermore we may assume that $c_1(t) = \exp(th(1))$ ($t \in [0, 2\pi]$) is one of the closed geodesics in $\mathbf{S}_{h(1)}^2$ of index at least 2. Thus

$$c_s(t) = \exp(th(s)) \text{ for } t \in [0, 2\pi], s \in [0, 1]$$

defines a homotopy of closed geodesics of length 2π satisfying all three conditions of Lemma 1.4 which is a contradiction. \square

1.4. M is a Blaschke manifold.

We now show that M is Blaschke i.e., $\text{inj}(M) \geq \pi$. By the previous subsection all geodesics of M are closed. This enables us to apply the following index parity theorem (cf. [Wil01]):

Theorem 1.9 (Wilking). *Let M^n be an oriented Riemannian manifold all of whose geodesics are closed, and let $c : [0, 1] \rightarrow M$ be a closed geodesic. Then the index of c in the free loop space of M is even if M is odd-dimensional and it is odd if M is even-dimensional.*

Proof that M is Blaschke. We argue by contradiction and assume that $\text{inj } M < \pi$. Then by the generalized injectivity radius estimate of Klingenberg, there exists a shortest closed geodesic α of length $2 \text{inj } M$. By the Long Homotopy Lemma (Lemma 1.1) we know that α is not freely null homotopic in the space of all curves of energy less than $2\pi^2$ (or all curves of length shorter than 2π). In particular, it follows that the curves of length less than 2π form a disconnected set such that α and the point curve lie in distinct components. On the component containing α , the energy functional attains a minimum at α and hence α has index 0 in the free loop space. However, if M is even-dimensional, then the index of α must be odd by the index parity theorem which leads to a contradiction. If M is odd dimensional consider the closed geodesic γ constructed in Section 1.2. By construction, γ has index exactly 1. Once again this contradicts the index parity theorem as the index of γ is required to be even. \square

2. A SPECIAL CASE OF THE BLASCHKE CONJECTURE

We have shown that if M is a complete, simply connected, Riemannian manifold with $\text{sec} \leq 1$ and with positive spherical rank, then $\text{inj } M = \text{diam } M = \pi$ i.e., M is a Blaschke manifold with extremal diameter (and injectivity radius). In this section we complete the proof of the main theorem by proving the following proposition which is special case of the Blaschke conjecture.

Proposition 2.1. *Let M be a simply connected Blaschke manifold with $\text{sec} \leq 1$ and extremal value of diameter (and injectivity radius) equal to π . Then M is isometric to a compact, rank one symmetric space.*

As we noted in the introduction, the above result has already been proved by Rovenskii and Toponogov in [RT96]; they also use Toponogov's theorem below. The proof given here has the slight virtue of being shorter. To be more precise we reduce the problem to two older theorems: one due to V. Toponogov (see [Top74]) and the other due to M. Berger (see [Ber78]).

Theorem 2.2 (Toponogov). *Let M be a complete, simply connected, Riemannian manifold such that $\text{sec}_M \leq 1$. Suppose M contains a closed geodesic γ of length 2π and index $k - 1$. Then γ is contained in an isometrically embedded, totally geodesic, sphere \mathbf{S}^k of constant curvature 1.*

It should be noted that the proof of Toponogov's theorem is not very hard in the special case that M is Blaschke. In that case the map

$f_p: T_p^1 M \rightarrow M$, $v \mapsto \exp(\pi v)$ has constant rank for all $p \in M$, namely the rank equals $n - k$, where $k - 1$ is the index of a closed geodesic of length 2π or equivalently the multiplicity of the conjugate point at π . Thus the fibers of f_p are submanifolds and using Proposition 1.3 it is easy to see that the fibers are great spheres of dimension $k - 1$. Furthermore one can use Proposition 1.3 to see that $\exp(\mathbf{R} \cdot f_p^{-1}(q))$ is a totally geodesic sphere of dimension k and of constant curvature 1.

Before we state Berger's theorem some notation is required. An SC_{2a} -manifold is one in which every geodesic is simply closed and periodic with period $2a$. It is well known that a Blaschke manifold

with $\text{inj} = \text{diam} = a$ is an SC_{2a} -manifold (cf. [Bes78, Chapter 7]). In our situation, we have a Blaschke manifold which happens to be an $SC_{2\pi}$ -manifold, so every geodesic is simply closed with period 2π .

Given two points p, q at distance π on a Riemannian manifold M , let $\Sigma_\pi(p, q)$ denote the set of all shortest geodesics from p to q . It is shown in [Bes78] that in this case $\Sigma_\pi(p, q)$ is homeomorphic to a sphere \mathbf{S}^k , where $k - 1$ is the index of a closed geodesic through p and q . If for all tuples (p, q)

with $d(p, q) = \pi$, the set $\Sigma_\pi(p, q)$ is totally geodesic, then, following Berger [Ber78], M is called a *totally geodesic Blaschke manifold*.

Theorem 2.3 (Berger). *Let M be a simply connected, totally geodesic Blaschke manifold. Then M is isometric to a compact, rank one symmetric space i.e., isometric to \mathbf{S}^n , $\mathbf{CP}^{\frac{n}{2}}$, $\mathbf{HP}^{\frac{n}{4}}$ or CaP^2 .*

Proof of Proposition 2.1. If a complete, simply connected Riemannian manifold has $\text{sec} \leq 1$ and positive spherical rank, then it is an $SC_{2\pi}$ Blaschke manifold. So every geodesic in M is simply closed, has length 2π and index at least 1.

Pick any geodesic γ of length 2π and index $k - 1 \geq 1$ and pick two points p, q on γ which are π apart. By Toponogov's theorem γ is contained in a totally geodesic, isometrically embedded \mathbf{S}^k of constant curvature 1. By construction we have $\mathbf{S}^k \subset \Sigma_\pi(p, q)$. But $\Sigma_\pi(p, q)$ is also a k -dimensional sphere because of the index estimate on γ . Moreover, $\Sigma_\pi(p, q)$ is

connected which implies $\mathbf{S}^k = \Sigma_\pi(p, q)$ and M is a totally geodesic Blaschke manifold. By Berger's theorem M must be isometric to a CROSS. \square

3. SOME EXAMPLES

In this section we explore another notion of spherical rank that is analogous to Hamenstädt's notion of hyperbolic rank. More precisely, consider the following:

Definition. Let M^n be a compact Riemannian manifold. Suppose along every geodesic $\gamma(t)$ in M there exists a normal Jacobi field $J(t)$ such that $\mathbf{sec}(\gamma'(t), J(t)) = 1$. If $\mathbf{sec} \leq 1$, we say that M^n has *weak upper spherical rank* at least 1. If $\mathbf{sec} \geq 1$, we say that M^n has *weak lower spherical rank* at least 1.

In the case of stronger notions of rank we have seen various rigidity results, most of them implying that the universal cover must be locally isometric to a symmetric space. We direct the reader to Table 1 for some of the known results. As is indicated there (metric) rigidity no longer holds for weak spherical rank (upper or lower). The main purpose of this section is to verify that claim.

Compact manifolds		
Curvature bound	$\forall \gamma$, there exists a Jacobi field J s.t. $\mathbf{sec}(J, \gamma')$ is extremal	$\forall \gamma$, there exists a parallel vector field E s.t. $\mathbf{sec}(E, \gamma')$ is extremal.
$\mathbf{sec} \leq 0$?	the universal cover of M is symmetric or isometric to a product; cf. [Bal85], [BS87].
$\mathbf{sec} \leq -1$	M is isometric to a locally symmetric space, cf. [Ham90].	\Rightarrow
$\mathbf{sec} \leq 1$	non-symmetric examples exist.	M is locally isometric to a CROSS. [ibid.]
$\mathbf{sec} \geq 1$	non-symmetric examples exist.	?
$\mathbf{sec} \geq 0$	\Leftarrow	there are simply connected, irreducible examples which are not homeomorphic to symmetric spaces, cf. [Heintze],[SS90].
$\mathbf{sec} \geq -1$?	?

TABLE 1. Rank rigidity for various curvature bounds.

3.1. The Berger spheres. We present here, briefly, the construction of the so called Berger spheres. This is the scaling of the round metric on a sphere; we will specifically look at \mathbf{S}^3 . The Berger spheres are important examples and originally were constructed by M. Berger in [Ber78] to show that in odd dimensions, Klingenberg’s injectivity radius estimate fails if the pinching is below $\frac{1}{9}$.

One may regard the round 3-sphere as the unit sphere in the quaternions \mathbf{H} . The Lie algebra is spanned i, j and k . These vectors are orthonormal with respect to a induced bi-invariant metric on \mathbf{S}^3 of constant curvature 1.

The Berger metric is obtained upon scaling the fibers of the Hopf fibration $\mathbf{S}^3 \rightarrow \mathbf{S}^3/\mathbf{S}^1 = \mathbf{S}^2$ where \mathbf{S}^1 is the image of the 1-parameter group $\exp(ti)$. More precisely, consider a family of left invariant metrics g_η on \mathbf{S}^3 which are defined by $g_\eta(i, j) = g_\eta(i, k) = g_\eta(k, j) = 0$, $\|j\|_{g_\eta} = \|k\|_{g_\eta} = 1$ and $\|i\|_{g_\eta} = \eta$.

It is then routine to check that $\mathbf{sec}(\frac{1}{\eta}X_1, sX_2 + \sqrt{1-s^2}X_3) = \eta^2$, $\mathbf{sec}(X_2, X_3) = 4 - 3\eta^2$, and η^2 and $4 - 3\eta^2$ are minimum and maximum of the sectional curvature; the Hopf fiber has length $2\pi\eta$ (for the calculation, see for instance [CE75], Example 3.35). Note that if $\eta > 1$, then the range of curvatures is $[4 - 3\eta^2, \eta^2]$.

In order to find a non-symmetric example with weak lower spherical rank take $\eta < 1$ i.e., shrink the Hopf fiber and then normalize the metric to make the lower bound 1. If γ is a vertical geodesic then all planes containing $\gamma'(t)$ have curvature 1. If γ is not vertical then the Killing field corresponding to the Hopf field i induces a Jacobi field J along γ with $\mathbf{sec}(J, \gamma') = 1$. Notice that J is not necessarily normal but one may replace J by its normal part.

For weak upper spherical rank, take $\eta > 1$ i.e., enlarge the Hopf fiber (we may choose $\eta < \frac{2}{\sqrt{3}}$ to ensure positive curvature). Normalize again to make the upper curvature bound 1 and as before it follows that the weak upper spherical rank is 1.

Of course the Berger spheres in higher dimensions also have positive weak upper or lower spherical rank. Thus there are non-symmetric examples in all odd dimensions above 2.

It remains unclear whether the assumption on weak spherical rank implies that the manifold is topologically a symmetric space. We leave that as a question for further study.

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