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Torus actions on homotopy complex projective spaces

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Abstract We prove that an effective action of a torus T on a homotopy $\mathbb{C}P^m$ is linear if $m < 4 \cdot \mathrm{rk}(T) - 1$. Examples show that the bound is optimal. Combining this with a theorem of Hattori we conclude that the total Pontrjagin class of such a manifold is given by the usual formula $(1 + x^2)^{m+1}$.

1 Introduction

In this paper we study torus actions of large rank on homotopy complex projective spaces. Let T be a torus and let M be a homotopy $\mathbb{C}P^m$ with smooth T-action. We fix a generator $x \in H^2(M; \mathbb{Z})$ and denote by γ a T-equivariant complex line bundle over M with first Chern class equal to x.

By restricting the tangent bundle TM and the line bundle γ to T-fixed points one obtains a set of T-representations. The action is called linear if for some linear T-action on $\mathbb{C}^{m+1}$ the induced action on the canonical line bundle over $\mathbb{C}P^m$ gives the same representations (see Section 2 for a precise definition).

In [6,8] Petrie constructed examples of S^1-actions on homotopy $\mathbb{C}P^{4r-1}$'s which are not linear. These "exotic actions" extend to effective actions by a torus of rank r. The linear actions and Petrie's exotic actions are the only known actions on homotopy complex projective spaces. In this paper we show

Theorem 1.1. *Let M be a homotopy $\mathbb{C}P^m$ with smooth effective action by a torus T of rank r. If $m < 4r - 1$ then the T-action is linear.*

Petrie conjectured that the total Pontrjagin class of a homotopy $\mathbb{C}P^m$ is standard (i.e. of the form $(1 + x^2)^{m+1}$) if the manifold admits a smooth effective S^1-action.

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He proved his conjecture for S^1-actions which extend to smooth effective actions by a torus of rank m [7]. Hattori has shown that the conjecture is true for linear S^1-actions [2]. Combining his result with Theorem 1.1 gives

Corollary 1.2. *Let M be a homotopy $\mathbb{C}P^m$ which admits a smooth effective action by a torus T of rank r. If $m < 4r - 1$ then the total Pontrjagin class of M is standard.* □

It follows from simply-connected surgery theory that for fixed $m \geq 3$ the set of diffeomorphism classes of homotopy $\mathbb{C}P^m$'s is infinite and partitioned into finite subsets by their total Pontrjagin class. Hence, Corollary 1.2 implies

Corollary 1.3. *For $m < 4r - 1$, $m \neq 2$, the class of homotopy $\mathbb{C}P^m$'s which admit a smooth effective action by a torus T of rank r contains only finitely many diffeomorphism types.* □

Remark 1.4. a) It is known that a compact manifold M which is homotopically equivalent to $\mathbb{C}P^m$ has a standard Pontrjagin class if and only if M is tangentially homotopically equivalent to M, i.e., there is a homotopy equivalence $h\colon M \to \mathbb{C}P^m$ such that the pull back bundle $h^*T\mathbb{C}P^m$ is stabily isomorphic to the tangent bundle TM. This in turn is equivalent to saying that for some $k > 0$ the manifolds $M \times \mathbb{R}^k$ and $\mathbb{C}P^m \times \mathbb{R}^k$ are diffeomorphic.

b) The paper was partly motivated by [11], where it is shown that a simply connected positively curved n–dimensional ($n \neq 7$) Riemannian manifold (M^n, g) that supports an isometric effective action of a r-dimensional torus with $r \geq \frac{n}{4} + 1$ is homeomorphic to $\mathbb{S}^n$ or $\mathbb{H}P^{n/4}$ or homotopically equivalent to $\mathbb{C}P^{n/2}$. By Corollary 1.2 the conclusion can be improved to tangentially homotopically equivalent.

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2 Basic properties

In this section we recall basic properties of torus actions on integral cohomology $\mathbb{C}P^m$'s. These are used in the next section to prove a slightly more general version of Theorem 1.1. As a general reference we recommend [6, 1, 4].

Recall that a smooth closed manifold M is an integral cohomology $\mathbb{C}P^m$ if $H^*(M;\mathbb{Z}) = \mathbb{Z}[x]/(x^{m+1})$, where x has degree 2. Any homotopy $\mathbb{C}P^m$ is an integral cohomology $\mathbb{C}P^m$. The converse is true for simply-connected manifolds.

Assume a torus T acts smoothly on M. Let $\gamma \to M$ be a complex line bundle over M with $c_1(\gamma) = x$ (the "Hopf bundle"). Hattori and Yoshida have shown that the T-action lifts to γ and any two lifts differ by a complex one-dimensional T-representation [3]. We fix a lift.

Let X be a connected component of the fixed point manifold M^T and let $pt \in X$. By restricting γ to pt one obtains a complex one-dimensional T-representation χ_X, the *Hopf representation* at X. At the fixed point pt the tangent bundle splits as a direct sum of the tangent space of X and the *normal representation* N_X which, by definition, is the normal bundle of X restricted to pt. Since X is a trivial T-space the isomorphism class of the real representation N_X and the isomorphism class of the complex representation χ_X are independent of the choice of the point pt in X. Note also that for two connected components $X, Y \subset M^T$ the isomorphism class of the complex representation $\chi_Y \cdot \chi_X^{-1}$ is independent of the lift by [3].

It will be convenient to use the following notation: Let W_1, W_2 be two T-representations (real or complex) and $\widetilde{T} \subset T$.

- $W_1 \cong_{\mathbb{R}} W_2$ if W_1 and W_2 are isomorphic as real T-representations.
- $W_1 \cong_{\mathbb{C}} W_2$ if W_1 and W_2 are complex representations which are isomorphic as complex T-representations.
- $W_1 \cong_{(\widetilde{T},\mathbb{R})} W_2$ (resp. $W_1 \cong_{(\widetilde{T},\mathbb{C})} W_2$) if W_1 and W_2 are isomorphic as real (resp. complex) $\widetilde{T}$-representations.

The localization theorem for cohomology or K-theory leads to strong relations between M and M^T (cf. [6]; [1], Ch. VII; [4], Ch. VI).

Proposition 2.1. *Let M be an integral cohomology $\mathbb{C}P^m$ with smooth T-action. Then the following holds:*

1. *The restriction of $x \in H^2(M;\mathbb{Z})$ to a connected component $X \subset M^T$ generates $H^*(X;\mathbb{Z})$. In particular, X is an integral cohomology complex projective space.*
2. $\dim_{\mathbb{C}} M + 1 = \sum_{X \subset M^T} (\dim_{\mathbb{C}} X + 1)$.
3. *Two connected components $X, Y \subset M^T$ are equal if and only if $\chi_X \cong_{\mathbb{C}} \chi_Y$.*
4. *The normal representation N_X, $X \subset M^T$, admits a T-equivariant complex structure such that*

$$\det N_X \cong_{\mathbb{C}} \det \left(\bigoplus_{Y \subset M^T, Y \neq X} (\dim_{\mathbb{C}} Y + 1) \cdot \chi_Y \cdot \chi_X^{-1} \right).$$

□

Here $\dim_{\mathbb{C}}$ denotes half of the dimension. If M is the complex projective space $\mathbb{C}P^m$ and the action is induced by a linear T-action on $\mathbb{C}^{m+1}$ then the normal representation N_X at $X \subset M^T$ is isomorphic to the direct sum of the representations $(\dim_{\mathbb{C}} Y+1)\cdot\chi_Y\cdot\chi_X^{-1}$, where the sum runs over the connected components $Y \subset M^T$ different from X. For an integral cohomology $\mathbb{C}P^m$ we make the

Definition 2.2. *The T-action is linear if for every $X \subset M^T$*

$$N_X \cong_{\mathbb{R}} \bigoplus_{Y \subset M^T, Y \neq X} (\dim_{\mathbb{C}} Y + 1) \cdot \chi_Y \cdot \chi_X^{-1}. \tag{1}$$

Remarks 2.3. 1. If (1) holds then the tangential representations $T_{pt}M$, $pt \in X \subset M^T$, and the Hopf representations χ_X are isomorphic to the ones for a T-action on the canonical line bundle over $\mathbb{C}P^m$ induced by a linear action on $\mathbb{C}^{m+1}$.

2. A smooth S^1-action on an integral cohomology projective space M is linear if M^{S^1} has less than 4 connected components [9,10,12].

3. A smooth torus action on an integral cohomology projective space M is linear if $\dim_{\mathbb{C}} M < 3$ (to see this apply the last remark and Proposition 2.1, Part 2, to a suitable S^1-subgroup).

Next we extend the notion of linearity to certain normal subspaces. Let $\widetilde{T}$ be a subtorus of T, $\widetilde{V} := (N_X)^{\widetilde{T}}$ and $\widetilde{F}$ the connected component of $M^{\widetilde{T}}$ which contains X.

Definition 2.4. *The T-action is linear on $\widetilde{V}$ if*

$$\widetilde{V} \cong_{\mathbb{R}} \bigoplus_{Y \subset \widetilde{F}^T, Y \neq X} (\dim_{\mathbb{C}} Y + 1) \cdot \chi_Y \cdot \chi_X^{-1}. \tag{2}$$

Note that T acts linearly on M if and only if T acts linearly on N_X for every connected component $X \subset M^T$.

We shall be interested in the case where $\widetilde{T}$ is the identity component of the kernel of an irreducible T-subrepresentation $\widetilde{R} \subset N_X$ (i.e. $\widetilde{T}$ is the maximal subtorus of T acting trivially on $\widetilde{R}$). Let $T_1, \dots, T_k$ denote the different subtori arising in this way and let $V_j := (N_X)^{T_j}$. Note that N_X is the direct sum of the V_j and that two irreducible representations R, $\widetilde{R}$ belong to the same V_j if and only if their kernels have the same identity components. The next lemma shows that linearity can be detected locally.

Lemma 2.5. *The T-action on M is linear if and only if the T-action is linear on $V_j \subset N_X$ for all X and all V_j.*

Proof. Assume the T-action on M is linear. By restricting to trivial T_j-representations in (1) one obtains

$$V_j \cong_{\mathbb{R}} \bigoplus_{Y \subset M^T, Y \neq X, \chi_Y \cong_{(T_j, \mathbb{C})} \chi_X} (\dim_{\mathbb{C}} Y + 1) \cdot \chi_Y \cdot \chi_X^{-1}.$$

Let F_j denote the connected component of M^{T_j} which contains X. By Proposition 2.1 $\chi_Y \cong_{(T_j, \mathbb{C})} \chi_X$ if and only if $Y \subset F_j$. Hence, T acts linearly on V_j.

Next assume that for all X and all $V_j \subset N_X$ the T-action is linear on V_j, i.e.

$$V_j \cong_{\mathbb{R}} \bigoplus_{Y \subset F_j^T, Y \neq X} (\dim_{\mathbb{C}} Y + 1) \cdot \chi_Y \cdot \chi_X^{-1}. \tag{3}$$

To show that the T-action on M is linear it suffices to show that T acts linearly on N_X. Consider a connected component $Y \subset M^T$ with $X \neq Y$. By Proposition 2.1 the representation $\chi_Y \cdot \chi_X^{-1}$ is nontrivial and hence the identity component of the kernel is a codimension one subtorus $\widetilde{T}$. Again by Proposition 2.1 X and Y are contained in the same component $\widetilde{F}$ of $M^{\widetilde{T}}$. This proves $\widetilde{T} = T_j$ and $\widetilde{F} = F_j$ for j suitable. Conversely if $Y \subset F_j$, then T_j is necessarily given by the identity component of the kernel of $\chi_Y \cdot \chi_X^{-1}$. In summary we can say that Y belongs to precisely one F_j. Also $N_X = \bigoplus_j V_j$. By summing up (3) it follows that T acts linearly on N_X. □

3 Proof of Theorem 1.1

In this section we prove Theorem 1.1 for integral cohomology $\mathbb{C}P^m$'s by induction on the rank of the action. In the induction step we will use the fact that a T-action is linear if some S^1-subgroup acts with low codimension.

Proposition 3.1. *Let M be an integral cohomology $\mathbb{C}P^m$ with smooth effective T-action. The T-action is linear if one of the following holds:*

1. $\operatorname{codim}_{\mathbb{C}} M^{S^1} < 3$ *for some* $S^1 \subset T$.
2. $\operatorname{codim}_{\mathbb{C}} M^{S^1} = 3$ *for some* $S^1 \subset T$ *and* $\dim_{\mathbb{C}} M \neq 3$.

Proof. For the S^1-subgroup itself linearity follows from work of Masuda, Tsukada-Washiyama, Wang, Yoshida and others: If M^{S^1} has at most 3 connected components then the S^1-action is linear [9,10,12]. By Proposition 2.1 this is the case if $\operatorname{codim}_{\mathbb{C}} M^{S^1} < 3$. If M^{S^1} has more than 3 connected components and if $\operatorname{codim}_{\mathbb{C}} M^{S^1} = 3$ then the number of connected components is 4 by Proposition 2.1. Masuda has shown that an S^1-action with 4 fixed point components is linear if the components don't have the same dimension (cf. [5], Lemma 5.4). Since $\dim_{\mathbb{C}} M \neq 3$ the fixed point component of complex codimension 3 has positive dimension. The other components are isolated fixed points by Proposition 2.1. Hence, T acts linearly by [5]. This completes the proof in the case that the rank of T is one.

So assume the rank of T is ≥ 2. Let $S^1 \subset T$ be as in the proposition and let $M_0^{S^1} \subset M^{S^1}$ be a component of minimal codimension. By the above S^1 acts linearly on M. To show linearity for T it suffices to show that the T-action is linear on $V_j \subset N_X$ for all $X \subset M^T$ and all V_j by Lemma 2.5.

We claim that T acts linearly on V_j if S^1 acts non-trivially on V_j or if $X \not\subset M_0^{S^1}$. Assume first that S^1 acts non-trivially on V_j. In particular, T is generated by T_j and S^1. Since S^1 acts linearly on M the S^1-action is linear on V_j by Lemma 2.5. Since T_j acts trivially on F_j (notation as in the proof of Lemma 2.5) it follows that T acts linearly on V_j. Next assume $V_j \subset N_X$ and $X \not\subset M_0^{S^1}$. By the previous case we may assume that S^1 acts trivially on V_j and hence by Proposition 2.1 $\dim_{\mathbb{C}} F_j \leq 2$. The claim now follows from Remark 2.3.

Next consider the representations $V_l \subset N_X$, where X is a component of $M_0^{S^1} \cap M^T$. By the above claim we may assume that V_l is tangential to the fixed point component $M_0^{S^1}$. Fix a connected component X_0 of M^T which is not contained in $M_0^{S^1}$, and fix the T-action on the Hopf bundle γ for which χ_{X_0} is a trivial T-representation.

Let $T_j \subset T$ be the identity component of the kernel of χ_X and let $V_j := N_X^{T_j}$. Since F_j contains X (by definition) and X_0 (apply Proposition 2.1, Part 3, to T_j) S^1 acts non-trivially on F_j. Let ν_{F_j} denote the normal bundle of $F_j \subset M$. By construction $V_l \subset \nu_{F_j}$. To understand the T-action on $V_l \subset N_X^{S^1} = (\nu_{F_j |pt})^{S^1}$ we will compare $\nu_{F_j |pt}$ with $\nu_{F_j |q_0}$, $q_0 \in X_0$, and use the established linearity at X_0.

Note that $\nu_{F_{j|pt}} \cong_{(T_j,\mathbb{R})} \nu_{F_{j|q_0}} \cong_{\mathbb{R}} N_{X_0} \ominus N_{X_0}^{T_j}$. Since T acts linearly on N_{X_0}

$$N_{X_0} \cong_{\mathbb{R}} \bigoplus_{Y\subset M^T, Y\neq X_0} (\dim_{\mathbb{C}} Y+1)\cdot\chi_Y\cdot\chi_{X_0}^{-1} \cong_{(T_j,\mathbb{C})} \bigoplus_{Y\subset M^T, Y\neq X_0} (\dim_{\mathbb{C}} Y+1)\cdot\chi_Y\cdot\chi_X^{-1}.$$

By Proposition 2.1, Part 3, $\chi_Y \cdot \chi_X^{-1}$ is a trivial T_j-representation if and only if $Y \subset F_j$. Hence,

$$\nu_{F_{j|pt}} \cong_{(T_j,\mathbb{R})} N_{X_0} \ominus N_{X_0}^{T_j} \cong_{(T_j,\mathbb{R})} \bigoplus_{Y\subset M^T, Y\not\subset F_j} (\dim_{\mathbb{C}} Y + 1)\cdot \chi_Y \cdot \chi_X^{-1}. \quad (4)$$

Recall that T acts linearly on all subrepresentations V_i of $\nu_{F_{j|pt}} \ominus (\nu_{F_{j|pt}})^{S^1}$. Hence,

$$\nu_{F_{j|pt}} \ominus (\nu_{F_{j|pt}})^{S^1} \cong_{\mathbb{R}} \bigoplus_{Y\subset F_i^T,\ i\neq j,\ Y\neq X,\ (V_i)^{S^1}\neq V_i} (\dim_{\mathbb{C}} Y + 1)\cdot \chi_Y \cdot \chi_X^{-1}. \quad (5)$$

Note that $\{Y \subset F_i^T \mid Y \neq X,\ V_i \subset (\nu_{F_{j|pt}})^{S^1}\} \subset \{Y \subset M^T \mid Y \not\subset F_j\}$ is the complement of the index set of the direct sum in (5). Thus (4) and (5) imply

$$(\nu_{F_{j|pt}})^{S^1} \cong_{(T_j,\mathbb{R})} \bigoplus_{Y\subset F_i^T, Y\neq X,\ V_i\subset(\nu_{F_{j|pt}})^{S^1}} (\dim_{\mathbb{C}} Y + 1)\cdot \chi_Y \cdot \chi_X^{-1}.$$

Since $\chi_Y \cdot \chi_X^{-1}$ is a trivial S^1-representation for $Y \subset F_i^T$, $V_i \subset (\nu_{F_{j|pt}})^{S^1}$, the isomorphism extends to an isomorphism of T-representations

$$(\nu_{F_{j|pt}})^{S^1} \cong_{\mathbb{R}} \bigoplus_{Y\subset F_i^T, Y\neq X,\ V_i\subset(\nu_{F_{j|pt}})^{S^1}} (\dim_{\mathbb{C}} Y + 1)\cdot \chi_Y \cdot \chi_X^{-1}. \quad (6)$$

By restricting to the trivial T_l-subrepresentations (for fixed $l \neq j$) on both sides of (6) it follows that T acts linearly on any $V_l \subset (\nu_{F_{j|pt}})^{S^1} = N_X^{S^1}$. This completes the proof of the proposition. □

Theorem 3.2. *Let M be an integral cohomology $\mathbb{C}P^m$ with effective smooth action by a torus T of rank r. If $m < 4r - 1$ then the T-action is linear.*

Proof. We prove the statement for almost effective actions (i.e. actions with finite kernel) by induction on the rank of the action. If $r = 1$ then $\dim_{\mathbb{C}} M \leq 2$ and the T-action is linear as pointed out before. So assume $r \geq 2$.

Let X be a connected component of M^T. By Lemma 2.5 it suffices to show that T acts linearly on every $V_i \subset N_X$.

For fixed $V_i \subset N_X$ let V_{max} be a maximal element (with respect to inclusion) of the set of representations

$$\{V \subset N_X \mid V_i \subset V,\ V^{S^1} = V \text{ for some subgroup } S^1 \subset T\}$$

and let S^1 denote the subtorus of T which acts trivially on V_{max}. Since V_{max} is maximal a complementary subtorus $\widetilde{T} \subset T$ of rank $r-1$ acts almost effectively on V_{max}.

Let $M_0^{S^1}$ denote the connected component of M^{S^1} which contains X. By Proposition 2.1 $M_0^{S^1}$ is an integral cohomology complex projective space. Note that $\widetilde{T}$ acts almost effectively on $M_0^{S^1}$.

If $\dim_{\mathbb{C}} M_0^{S^1} < 4r-5$ then $\widetilde{T}$ acts linearly on $M_0^{S^1}$ by the induction hypothesis. In this case $\widetilde{T}$ acts linearly on V_i which implies the same for the T-action on V_i. If $\mathrm{codim}_{\mathbb{C}} M_0^{S^1} < 3$ then T acts linearly on M by Proposition 3.1. The remaining case ($\dim_{\mathbb{C}} M = 4r-2 \neq 3$ and $\mathrm{codim}_{\mathbb{C}} M_0^{S^1} = 3$) also follows from Proposition 3.1. □

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