Torus actions on homotopy complex projective spaces

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Abstract We prove that an effective action of a torus $T$ on a homotopy $\mathbb{C}P^m$ is linear if $m < 4 \cdot \text{rk}(T) - 1$. Examples show that the bound is optimal. Combining this with a theorem of Hattori we conclude that the total Pontrjagin class of such a manifold is given by the usual formula $(1 + x^2)^{m+1}$.

1 Introduction

In this paper we study torus actions of large rank on homotopy complex projective spaces. Let $T$ be a torus and let $M$ be a homotopy $\mathbb{C}P^m$ with smooth $T$-action. We fix a generator $x \in H^2(M; \mathbb{Z})$ and denote by $\gamma$ a $T$-equivariant complex line bundle over $M$ with first Chern class equal to $x$.

By restricting the tangent bundle $TM$ and the line bundle $\gamma$ to $T$-fixed points one obtains a set of $T$-representations. The action is called linear if for some linear $T$-action on $\mathbb{C}P^m$ the induced action on the canonical line bundle over $\mathbb{C}P^m$ gives the same representations (see Section 2 for a precise definition).

In [6,8] Petrie constructed examples of $S^1$-actions on homotopy $\mathbb{C}P^{4r-1}$s which are not linear. These “exotic actions” extend to effective actions by a torus of rank $r$. The linear actions and Petrie’s exotic actions are the only known actions on homotopy complex projective spaces. In this paper we show

**Theorem 1.1.** Let $M$ be a homotopy $\mathbb{C}P^m$ with smooth effective action by a torus $T$ of rank $r$. If $m < 4r - 1$ then the $T$-action is linear.

Petrie conjectured that the total Pontrjagin class of a homotopy $\mathbb{C}P^m$ is standard (i.e. of the form $(1 + x^2)^{m+1}$) if the manifold admits a smooth effective $S^1$-action.

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He proved his conjecture for $S^1$-actions which extend to smooth effective actions by a torus of rank $m$ [7]. Hattori has shown that the conjecture is true for linear $S^1$-actions [2]. Combining his result with Theorem 1.1 gives

**Corollary 1.2.** Let $M$ be a homotopy $\mathbb{C}P^m$ which admits a smooth effective action by a torus $T$ of rank $r$. If $m < 4r - 1$ then the total Pontrjagin class of $M$ is standard. □

It follows from simply-connected surgery theory that for fixed $m \geq 3$ the set of diffeomorphism classes of homotopy $\mathbb{C}P^m$’s is infinite and partitioned into finite subsets by their total Pontrjagin class. Hence, Corollary 1.2 implies

**Corollary 1.3.** For $m < 4r - 1$, $m \neq 2$, the class of homotopy $\mathbb{C}P^m$’s which admit a smooth effective action by a torus $T$ of rank $r$ contains only finitely many diffeomorphism types. □

**Remark 1.4.** a) It is known that a compact manifold $M$ which is homotopically equivalent to $\mathbb{C}P^m$ has a standard Pontrjagin class if and only if $M$ is tangentially homotopically equivalent to $\mathbb{C}P^m$, i.e., there is a homotopy equivalence $h: M \to \mathbb{C}P^m$ such that the pull back bundle $h^*T\mathbb{C}P^m$ is stably isomorphic to the tangent bundle $TM$. This in turn is equivalent to saying that for some $k > 0$ the manifolds $M \times \mathbb{R}^k$ and $\mathbb{C}P^m \times \mathbb{R}^k$ are diffeomorphic.

b) The paper was partly motivated by [11], where it is shown that a simply connected positively curved $n$–dimensional ($n \neq 7$) Riemannian manifold $(M^n, g)$ that supports an isometric effective action of a $r$-dimensional torus with $r \geq \frac{n}{2} + 1$ is homeomorphic to $S^n$ or $\mathbb{H}P^{n/4}$ or homotopically equivalent to $\mathbb{C}P^{n/2}$. By Corollary 1.2 the conclusion can be improved to tangentially homotopically equivalent.

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2 Basic properties

In this section we recall basic properties of torus actions on integral cohomology $\mathbb{C}P^m$’s. These are used in the next section to prove a slightly more general version of Theorem 1.1. As a general reference we recommend [6,1,4].

Recall that a smooth closed manifold $M$ is an integral cohomology $\mathbb{C}P^m$ if $H^*(M; \mathbb{Z}) = \mathbb{Z}[x]/(x^{m+1})$, where $x$ has degree 2. Any homotopy $\mathbb{C}P^m$ is an integral cohomology $\mathbb{C}P^m$. The converse is true for simply-connected manifolds.

Assume a torus $T$ acts smoothly on $M$. Let $\gamma \to M$ be a complex line bundle over $M$ with $c_1(\gamma) = x$ (the “Hopf bundle”). Hattori and Yoshida have shown that the $T$-action lifts to $\gamma$ and any two lifts differ by a complex one-dimensional $T$-representation [3]. We fix a lift.

Let $X$ be a connected component of the fixed point manifold $M^T$ and let $pt \in X$. By restricting $\gamma$ to $pt$ one obtains a complex one-dimensional $T$-representation $\chi_X$. 

□
the Hopf representation at \( X \). At the fixed point \( pt \) the tangent bundle splits as a direct sum of the tangent space of \( X \) and the normal representation \( N_X \) which, by definition, is the normal bundle of \( X \) restricted to \( pt \). Since \( X \) is a trivial \( T \)-space the isomorphism class of the real representation \( N_X \) and the isomorphism class of the complex representation \( \chi_X \) are independent of the choice of the point \( pt \) in \( X \).

Note also that for two connected components \( X, Y \subset M^T \) the isomorphism class of the complex representation \( \chi_Y \cdot \chi_X^{-1} \) is independent of the lift by \( \beta \).

It will be convenient to use the following notation: Let \( W_1, W_2 \) be two \( T \)-representations (real or complex) and \( \tilde{T} \subset T \).

- \( W_1 \cong_R W_2 \) if \( W_1 \) and \( W_2 \) are isomorphic as real \( T \)-representations.
- \( W_1 \cong_C W_2 \) if \( W_1 \) and \( W_2 \) are complex representations which are isomorphic as complex \( T \)-representations.
- \( W_1 \cong_{(\tilde{T},R)} W_2 \) (resp. \( W_1 \cong_{(\tilde{T},C)} W_2 \)) if \( W_1 \) and \( W_2 \) are isomorphic as real (resp. complex) \( \tilde{T} \)-representations.

The localization theorem for cohomology or \( K \)-theory leads to strong relations between \( M \) and \( M^T \) (cf. [6]; [1], Ch. VII; [4], Ch. VI).

**Proposition 2.1.** Let \( M \) be an integral cohomology \( \mathbb{C}P^m \) with smooth \( T \)-action. Then the following holds:

1. The restriction of \( x \in H^2(M; \mathbb{Z}) \) to a connected component \( X \subset M^T \) generates \( H^*(X; \mathbb{Z}) \). In particular, \( X \) is an integral cohomology complex projective space.
2. \( \dim_{\mathbb{C}} M + 1 = \sum_{X \subset M^T} (\dim_{\mathbb{C}} X + 1) \).
3. Two connected components \( X, Y \subset M^T \) are equal if and only if \( \chi_X \cong_{\mathbb{C}} \chi_Y \).
4. The normal representation \( N_X, X \subset M^T \), admits a \( T \)-equivariant complex structure such that

\[
\det N_X \cong_{\mathbb{C}} \det \left( \bigoplus_{Y \subset M^T, Y \neq X} (\dim_{\mathbb{C}} Y + 1) \cdot \chi_Y \cdot \chi_X^{-1} \right).
\]

\[\square\]

Here \( \dim_{\mathbb{C}} \) denotes half of the dimension. If \( M \) is the complex projective space \( \mathbb{C}P^m \) and the action is induced by a linear \( T \)-action on \( \mathbb{C}^{m+1} \) then the normal representation \( N_X \) at \( X \subset M^T \) is isomorphic to the direct sum of the representations \( (\dim_{\mathbb{C}} Y + 1) \cdot \chi_Y \cdot \chi_X^{-1} \), where the sum runs over the connected components \( Y \subset M^T \) different from \( X \). For an integral cohomology \( \mathbb{C}P^m \) we make the

**Definition 2.2.** The \( T \)-action is linear if for every \( X \subset M^T \)

\[
N_X \cong_{\mathbb{R}} \bigoplus_{Y \subset M^T, Y \neq X} (\dim_{\mathbb{C}} Y + 1) \cdot \chi_Y \cdot \chi_X^{-1}. \tag{1}
\]

**Remarks 2.3.**

1. If (1) holds then the tangential representations \( T_{\beta}M, pt \in X \subset M^T \), and the Hopf representations \( \chi_X \) are isomorphic to the ones for a \( T \)-action on the canonical line bundle over \( \mathbb{C}P^m \) induced by a linear action on \( \mathbb{C}^{m+1} \).
2. A smooth \( S^1 \)-action on an integral cohomology projective space \( M \) is linear if \( M^{S^1} \) has less than 4 connected components [9, 10, 12].
3. A smooth torus action on an integral cohomology projective space $M$ is linear if $\dim \mathbb{C} M < 3$ (to see this apply the last remark and Proposition 2.1, Part 2, to a suitable $S^1$-subgroup).

Next we extend the notion of linearity to certain normal subspaces. Let $\tilde{T}$ be a subtorus of $T$, $\tilde{V} := (N_X)^\tilde{T}$ and $\tilde{F}$ the connected component of $M^\tilde{T}$ which contains $X$.

**Definition 2.4.** The $T$-action is linear on $\tilde{V}$ if
\[
\tilde{V} \cong_{\mathbb{R}} \bigoplus_{Y \subset \tilde{F}, Y \neq X} (\dim \mathbb{C} Y + 1) \cdot \chi_Y \cdot \chi_X^{-1}. \tag{2}
\]

Note that $T$ acts linearly on $M$ if and only if $T$ acts linearly on $N_X$ for every connected component $X \subset M^T$.

We shall be interested in the case where $\tilde{T}$ is the identity component of the kernel of an irreducible $T$-subrepresentation $\tilde{R} \subset N_X$ (i.e. $\tilde{T}$ is the maximal subtorus of $T$ acting trivially on $\tilde{R}$). Let $T_1, \ldots, T_k$ denote the different subtori arising in this way and let $V_j := (N_X)^{T_j}$. Note that $N_X$ is the direct sum of the $V_j$ and that two irreducible representations $R, \tilde{R}$ belong to the same $V_j$ if and only if their kernels have the same identity components. The next lemma shows that linearity can be detected locally.

**Lemma 2.5.** The $T$-action on $M$ is linear if and only if the $T$-action is linear on $V_j \subset N_X$ for all $X$ and all $V_j$.

**Proof.** Assume the $T$-action on $M$ is linear. By restricting to trivial $T_j$-representations in (1) one obtains
\[
V_j \cong_{\mathbb{R}} \bigoplus_{Y \subset M^T, Y \neq X, Y \cong (T_j, \mathbb{C}) \chi_X} (\dim \mathbb{C} Y + 1) \cdot \chi_Y \cdot \chi_X^{-1}. \tag{3}
\]

Let $F_j$ denote the connected component of $M^{T_j}$ which contains $X$. By Proposition 2.1 $\chi_Y \cong (T_j, \mathbb{C}) \chi_X$ if and only if $Y \subset F_j$. Hence, $T$ acts linearly on $V_j$.

Next assume that for all $X$ and all $V_j \subset N_X$ the $T$-action is linear on $V_j$, i.e.
\[
V_j \cong_{\mathbb{R}} \bigoplus_{Y \subset F_j, Y \neq X} (\dim \mathbb{C} Y + 1) \cdot \chi_Y \cdot \chi_X^{-1}. \tag{3}
\]

To show that the $T$-action on $M$ is linear it suffices to show that $T$ acts linearly on $N_X$. Consider a connected component $Y \subset M^T$ with $X \neq Y$. By Proposition 2.1 the representation $\chi_Y \cdot \chi_X^{-1}$ is nontrivial and hence the identity component of the kernel is a codimension one subtorus $\tilde{T}$. Again by Proposition 2.1 $X$ and $Y$ are contained in the same component $\tilde{F}$ of $M^\tilde{T}$. This proves $\tilde{T} = T_j$ and $\tilde{F} = F_j$ for $j$ suitable. Conversely if $Y \subset F_j$, then $T_j$ is necessarily given by the identity component of the kernel of $\chi_Y \cdot \chi_X^{-1}$. In summary we can say that $Y$ belongs to precisely one $F_j$. Also $N_X = \bigoplus_j V_j$. By summing up (3) it follows that $T$ acts linearly on $N_X$. \qed
3 Proof of Theorem 1.1

In this section we prove Theorem 1.1 for integral cohomology $\mathbb{C}P^m$’s by induction on the rank of the action. In the induction step we will use the fact that a $T$-action is linear if some $S^1$-subgroup acts with low codimension.

Proposition 3.1. Let $M$ be an integral cohomology $\mathbb{C}P^m$ with smooth effective $T$-action. The $T$-action is linear if one of the following holds:

1. $\text{codim}_\mathbb{C} M^{S^1} < 3$ for some $S^1 \subset T$.
2. $\text{codim}_\mathbb{C} M^{S^1} = 3$ for some $S^1 \subset T$ and $\dim_\mathbb{C} M \neq 3$.

Proof. For the $S^1$-subgroup itself linearity follows from work of Masuda, Tsukada-Washiyama, Wang, Yoshida and others: If $M^{S^1}$ has at most 3 connected components then the $S^1$-action is linear [9,10,12]. By Proposition 2.1 this is the case if $\text{codim}_\mathbb{C} M^{S^1} < 3$. If $M^{S^1}$ has more than 3 connected components and $\text{codim}_\mathbb{C} M^{S^1} = 3$ then the number of connected components is 4 by Proposition 2.1. Masuda has shown that an $S^1$-action with 4 fixed point components is linear if the components don’t have the same dimension (cf. [5], Lemma 5.4). Since $\dim_\mathbb{C} M \neq 3$ the fixed point component of complex codimension 3 has positive dimension. The other components are isolated fixed points by Proposition 2.1. Hence, $T$ acts linearly by [5]. This completes the proof in the case that the rank of $T$ is one.

So assume the rank of $T$ is $\geq 2$. Let $S^1 \subset T$ be as in the proposition and let $M_0^{S^1} \subset M^{S^1}$ be a component of minimal codimension. By the above $S^1$ acts linearly on $M$. To show linearity for $T$ it suffices to show that the $T$-action is linear on $V_j \subset N_X$ for all $X \subset M^T$ and all $V_j$ by Lemma 2.5.

We claim that $T$ acts linearly on $V_j$ if $S^1$ acts non-trivially on $V_j$ or if $X \not\subset M_0^{S^1}$. Assume first that $S^1$ acts non-trivially on $V_j$. In particular, $T$ is generated by $T_j$ and $S^1$. Since $S^1$ acts linearly on $M$ the $S^1$-action is linear on $V_j$ by Lemma 2.5. Since $T_j$ acts trivially on $F_j$ (notation as in the proof of Lemma 2.5) it follows that $T$ acts linearly on $V_j$. Next assume $V_j \subset N_X$ and $X \not\subset M_0^{S^1}$. By the previous case we may assume that $S^1$ acts trivially on $V_j$ and hence by Proposition 2.1 $\dim_\mathbb{C} F_j \leq 2$. The claim now follows from Remark 2.3.

Next consider the representations $V_j \subset N_X$, where $X$ is a component of $M_0^{S^1} \cap M^T$. By the above claim we may assume that $V_j$ is tangential to the fixed point component $M_0^{S^1}$. Fix a connected component $X_0$ of $M^T$ which is not contained in $M_0^{S^1}$, and fix the $T$-action on the Hopf bundle $\gamma$ for which $\chi_{X_0}$ is a trivial $T$-representation.

Let $T_j \subset T$ be the identity component of the kernel of $\chi_X$ and let $V_j := N_X^{T_j}$. Since $F_j$ contains $X$ (by definition) and $X_0$ (apply Proposition 2.1, Part 3, to $T_j$) $S^1$ acts non-trivially on $F_j$. Let $v_{F_j}$ denote the normal bundle of $F_j \subset M$. By construction $V_j \subset v_{F_j}$. To understand the $T$-action on $V_j \subset N_X^{S^1} = (v_{F_j} |_{\text{pt}})^{S^1}$ we will compare $v_{F_j} |_{\text{pt}}$ with $v_{F_j} |_{q_0}$, $q_0 \in X_0$, and use the established linearity at $X_0$. 

Note that \( \nu_{F_j|PT} \cong (T_j, R) \). Since \( T \) acts linearly on \( N_{X_0} \),

\[
N_{X_0} \cong \bigoplus_{Y \subset MT, Y \neq X_0} (\dim Y + 1) \cdot \chi_Y \cdot \chi_X^{-1}.
\]

By Proposition 2.1, Part 3, \( \chi_Y \cdot \chi_X^{-1} \) is a trivial \( T_j \)-representation if and only if \( Y \subset F_j \). Hence,

\[
u_{F_j|PT} \cong (T_j, R) \bigoplus_{Y \subset MT, Y \neq F_j} (\dim Y + 1) \cdot \chi_Y \cdot \chi_X^{-1}. \tag{4}
\]

Recall that \( T \) acts linearly on all subrepresentations \( V_i \) of \( \nu_{F_j|PT} \). Hence,

\[
u_{F_j|PT} \cong (T_j, R) \bigoplus_{Y \subset MT, Y \neq F_j} (\dim Y + 1) \cdot \chi_Y \cdot \chi_X^{-1}. \tag{5}
\]

Note that \( \{ Y \subset F_T^j \mid Y \neq X, V_i \subset (\nu_{F_j|PT})^S \} \subset \{ Y \subset MT \mid Y \neq F_j \} \) is the complement of the index set of the direct sum in (5). Thus (4) and (5) imply

\[
(\nu_{F_j|PT})^S \cong (T_j, R) \bigoplus_{Y \subset F_T^j, Y \neq X, V_i \subset (\nu_{F_j|PT})^S} (\dim Y + 1) \cdot \chi_Y \cdot \chi_X^{-1}.
\]

Since \( \chi_Y \cdot \chi_X^{-1} \) is a trivial \( S^1 \)-representation for \( Y \subset F_T^j, V_i \subset (\nu_{F_j|PT})^S \), the isomorphism extends to an isomorphism of \( T \)-representations

\[
(\nu_{F_j|PT})^S \cong (T_j, R) \bigoplus_{Y \subset F_T^j, Y \neq X, V_i \subset (\nu_{F_j|PT})^S} (\dim Y + 1) \cdot \chi_Y \cdot \chi_X^{-1}. \tag{6}
\]

By restricting to the trivial \( T_l \)-subrepresentations (for fixed \( l \neq j \)) on both sides of (6) it follows that \( T \) acts linearly on any \( V_i \subset (\nu_{F_j|PT})^S \) = \( N_X^S \). This completes the proof of the proposition. \( \square \)

**Theorem 3.2.** Let \( M \) be an integral cohomology \( \mathbb{C}^m \) with effective smooth action by a torus \( T \) of rank \( r \). If \( m < 4r - 1 \) then the \( T \)-action is linear.

**Proof.** We prove the statement for almost effective actions (i.e. actions with finite kernel) by induction on the rank of the action. If \( r = 1 \) then \( \dim M \leq 2 \) and the \( T \)-action is linear as pointed out before. So assume \( r \geq 2 \).

Let \( X \) be a connected component of \( MT \). By Lemma 2.5 it suffices to show that \( T \) acts linearly on every \( V_i \subset N_X \).

For fixed \( V_i \subset N_X \) let \( V_{max} \) be a maximal element (with respect to inclusion) of the set of representations

\[
\{ V \subset N_X \mid V_i \subset V, V^S = V \} \text{ for some subgroup } S^1 \subset T\}
\]
and let $S^1$ denote the subtorus of $T$ which acts trivially on $V_{\text{max}}$. Since $V_{\text{max}}$ is maximal a complementary subtorus $\tilde{T} \subset T$ of rank $r - 1$ acts almost effectively on $V_{\text{max}}$.

Let $M_0^{S^1}$ denote the connected component of $M^{S^1}$ which contains $X$. By Proposition 2.1 $M_0^{S^1}$ is an integral cohomology complex projective space. Note that $\tilde{T}$ acts almost effectively on $M_0^{S^1}$.

If $\dim \mathbb{C} M_0^{S^1} < 4r - 5$ then $\tilde{T}$ acts linearly on $M_0^{S^1}$ by the induction hypothesis. In this case $\tilde{T}$ acts linearly on $V_i$ which implies the same for the $T$-action on $V_i$.

If $\text{codim} \mathbb{C} M_0^{S^1} < 3$ then $T$ acts linearly on $M$ by Proposition 3.1. The remaining case ($\dim \mathbb{C} M = 4r - 2 \neq 3$ and $\text{codim} \mathbb{C} M_0^{S^1} = 3$) also follows from Proposition 3.1.

□

References


