Regularity questions for polyharmonic maps

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partially based on joint work with Christoph Scheven, Andreas Nerf, Felix Zorn, and Frédéric Louis de Longueville

$u : M \to N$ $\quad$ M, N Riem mfds., compact,

$N \subseteq \mathbb{R}^{n+\tilde{k}}$, $\dim M = m$

harmonic maps are stationary points of

$$E(u) = \frac{1}{2} \int_M |Du|^2 \, dx$$

here for simplicity $M = \Omega \subseteq \mathbb{R}^m$ bdd domain

$N = S^n \subseteq \mathbb{R}^{n+1}$

$u_t : \Omega \to S^n$, $u_0 = u$, $\frac{\partial}{\partial t}\big|_{t=0} u_t(x) = \varphi(x) \in T_x S^n$.

$$0 = \delta E(u; \varphi) := \frac{d}{dt}\Big|_{t=0} E(u_t) = \int_M Du \cdot D\varphi \, dx$$

$$= -\int_M \Delta u \cdot \varphi \, dx \qquad \forall \varphi \text{ with } \varphi(x) = T_x S^n$$

$\Rightarrow$ $\Delta u \perp T S^n$, $\quad \Delta u = \lambda(x) u(x)$

$\underset{|u| \equiv 1}{\Rightarrow}$ $\lambda = u \cdot \Delta u = \operatorname{div}(u \cdot Du) - Du \cdot Du = -|Du|^2$

$$\boxed{\Delta u + |Du|^2 u = 0}$$

$$\Delta u - 2\,\mathrm{tr}(\mathrm{II}^N \circ u)(Du, Du) = 0 \quad \text{for general } N.$$

bi-harmonic maps crit. points of $(u: M \to N \subseteq \mathbb{R}^{n+\tilde{k}})$

$$E^2(u) := \frac{1}{2}\int_M |D^2 u|^2 dx \qquad \left(\text{or } \int_M |\Delta u|^2 dx, \right.$$

$$\left.\text{or } \int_M |\nabla D u|^2 dx\right)$$

from $\Omega \to S^n$

$$\boxed{\Delta^2 u + (\underbrace{|\Delta u|^2 + 2|D^2 u|^2}_{D^2u \# D^2 u} + 4\underbrace{Du \cdot D\Delta u}_{Du \# D^3 u})u = 0}$$

(extrinsically) polyharmonic maps crit. pts. of

$$E^k(u) = \frac{1}{2}\int |D^k u|^2 dx$$

$$\Delta^k u + \left(\sum_{j=1}^{k} D^j u \# D^{2k-j} u\right) u = 0$$

natural Sobolev space for that problem:

$$W^{k,2}(\Omega, N) = \{u \in L^2(\Omega, \mathbb{R}^{n+\tilde{k}}) : u(x) \in N \text{ a.e. } x,$$
$$D^j u \in L^2 \text{ for } 1 \le j \le k\}$$

Hélein's trick (Evans' trick) $\boxed{\Delta u + |Du|^2 u = 0}$

rewrite that as

$$\Delta u^k + \underbrace{\sum_j (\overbrace{u^k \nabla u^j - \underbrace{u^j \nabla u^k) \cdot \overbrace{\nabla u^j}^{\in L^2}}_{\sum_j \ldots \equiv 0 \text{ as } |u|^2 = 1}}^{\in L^2}}_{\text{a divergence-free vector field}}$$

$$\mathrm{div}(u^k \nabla u^j - u^j \nabla u^k) = \cancel{\nabla u^k \cdot \nabla u^j} + \cancel{u^k \underbrace{\Delta u^j}_{= |Du|^2 u^j}} - \cancel{\nabla u^k \cdot \nabla u^j} - \cancel{u^j \underbrace{\Delta u^k}_{= |Du|^2 u^k}}$$

$$= 0$$

Regularity results for harmonic maps ($k = 1$), biharmonic maps ($k = 2$), and polyharmonic maps ($k \geq 3$). For $u \in W^{k,2}(M, N)$, M open, $\dim M = m$, N closed submanifold of some $\mathbb{R}^n$.

$\boxed{\textbf{Weak solutions} \text{ are smooth are smooth if } m \leq 2k.}$

harmonic: *Hélein, Grüter, ...*
biharmonic: *Chang/Wang/Yang, Wang*
polyharmonic: *G./Scheven*

What if $m > 2k$?

$\boxed{\textbf{Minimizers} \text{ are smooth outside a closed set of dimension } \leq m - 2k - 1.}$

harmonic: *Schoen/Uhlenbeck, Giquinta/Giusti*
biharmonic: *Wang*
(polyharmonic: *G.*, very partial results)

$\boxed{\textbf{Stationary weak solutions} \text{ are smooth outside a closed set of dimension } \leq m - 2k.}$

harmonic: *Bethuel*
biharmonic: *Wang, Angelsberg, Struwe*

Let $\pi : E \to M$ be a vector bundle over M. For any connection $d + A$ on E, the Euclidean norm of the curvature $F_A := dA + \frac{1}{2}[A, A]$ is invariant under pointwise orthonormal changes of coordinates in the bundle fibres E_x. ("gauge invariance")

Uhleneck's gauge theorem. *If $A \in W^{1,m/2}$, and $\|F_A\|_{L^{m/2}} < \varepsilon$, there is a gauge transformation $g : M \to SO(n)$ such that $g^{-1}(d + A)g =: d + \Omega$ satisfies*

$$\delta\Omega = 0 \qquad \textit{and} \qquad \|\Omega\|_{W^{1,m/2}} \leq C\|F_\Omega\|_{L^{m/2}}.$$

apply it on $u_* TN$

on $A = (\langle e_\alpha , de_\beta \rangle)_{\alpha\beta}$

Conservation laws. Assume $m = 2k$.

Harmonic map type equations. *(Rivière)*

$$\boxed{-\Delta u = \Omega \cdot du}$$

Ω an $so(n)$-valued 1-form. If one can find $A \in W^{1,2} \cap L^\infty(U, GGL(n))$ and $B \in W^{1,2}(U, \mathbb{R}^{n\times n} \otimes \wedge^2\mathbb{R}^2$ such that

$$\boxed{dA - A\Omega = -\delta B,}$$

then the equation is equivalent to

$$\boxed{d(*A\, du - (*B) \wedge du) = 0.}$$

Biharmonic map type equations. *(Lamm/Rivière)*

$$\boxed{\Delta^2 u = \Delta\langle V, du\rangle + \delta(w\, du) + \langle W, du\rangle}$$

where $V \in W^{1,2}$, $w \in L^2$ and $W \in W^{-1,2}$

If $W = d\eta + F$ with $F \in L^{4/3,1}$ and $\eta \in L^2$ and η skew-symmetric, and if there are $A \in W^{2,2} \cap L^\infty(U, GL(n))$ and $B \in W^{1,\frac{4}{3}}(U, \mathbb{R}^{n\times n} \otimes \wedge^2\mathbb{R}^4)$ for which

$$\boxed{\Delta dA + (\Delta A)V - (dA)w + AW = \delta B,}$$

then the equation is equivalent to

$$\boxed{\delta\Big[d(A\Delta u) - 2dA\,\Delta u + \Delta A\, du - Aw\, du + dA\langle V, du\rangle - Ad\langle V, du\rangle - \langle B, du\rangle\Big] = 0.}$$

Theorem (de Longueville/G.) *Assume $m \geq 3$, $n \in \mathbb{N}$. Let coefficient functions be given as*

$$\begin{aligned}
w_k &\in W^{2k+2-m,2}(B^{2m}, \mathbb{R}^{n\times n}) \qquad \text{for } k \in \{0,\dots,m-2\},\\
V_k &\in W^{2k+1-m,2}(B^{2m}, \mathbb{R}^{n\times n} \otimes \wedge^1\mathbb{R}^{2m}) \qquad \text{for } k \in \{0,\dots,m-1\}, \text{ where}\\
V_0 &= d\eta + F,\\
\eta &\in W^{2-m,2}(B^{2m}, so(n)), \quad F \in W^{2-m,\frac{2m}{m+1},1}(B^{2m}, \mathbb{R}^{n\times n} \otimes \wedge^1\mathbb{R}^{2m}).
\end{aligned}$$

We consider the equation

$$\boxed{\Delta^m u = \sum_{k=0}^{m-1} \Delta^k \langle V_k, du\rangle + \sum_{k=0}^{m-2} \Delta^k \delta(w_k\, du).} \tag{1}$$

For this equation, the following statements hold.

(i) Let

$$\begin{aligned}
\theta \;:=\; & \sum_{k=0}^{m-2} \|w_k\|_{W^{2k+2-m,2}(B^{2m})} + \sum_{k=1}^{m-1} \|V_k\|_{W^{2k+1-m,2}(B^{2m})}\\
& + \|\eta\|_{W^{2-m,2}(B^{2m})} + \|F\|_{W^{2-m,\frac{2m}{m+1},1}(B^{2m})}.
\end{aligned}$$

There is $\theta_0 > 0$ such that whenever $\theta < \theta_0$, there are a function $A \in W^{m,2} \cap L^\infty(B_{1/4}; GL(n))$ and a distribution $B \in W^{2-m,2}(B_{1/4}, \mathbb{R}^{n\times n} \otimes \wedge^2\mathbb{R}^{2m})$ that solve

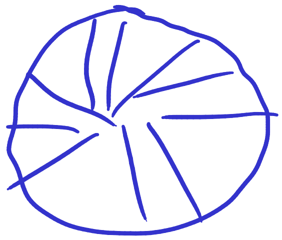
$$\boxed{\Delta^{m-1} dA + \sum_{k=0}^{m-1} (\Delta^k A) V_k - \sum_{k=0}^{m-2} (\Delta^k dA) w_k = \delta B.} \tag{2}$$

(ii) A function $u \in W^{m,2}(B_{1/2}, \mathbb{R}^n)$ solves (1) weakly on $B_{1/4}$ if and only if it is a distributional solution of the conservation law

$$\boxed{\begin{aligned}
0 \;=\; \delta\Big[& \sum_{\ell=0}^{m-1} (\Delta^\ell A)\Delta^{m-\ell-1} du - \sum_{\ell=0}^{m-2} (d\Delta^\ell A)\Delta^{m-\ell-1} u\\
& - \sum_{k=0}^{m-1}\sum_{\ell=0}^{k-1} (\Delta^\ell A)\Delta^{k-\ell-1} d\langle V_k, du\rangle + \sum_{k=0}^{m-1}\sum_{\ell=0}^{k-1} (d\Delta^\ell A)\Delta^{k-\ell-1}\langle V_k, du\rangle\\
& - \sum_{k=0}^{m-2}\sum_{\ell=0}^{k-2} (\Delta^\ell A) d\Delta^{k-\ell-1}\delta(w_k\, du) + \sum_{k=0}^{m-2}\sum_{\ell=0}^{k-2} (d\Delta^\ell A)\Delta^{k-\ell-1}\delta(w_k\, du)\\
& - \langle B, du\rangle \Big].
\end{aligned}} \tag{3}$$

(Here $d\Delta^{-1}\delta$ means the identity map.)

(iii) Every weak solution of (1) on B^{2m} is continuous on $B_{1/16}$ if the smallness condition $\theta < \theta_0$ holds.



$\mathbb{R}^n$

problem for N-values

"monontonicity formula"

$$\rho^{2-m} \int_{B_\rho(x_0)} |Du|^2 \, dx \le C$$

$$\rho^{4-m} \int_{B_\rho(x_0)} |D^2 u|^2 \, dx \le C + \dots$$

$|D^3 u|^2 \, dx$ Simon Blatt $m \le 20$.