Stable and unstable spectral inequalities

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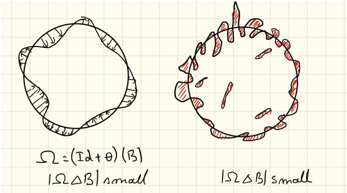
Isoperimetric inequalities in quantitative form

Fusco - Maggi - Pratelli 2008, $\Omega \subseteq \mathbb{R}^N$, finite measure

$$Per(\Omega) - Per(B) \geq C\,\mathcal{A}(\Omega)^2$$

The Fraenkel asymmetry

$$\mathcal{A}(\Omega) = \inf\{\frac{|\Omega \Delta B_x|}{|\Omega|} : x \in \mathbb{R}^N, |B_x| = |\Omega|\}.$$



The exponent 2 is sharp.

Isoperimetric inequalities in quantitative form

Long history

- Bernstein 1905 and Bonnesen 1924 : two dimensional convex sets - inner and outer radius
- Fuglede, 1989 : convex sets in higher dimension or near spherical domains (perturbation argument)
- Hall, 1992 : power 4 on the Fraenkel asymmetry and conjectures power 2
- New proof : Figalli - Maggi - Pratelli, 2010 : new proof by mass transportation techniques
- New proof : Cicalese - Leonardi, 2013 : new proof by using the selection principle

Quantitative spectral inequalities

- Neumann (Szegö-Weinberger) [Brasco - Pratelli, 2012]

$$\begin{cases} -\Delta u & = \mu u \text{ in } \Omega \\ \frac{\partial u}{\partial n} & = 0 \text{ on } \partial\Omega \end{cases}$$

$$\mu_1(B) - \mu_1(\Omega) \geq C\,\mathcal{A}(\Omega)^2$$

- Steklov eigenvalue (Brock-Weinstock) [Brasco - De Philippis - Ruffini, 2012]

$$\begin{cases} -\Delta u & = 0 \text{ in } \Omega \\ \frac{\partial u}{\partial n} & = \sigma u \text{ on } \partial\Omega \end{cases}$$

$$\sigma_1(B) - \sigma_1(\Omega) \geq C\,\mathcal{A}(\Omega)^2$$

The Dirichlet eigenvalue

- Dirichlet Laplacian (Faber-Krahn)

$$\begin{cases} -\Delta u & = \lambda u \text{ in } \Omega \\ u & = 0 \text{ on } \partial\Omega \end{cases}$$

$$\lambda_1(\Omega) - \lambda_1(B) \geq C\,\mathcal{A}(\Omega)^2$$

$$\lambda_1(\Omega) = \min_{u \in H_0^1(\Omega)} \frac{\int_\Omega |\nabla u|^2 dx}{\int_\Omega |u|^2 dx}, \quad \lambda_{1,q}(\Omega) = \min_{u \in H_0^1(\Omega)} \frac{\int_\Omega |\nabla u|^2 dx}{\left(\int_\Omega |u|^q dx\right)^{\frac{2}{q}}}.$$

Fundamental difference : this is a minimization problem !

The use of fixed test functions is useless...

The Dirichlet eigenvalue

- Melas, 1992, Hansen - Nadirashvili, 1994, simply connected $\mathbb{R}^2$, convex $\mathbb{R}^N$
- Fusco - Maggi - Pratelli, 2009, non-sharp power
- Brasco - De Philippis - Velichkov, 2015
  - it suffices to know a quantitative form for the torsional rigidity (Saint-Venant). By Kohler-Jobin, it works for all semi-linear eigenvalues, including Faber-Krahn

$$\frac{\lambda_1(\Omega)}{\lambda_1(B)} \geq \Big(\frac{\lambda_{1,1}(\Omega)}{\lambda_{1,1}(B)}\Big)^{\frac{2}{N+2}}$$

where $T(\Omega) = \frac{1}{\lambda_{1,1}(\Omega)} = -2E(\Omega)$ is the torsion

$$E(\Omega) = \min\{\frac{1}{2}\int_\Omega |\nabla u|^2 dx - \int_\Omega u dx : u \in H^1_0(\Omega)\}$$

  - use of the selection principle : solve auxiliary free boundary problem

A. Girouard, I. Polterovich, *Spectral geometry of the Steklov problem.* J. Spectr. Theory 7 (2017), no. 2, 321–359.

Open Problem 3. *Let Ω be a planar simply–connected domain such that the difference $2\pi - \sigma_1(\Omega)L(\partial\Omega)$ is small. Show that Ω must be close to a disk* (*in the sense of Fraenkel asymmetry or some other measure of proximity*).

L. Brasco, G. ; De Philippis, *Spectral inequalities in quantitative form.* Shape optimization and spectral theory, 201–281, De Gruyter Open, Warsaw, 2017.

4 BRASCO AND DE PHILIPPIS

1.3. **An open issue.** We conclude the Introduction by pointing out that at present no quantitative stability results are available for the case of the Bossel-Daners inequality. We thus start by formulating the following

Open problem 1. *Prove a quantitative stability estimate of the type* (1.5) *for the Bossel-Daners inequality for the first eigenvalue of the Robin Laplacian $\lambda_1(\Omega, \alpha)$.*

Weinstock inequality

$\Omega \subseteq \mathbb{R}^2$ smooth, simply connected

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial n} = \sigma u & \text{on } \partial\Omega. \end{cases}$$

Weinstock, *J. Rational Mech. Anal.*, 1954.

$$|\partial\Omega|\sigma_1(\Omega) \leq 2\pi = |\partial\mathbb{D}|\sigma_1(\mathbb{D}).$$

Proof : test functions of the disc $\mathbb{D}$, transplanted on Ω by conformal mapping.

Weinstock inequality

Weinstock, *Department of Math., Stanford Univ.*, Tech. Rep., 37, 1954.

If $\Omega \subseteq \mathbb{R}^2$ is convex,

$$2\pi - |\partial\Omega|\sigma_1(\Omega) \geq \frac{|\partial\Omega|}{\int_{\partial\Omega} |x|^2 dx} \int_{S^1} (h - \overline{h})^2 d\sigma.$$

Stability of Weinstock inequality

Gavitone, La Manna, Paoli,Trani 2019

- In $\mathbb{R}^2$, convex sets

$$2\pi - |\partial\Omega|\sigma_1(\Omega) \geq C\mathcal{A}^{\frac{5}{2}}(\Omega)$$

- in $\mathbb{R}^N$, convex sets

$$|\partial B|^{\frac{1}{N}}\sigma_1(B) - |\partial\Omega|^{\frac{1}{N}}\sigma_1(\Omega) \geq C\mathcal{A}^p(\Omega)$$

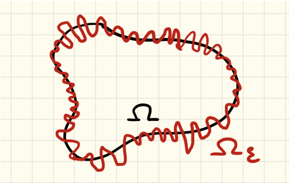
$p = 2$ for $N = 3$ and $p = \frac{N+1}{2}$ for $N \geq 4$.

The Weinstock inequality is genuinely unstable

Theorem (B. - Nahon, 2020)

Let $\Omega \subseteq \mathbb{R}^2$ open, smooth, simply connected. Then, there exists a perturbation Ω_ε such that

- *Ω_ε smooth and simply connected*
- *Ω_ε converges "uniformly" to Ω*
- *$|\partial\Omega_\varepsilon|$ is uniformly bounded*
- *$|\partial\Omega_\varepsilon|\sigma_1(\Omega_\varepsilon) \to 2\pi$.*



Ideas of the proof

Lemma (geometric perturbation)

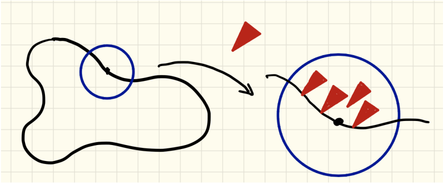
Let $\Omega \subseteq \mathbb{R}^2$, *smooth, simply connected, open and a perturbation* Ω_ε *such that*

- Ω_ε *satisfy a uniform cone condition*
- $\partial\Omega_\varepsilon$ *converges to* $\partial\Omega$ *(in the Hausdorff metric)*
- $\mathcal{H}^1\lfloor_{\partial\Omega_\varepsilon} \rightharpoonup \theta\mathcal{H}^1\lfloor_{\partial\Omega}$, *weakly-*$*$ *in the sense of measures*

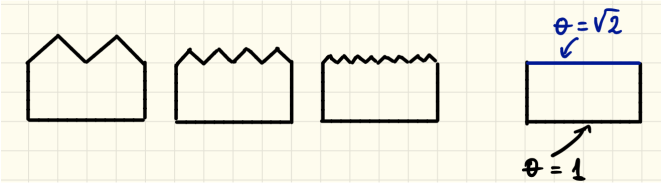
Then $\sigma_k(\Omega_\varepsilon) \to \sigma_k(\Omega, \theta)$, *where*

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial n} = \sigma\theta u & \text{on } \partial\Omega. \end{cases}$$

Uniform cone condition



Weak-∗ convergence of boundary measures



Key ingredient of the proof

- Control of the boundary trace along weak convergent sequences in H^1

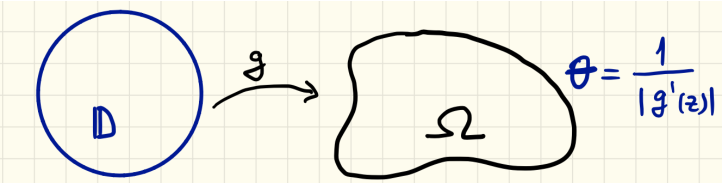
$$u_\varepsilon \in H^1(\mathbb{R}^2), u_\varepsilon \to u \text{ weakly in } H^1(\mathbb{R}^2)$$

$$\Longrightarrow \int_{\partial\Omega_\varepsilon} u_\varepsilon^2 d\sigma \to \int_{\partial\Omega} \theta u^2 d\sigma$$

From $\mathbb{D}$ to (Ω, θ)

Let $\Omega \subseteq \mathbb{R}^2$ be open, smooth, bounded, simply connected and $g : \mathbb{D} \to \Omega$ a conformal mapping. Let $\theta : \partial\Omega \to \mathbb{R}^+$

$$\theta(g(z)) = \frac{1}{|g'(z)|}.$$



Then

$$\sigma_k(\Omega, \theta) = \sigma_k(\mathbb{D}).$$

Existence of geometric perturbations

Lemma

Let $\Omega \subseteq \mathbb{R}^2$ *be open, smooth, bounded, simply connected and* $\theta : \partial\Omega \to [1, M]$ *continuous. Then there exists a perturbation* (Ω_ε) *of* Ω *satisfying :*

1. $\partial\Omega_\varepsilon$ *converges to* $\partial\Omega$ *(in the Hausdorff metric)*
2. *uniform cone condition*
3. *weak-$*$ convergence of the boundary measures* $\mathcal{H}^1\lfloor_{\partial\Omega_\varepsilon} \rightharpoonup \theta\mathcal{H}^1\lfloor_{\partial\Omega}$

Consequence

Theorem

$$|\partial\Omega_\varepsilon|\sigma_k(\Omega_\varepsilon) \to |\partial\mathbb{D}|\sigma_k(\mathbb{D}).$$

Indeed, we know

$$\sigma_k(\Omega_\varepsilon) \to \sigma_k(\Omega, \theta) = \sigma_k(\mathbb{D})$$

$$|\partial\Omega_\varepsilon| = \int_{\partial\Omega_\varepsilon} 1 d\mathcal{H}^1 \to \int_{\partial\Omega} \theta d\mathcal{H}^1 = \int_{\partial\Omega} \frac{1}{|g'(g^{-1}(x))|} d\mathcal{H}^1 = |\partial\mathbb{D}|.$$

Consequence

Theorem

Let $\Omega, \omega \subseteq \mathbb{R}^2$ be two smooth, simply connected open sets. Then there exists a sequence of smooth open sets $(\Omega_\epsilon)_{\epsilon>0}$ with uniformly bounded perimeter such that

$$\Omega_\varepsilon \to \Omega \text{ and } \quad \forall k \in \mathbb{N}, \lim_{\epsilon \to 0} |\partial \Omega_\epsilon| \sigma_k(\Omega_\epsilon) = |\partial \omega| \sigma_k(\omega).$$

The result remains true if $\Omega, \omega \subseteq \mathbb{R}^2$ are conformal. Moreover, Ω_ε is homeomorphic to Ω and ω !

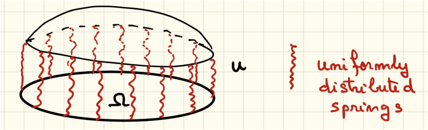
Robin boundary conditions

For $\beta > 0$ and $\Omega \subseteq \mathbb{R}^N$ *bounded, Lipschitz*

$$\begin{cases} -\Delta u = & \lambda u \ \text{ in } \ \Omega \\ \frac{\partial u}{\partial n} + \beta u = & 0 \ \ \partial\Omega \end{cases}$$

Rayleigh quotient :

$$\lambda_1(\Omega) = \min_{u \in H^1(\Omega)} \frac{\displaystyle\int_\Omega |\nabla u|^2 dx + \beta \int_{\partial\Omega} |u|^2 d\mathcal{H}^{N-1}}{\displaystyle\int_\Omega u^2 dx}$$



1986-2005 Bossel, Daners for Lipschitz sets by the use of the H-function !

Open problem of Brasco and De Philippis : quantitative form of the inequality

$$\lambda_1(\Omega) - \lambda_1(B) \geq C\,\mathcal{A}(\Omega)^2.$$

B. - Ferone - Nitsch - Trombetti, 2018.

Step 1. Intermediate inequality !

$$\lambda_1(\Omega) - \lambda_1(B) \geq \frac{\beta}{2} \inf_{x \in \partial\Omega} u^2(x)(|\partial\Omega| - |\partial B|)$$

$$\left[\geq \frac{\beta}{2} \inf_{x \in \partial\Omega} u^2(x) \frac{C_N}{|\Omega|^{\frac{N-1}{N}}} \mathcal{A}(\Omega)^2\right]$$

Quantitative form of the inequality

Step 2. Use the selection principle to replace Ω by a better set A

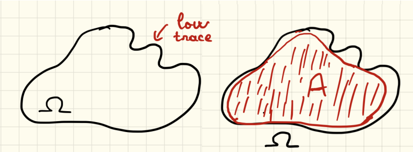
- $\lambda_1(\Omega) \geq \lambda_1(A)$
- $\mathcal{A}(\Omega)$ is comparable to $\mathcal{A}(A)$
- $\inf_{x \in \partial A} u^2(x)$ is controlled independently of Ω
- Get the quantitative inequality for Ω.

The selection of a "good" set.

Given Ω we solve the auxiliary free discontinuity problem

$$\min\{\lambda_1(A) + k|A| : A \subset \Omega, A \text{ open}\},$$

for a well chosen value of k.



The set A satisfies

- is (lightly) smooth
- $ess\inf_{x \in A} u_A(x) \geq \alpha$, with $\alpha > 0$ independent on Ω.
- $\mathcal{A}(A) + |B_{|\Omega|}| - |B_{|A|}| \geq C\mathcal{A}(\Omega)$

Robin boundary conditions : general isoperimetric inequality, $1 \leq q < \frac{2N}{N-1}$

$$\lambda_{1,q}(\Omega) = \min_{u \in H^1(\Omega)} \frac{\displaystyle\int_\Omega |\nabla u|^2 dx + \beta \int_{\partial\Omega} |u|^2 d\mathcal{H}^{N-1}}{\left(\displaystyle\int_\Omega |u|^q dx \right)^{\frac{2}{q}}}$$

Then

$$\lambda_{1,q}(\Omega) \geq \lambda_{1,q}(B)$$

- 2012, Bandle, Wagner, $q = 1$ (torsional rigidity), local minimizer, absence of an H-function
- 2015 B. Giacomini, $q \in [1, \frac{2N}{N-1})$ by a free discontinuity approach
- 2019 Alvino, Nitsch, Trombetti, $q = 1$ in $\mathbb{R}^N$ and $q = 2$ in $\mathbb{R}^2$, by Talenti type approach.

How to prove the quantitative inequality ?

KEY STEP : the intermediate inequality !

- $q = 1$ corresponds to the torsional rigidity !

$$\begin{cases} -\Delta u = & 1 \text{ in } \Omega \\ \frac{\partial u}{\partial n} + \beta u = & 0 \;\; \partial\Omega \end{cases}$$

We know that

$$E(\Omega) \geq E(B),$$

where

$$E(\Omega) = \min_{u \in H^1(\Omega)} \frac{1}{2} \int_\Omega |\nabla u|^2 dx + \frac{\beta}{2} \int_{\partial\Omega} u^2 d\sigma - \int_\Omega u dx$$

and

$$\lambda_{1,1}(\Omega) = -\frac{1}{2E(\Omega)}.$$

How to prove the intermediate inequality ?

$$E(\Omega) - E(B) \geq \frac{\beta}{2} \inf_{x \in \partial\Omega} u^2(x)(|\partial\Omega| - |\partial B|)$$

A new PDE/geometric functional, $c \geq 0$

$$E_c(\Omega) = \min\{\frac{1}{2}\int_\Omega |\nabla u|^2 dx + \frac{\beta}{2}\int_{\partial\Omega}(u^2 - c^2)d\sigma - \int_\Omega (u - c)dx, u \in H^1(\Omega), u \geq c\}$$

so that

$$E_c(\Omega) = E_{obstacle}(\Omega) - \frac{\beta}{2}c^2|\partial\Omega| + c|\Omega|.$$

Minimizer of the obstacle energy

Theorem (B., Giacomini, Nahon, 2020)

The minimizer of E_c among sets of prescribed measure is the ball!

Consequence

$$E_c(\Omega) \geq E_c(B) \Longrightarrow$$

But

$$E_c(\Omega) = E_{obstacle}(\Omega) - \frac{\beta}{2} c^2 |\partial\Omega| + c|\Omega|.$$

So

$$E_{obstacle}(\Omega) - E_{obstacle}(B) \geq \frac{\beta}{2} c^2 (|\partial\Omega| - |\partial B|).$$

Consequence

What is the optimal c ?

In order to get the best inequality, take

$$c = \inf_{\partial\Omega} u_\Omega(x) \le \inf_{\partial B} u_B(x),$$

such that

$$E_{obstacle}(\Omega) = E(\Omega) \text{ and } E_{obstacle}(B) = E(B)$$

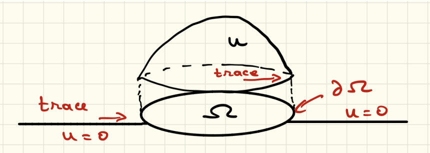
So

$$E(\Omega) - E(B) \ge \frac{\beta}{2} \inf_{\partial\Omega} u_\Omega^2(x)(|\partial\Omega| - |\partial B|).$$

How to prove the minimality of the ball

$$E_c(\Omega) \geq E_c(B).$$

B. - Giacomini : take the first eigenfunction of the Robin problem in Lipschitz set and extend it by zero.



Free discontinuity approach

The (square of) the new function seen in $\mathbb{R}^N$ has a distributional derivative

$$Du^2 = \nabla u^2 dx|_\Omega + u^2 \nu_{in} \mathcal{H}^{N-1}|_{\partial\Omega}.$$

So $u^2 \in SBV(\mathbb{R}^N)$!

$v \in L^1(\mathbb{R}^N), Dv = \nabla v dx + (v^+ - v^-) \nu_v \mathcal{H}^{N-1}|_{J_v}$ finite Radon measure

Free discontinuity approach

$$\min_{|\Omega|=m} E_c(\Omega) =$$

$$\min_{|\Omega|=m} \min_{u\in H^1(\Omega), u\geq c} \frac{1}{2}\int_\Omega |\nabla u|^2 dx + \frac{\beta}{2}\int_{\partial\Omega} u^2 - c^2 d\sigma - \int_\Omega (u-c)dx$$

becomes (with $v = u - c$)

$$\min_{v\in SBV(\mathbb{R}^N,\mathbb{R}_+), |\{v\neq 0\}=m|} \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2 dx + \frac{\beta}{2}\int_{J_v} [v_+^2 + v_-^2 + 2c(v_+ + v_-)] d\sigma - \int_{\mathbb{R}^N} v dx$$

• Approximation : Replace the jump term $v_+ + v_-$ by $v_+^{1+\varepsilon} + v_-^{1+\varepsilon}$.
• Existence of a solution : concentration - compactness argument and Ambrosio lower semicontinuity theorem.

• Regularity : non degeneracy $v(x) \geq \alpha > 0$ a.e. $v(x) > 0$ and monotonicity formula (B.-Luckhaus 2014) $\Longrightarrow$ closedness and Ahlfors regularity of J_u.

• Existence+regularity $\Longrightarrow$ ball ! Use of the reflection principle.

Quantitative form for full range of inequalities

Theorem (B., Giacomini, Nahon, 2020)

For every $q \in [1, 2)$ it holds

$$\lambda_{1,q}(\Omega) - \lambda_{1,q}(B) \geq C\mathcal{A}^2(\Omega).$$

Thank you for your attention !