ON THE LONG TIME BEHAVIOR OF HOMOGENEOUS RICCI FLOWS

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A family \((g(t))_{t \in [0,T)}\) of smooth, complete, Riemannian metrics on a smooth manifold \(M^n\) is called a solution to Hamilton’s Ricci flow [H1], if it satisfies the geometric evolution equation

\[
\frac{\partial}{\partial t} g(t) = -2 \text{ric}(g(t)) \quad \text{and} \quad g(0) = g_0.
\]

We call a Ricci flow solution a homogeneous Ricci flow, if the initial metric \(g_0\) is homogeneous. In this case the evolved metrics are homogeneous as well, in fact the isometry groups do not change [Kot]. The Ricci flow on homogeneous spaces has been investigated by many authors, in particular in low dimensions and on Lie groups (see e.g. [IJ], [I JL], [Lo], [BW], [Bu], [CSC], [Pa], [AC], [L11], [L13], [Ar]).

Still, in general the long time behavior of homogeneous Ricci flows is completely understood only in very special cases.

If a solution to the Ricci flow cannot be extended smoothly past time \(T\), then we call \(T \in (0, \infty]\) a singular time. If the singular time \(T\) is finite, the Ricci flow solution is said to have finite extinction time. A Ricci flow solution with finite extinction time is said to develop a Type I singularity, if there exists a constant \(C_{g_0} > 0\), such that

\[
\sup_{M^n} \|R(g(t))\|_{g(t)} \cdot (T - t) \leq C_{g_0}
\]

for all \(t \in [0,T)\). Here \(R(g(t))\) denotes the curvature tensor of the metric \(g(t)\).

By a recent result of Lafuente [La] a homogeneous Ricci flow has finite extinction time if and only if the scalar curvature of the evolved metrics becomes positive close to extinction time.

Our first main result is

**Theorem 1.** A homogeneous Ricci flow with finite extinction time develops a Type I singularity.

We will also show that for such homogeneous Ricci flows the norm of the curvature tensor can be controlled by the scalar curvature as soon as the scalar curvature is positive (Remark 1.2). By [Ki], [DH], [GZ] the homogeneity assumption in Theorem 1 cannot be dropped, since on the Euclidean plane and on spheres there exist rotationally invariant metrics, which lead to Type II singularities.

Our second main result is

**Theorem 2.** Let \(M^n\) be a compact homogeneous space not diffeomorphic to the torus \(T^n\). Then any homogeneous Ricci flow solution has finite extinction time.

A compact homogeneous space admits in general homogeneous metrics with negative scalar curvature; the spaces which do not have essentially been classified by Wang and Ziller [WZ] (see also [Boe1]). Notice that any homogeneous metric on a torus is flat.
By general results of Naber [Na] and Enders, Müller, Topping [EMT] on Type I singularities of the Ricci flow, it follows that along any sequence of times converging to the finite extinction time $T$, parabolic rescalings will subconverge to a nonflat homogeneous gradient shrinking soliton. By work of Petersen and Wylie [PW] such a shrinking soliton is in our situation a finite quotient of a nonflat product metric of a homogeneous Einstein metric with positive scalar curvature and a flat metric on Euclidean space. Notice that the flat factor might be absent.

We turn to the question whether the compact homogeneous Einstein space $E_\infty$ appearing in the limit soliton can be related to the homogeneous space considered. Recall that a homogeneous space is diffeomorphic to a coset space $G/H$, where $G$ is a Lie group acting isometrically and transitively on $M^n$ and $H$ is the compact isotropy subgroup of a point.

**Theorem 3.** Let $M^n = G/H$ be a compact homogeneous space not diffeomorphic to the torus $T^n$. Suppose that the isotropy representation decomposes into pairwise inequivalent summands. Then for any homogeneous Ricci flow on $G/H$ there exists a compact intermediate subgroup $K$, such that $E_\infty = K/H$.

The intermediate subgroup $K$ corresponds to the most shrinking direction of the metrics $g(t)$ (see section 4) and depends only on the initial metric $g_0$. Notice though, that for different initial metrics the group $K$ may vary as can be seen easily from considering homogeneous product metrics on $S^2 \times S^2$.

Since there exist homogeneous spaces $K/H$ not admitting any $K$-invariant Einstein metrics (see [WZ], [Boe2]), in general not all intermediate subgroups can occur. For instance, let $G = \text{SO}(2p + q)$, $L = \text{SO}(2p)\text{SO}(q)$ and $H = \text{SO}(p)\text{U}(1)\text{SO}(q)$, where $\text{SO}(p)\text{U}(1) \subset \text{U}(p) \subset \text{SO}(2p)$. Then, if $p \geq 3$, $p \neq 4$ and $q = 3$, the spaces $G/H$ and $L/H$ do not admit homogeneous Einstein metrics by [PS], [WZ]. As a consequence the only possible intermediate subgroup is $K = \text{U}(p)\text{SO}(q)$. If $p = 3$ and $q \geq 4$, the space $G/H$ does admit homogeneous Einstein metrics, that is one can also have $K = G$ for appropriate initial metrics.

We will show in Theorem 4.6, that for any sequence of times converging to $T$, the restriction of appropriately rescaled metrics $g(t)$ to $K/H$ subconverges to an Einstein metric of positive scalar curvature. The limit Einstein metric depends only on the initial metric and not on any subsequences chosen, if on $K/H$ there exist only finitely many solutions to the homogeneous Einstein equation of fixed volume (cp. finiteness conjecture of [BWZ]). Since in the first of the above two examples the space $K/H$ is isotropy irreducible, we get for any initial metric the same limit soliton. Let us mention, that if the isotropy representation has two inequivalent summands these results where obtained in [Bu].

We expect Theorem 3 to be true for arbitrary homogeneous spaces. In general the most shrinking direction of the evolved metrics corresponds to a distribution, which becomes integrable only in the limit. As a consequence, the Ricci flow on such homogeneous spaces is much more difficult to deal with.

We turn to homogeneous Ricci flows on noncompact homogeneous spaces. Bérard Bergery [BB] has shown that a homogeneous space admits a homogeneous metric of positive scalar curvature if and only if the universal covering space is not diffeomorphic to Euclidean space. By [La] it follows that on homogeneous spaces with Euclidean universal covering space any homogeneous Ricci flow solution will be immortal, that is $T = \infty$. Recall that a homogeneous Ricci flat metric is flat by
As a consequence, for an immortal homogeneous Ricci flow solution, which is not flat, the scalar curvature is negative and must converge to zero.

An immortal solution to the Ricci flow is said to develop a Type III singularity, if there exists a constant $C_{g_0} > 0$, such that for all $t \in [0, \infty)$

$$\sup_{M^n} \|R(g(t))\| : t \leq C_{g_0}.$$ 

Our third main result is

**Theorem 4.** An immortal homogeneous Ricci flow develops a Type III singularity.

As an immediate consequence of the above results we obtain

**Corollary 5.** For homogeneous spaces with compact or Euclidean universal covering space the following holds: The homogeneous Ricci flows on these spaces develop either a Type I or a Type III singularity, irrespectively of the chosen initial metric.

It is an open problem, whether this dichotomy holds for an arbitrary homogeneous space. If true, this would imply the long standing conjecture of Alekseevski on noncompact homogeneous Einstein spaces (see [Bes], 7.57).

We turn to the question to which extent there should be counterparts of the above mentioned results of Naber and Enders, Müller, Topping for Type III singularities of homogeneous Ricci flows on noncompact homogeneous spaces.

We consider for $s > 0$ the immortal solution $g_s(t) := g(st)$. It follows from Hamilton's compactness theorem that if the injectivity radius of $(M^n, g(t))$ is bounded from the below by $C_{g_0} \sqrt{t}$, then for any sequence $\{s_i\}_{i=1}^\infty$ converging to infinity the sequence $(M^n, g_{s_i}(t))$ of blow downs subconverges to a homogeneous immortal limit Ricci flow $(M^n_\infty, g_\infty(t))$ on a possibly different homogeneous space $M^n_\infty$. In general, by work of Glickenstein [Gl] and Lott ([Lo], Corollary 5.14), one obtains subconvergence to a limit flow on an $n$-dimensional, etale groupoid. In our situation such a groupoid is nothing but a locally homogeneous space, which in general will be incomplete (see section 5).

Our fourth main result is

**Theorem 6.** For any immortal homogeneous Ricci flow solution the above defined blow downs subconverge to an immortal locally homogeneous Ricci flow solution.

By [Sp] a locally homogeneous space with nonpositive Ricci curvature can be extended to a (complete) homogeneous space. In general this is not true anymore. There exist even Einstein metrics of positive scalar curvature on locally homogeneous spaces, which do not extend to a complete Einstein metric [Ro], [Kow].

Lott proved in [Lo], that if the sequence $(M^n, g_s(t))$ of blow downs has a limit for $s \to \infty$, then this limit Ricci flow is an expanding Ricci soliton. In special cases such as in dimension three and four [Lo] and for homogeneous metrics on nilpotent or certain solvable Lie groups this is known to be true [L11], [Ar].

**Problem.** Show that for any immortal homogeneous Ricci flow solution any blow down subconverges to an expanding Ricci soliton on a locally homogeneous space.

Notice first, that such an expanding limit soliton might be flat even if the scalar curvature of the approximating Ricci flow solution is negative for all times. For instance on the isometry group $E(2)$ of the Euclidean plane there exists a homogeneous immortal solution to the Ricci flow such that $\|\text{R}(g(t))\|_{g(t)} \approx \exp(-ct)$ and $\text{scal}(g(t)) \approx -\exp(-2ct)$ for $c > 0$ (see [IJ]). It follows, that the curvature is
so rapidly decreasing that any geometric limit solution must be flat. Notice also that the norm of the curvature tensor is not controlled by the absolute value of the scalar curvature in this example in contrast to Type I singularities.

Let us now turn to immortal homogeneous solutions, for which there exists a constant \(c_{g_0} > 0\), such that \(c_{g_0} \leq \sup_{M^n} \| R(g(t)) \| \cdot t \) for all \(t \in [0, \infty)\). At first hand, one might hope that the bracket flow introduced by Lauret in [L13] might be helpful establishing the existence of a nonflat expanding limit soliton. Recall that the bracket flow is a geometric flow on the set of Lie brackets, which is equivalent to the Ricci flow. In example 4.4 in [LL] Lafuente and Lauret provide an example of a Type III solution (in fact an expanding soliton), such that the norm of the corresponding bracket flow solution tends to infinity. It follows of course that the above blow downs will converge to the nonflat expanding soliton given. This limit soliton can however never be obtained by considering any normalized bracket flow.

The paper is organized as follows: In section 1 we prove Theorem 1, in section 2 Theorem 2 and in section 3 Theorem 4. Theorem 3 is proven in section 4 and in section 5 we present a proof a Spiro's result.

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1. Finite time singularities of homogeneous Ricci flows

In this section we will provide the proof of Theorem 1. Furthermore we will show that the norm of the curvature tensor is controlled by the scalar curvature.

Recall that by [La] we may assume that for a homogeneous Ricci flow \(g(t)\) with finite extinction time \(T < \infty\) we have \(s(g(0)) = 1\).

**Theorem 1.1.** A homogeneous Ricci flow \((M^n, g(t))_{t \in [0,T)}\) with finite extinction time \(T\) develops a Type I singularity.

**Proof.** From the evolution equation for the scalar curvature along a solution to the Ricci flow we know 
\[
s'(t) \geq \frac{n}{2} \cdot s^2(t),
\]
where we have set \(s(t) := \text{scal}(g(t))\). Let \(t_0 \in [0,T)\). As is well known, if \(s(t_0) \neq 0\), this implies
\[
s(t) \geq \frac{1}{s(t_0) - \frac{2}{n} \cdot (t - t_0)}.
\]

Since by assumption \(s(0) = 1\), we conclude \(T \leq \frac{n}{2} T - t_0\). Furthermore, we get
\[
s(t) \leq \frac{n}{2(T-t)}
\]
for all \(t \in [0, T)\). If not, then there exists \(t_0 \in [0, T)\) such that \(s(t_0) = \frac{(1+\varepsilon)n}{2(T-t_0)}\) for some \(\varepsilon > 0\). From (1) we deduce for all \(t \in [t_0, T)\) that
\[
t < \frac{T-t_0}{1+\varepsilon} + t_0 = \frac{T+t_0}{1+\varepsilon} = T - \frac{\varepsilon(T-t_0)}{1+\varepsilon}.
\]

Contradiction.

Let now \(K(t) := \| R(g(t)) \|_{g(t)}\) denote the norm of the curvature tensor at time \(t \in [0, T)\). We will show below that there exists a constant \(C > 0\), such that \(\frac{K(t)}{s(t)} \leq C\) for all \(t \in [0, T)\). This implies by (2)
\[
(T - t) \cdot K(t) \leq (T - t) \cdot s(t) \cdot C \leq \frac{C \cdot n}{2},
\]
which shows that we have a Type I singularity.
It remains to show that $\frac{K(t)}{s(t)}$ is bounded for $t \in [0, T)$. Suppose the contrary: Then there exist times $t_i \in [0, T)$ with $t_i \to T$ and $\frac{K(t_i)}{s(t_i)} = i$. Moreover, we can assume that

$$i = \max \left\{ \frac{K(t)}{s(t)\mid t \in [0, t_i]} \right\}.$$

Since the scalar curvature, i.e. $s(t)$, is not decreasing, $K(t_i) \geq K(t)$ for all $t \in [0, t_i]$. We set $Q_i := K(t_i)$ and rescale parabolically at $t = t_i$ by setting

$$g_i(t) := Q_i \cdot g(t_i + \frac{t}{t_i}).$$

The solutions $g_i$ live on $(-Q_i \cdot t_i, Q_i \cdot (w - t_i))$. Moreover the norm of the curvature tensor of $g_i(0)$ equals to 1 for all $i$. By the doubling time property of the norm of the curvature tensor of a solution to the Ricci flow (cf. [CLN], p. 213) this implies that any of the solutions $g_i(t)$ exist as long as $t \in [0, \frac{1}{Q_i}]$. It follows that the above intervals converge (along a subsequence possibly) to $(-\infty, T_\infty)$ with $T_\infty > 0$.

By the choice of the $t_i$ we have $\|R(g_i(t))\| \leq 1$ for $t \in (-t_i \cdot Q_i, 0]$. In general, we do have of course no injectivity radius bound from the below. But due to Theorem 5.12 in [Lo] there exists a convergent subsequence converging to a solution on an etale groupoid. This groupoid is a locally homogeneous, ancient solution $g_\infty(t)$ to the Ricci flow with nonnegative scalar curvature. At time $t = 0$ we have $\|R(g_\infty(0))\| = 1$, but $\text{scal}(g_\infty(0)) = 0$, since the function $\frac{K(t)}{s(t)}$ is scale invariant and by assumption we had $i = \frac{K(t_i)}{s(t)} = \frac{1}{\text{scal}(g_i(0))}$. Hence the limit solution is locally homogeneous and Ricci flat, hence by Theorem 5.2 flat. This is a contradiction to the fact that the norm of limit curvature tensor is 1 at $t = 0$.

\textbf{Remark 1.2.} We have shown above that there for any homogeneous Ricci flow solution $(g(t))_{t \in [0, T)}$ with $\text{scal}(g(0)) > 0$ there exists a constant $C_{g_0} > 0$ such that for all $t \in [0, T)$ we have

$$\|R(g(t))\|_{g(t)} \leq C_{g_0} \cdot \text{scal}(g(t)).$$

\section{An algebraic proof of Bochner's theorem}

As is well known by a theorem of Bochner [Bo] a compact homogeneous manifold cannot admit a Riemannian metric of nonpositive Ricci curvature unless it is flat. Moreover, the only compact homogeneous manifold admitting flat homogeneous metrics is the torus ([Bes], 7.61). Hence Theorem 2 follows from Theorem 2.2.

In this section we will provide an algebraic proof of the above result for compact, locally homogeneous spaces (see section 5). A locally homogeneous space $G/H$ is called compact, if the Lie algebra $\mathfrak{g}$ of $G$ is the Lie algebra of a compact Lie group $\hat{G}$. Recall that $\hat{G} = (G_1 \times \cdots \times G_s \times T^\circ)/\Gamma$, where $G_1, \ldots, G_s$ are compact, simply connected, simple Lie groups and $\Gamma$ is a finite subgroup of the center of $G$. As a consequence $\hat{G} = G_1 \times \cdots \times G_s \times \mathbb{R}^n$.

Since the isotropy group $H$ is not assumed to be a compact subgroup of $G$, in general the locally homogeneous space $G/H$ cannot be extended to a globally homogeneous space (see [Kow]). In particular, Stoke’s theorem is not applicable.

\textbf{Theorem 2.1.} Let $G/H$ be a connected, compact, locally homogeneous space. Let $g$ be a homogeneous metric on $G/H$, which is not flat. Then the Ricci curvature of $g$ is not nonpositive.
Proof: Since $G/H$ is a compact, locally homogeneous space, there exists an $\text{Ad}(G)$-invariant scalar product $Q$ on $\mathfrak{g}$. Let $\mathfrak{p}$ denote the orthogonal complement of the Lie algebra $\mathfrak{h}$ of $H$ in $\mathfrak{g}$. Then $\mathfrak{p}$ is $\text{Ad}(H)$-invariant. Let $B$ denote the Killing form of $G$ and let $\mathfrak{g} = \mathfrak{g}_s \oplus \mathfrak{a}$ be the decomposition of $\mathfrak{g}$ into its semisimple part $\mathfrak{g}_s = [\mathfrak{g}, \mathfrak{g}]$ and its center $\mathfrak{a}$. Notice that $\mathfrak{a}$ is the kernel of $B$, whereas on $\mathfrak{g}_s$ the Killing form $B$ is negative definite. Let $\mathfrak{p}_s := \mathfrak{p} \cap \mathfrak{a}$ and let $\mathfrak{p}_s$ denote the $\mathfrak{Q}$-orthogonal complement of $\mathfrak{p}_s$ in $\mathfrak{p}$. Then, since $\mathfrak{p}_s$ is $\text{Ad}(H)$-invariant, so is $\mathfrak{p}_s$. Notice that on $\mathfrak{p}_s$ the Killing form $B$ is negative definite.

Any $G$-invariant metric on $G/H$ corresponds to an $\text{Ad}(H)$-invariant scalar product $g$ on $\mathfrak{p}$. Using $Q := Q|_\mathfrak{p}$ we may write
\[ g(v, w) = \bar{Q}(P \cdot v, w), \]
where $P$ is an $\text{Ad}(H)$-equivariant endomorphisms of $\mathfrak{p}$, which is positive definite. Using the decomposition $\mathfrak{p} = \mathfrak{p}_s \oplus \mathfrak{p}_a$ we write
\[ P = \begin{pmatrix} P_{ss} & P_{sa} \\ P^T_{sa} & P_{aa} \end{pmatrix}. \]
The endomorphism $P_{ss}$ of $\mathfrak{p}_s$ is positive definite. Let $(\hat{e}_1, \ldots, \hat{e}_l)$ denote an $\mathfrak{Q}$-orthonormal basis of $\mathfrak{P}_{ss}$ corresponding to eigenvalues $p_1, \ldots, p_n > 0$. We set $e_i := \hat{e}_i/\sqrt{p_i}$ for $1 \leq i \leq n$. Then $(e_1, \ldots, e_n)$ is a $g$-orthonormal basis of $\mathfrak{p}_s$. We extend $(e_1, \ldots, e_n)$ to an $g$-orthonormal basis $(e_1, \ldots, e_n, e_{n+1}, \ldots, e_{n+l})$ of $\mathfrak{p}$. Then by [Bes], (7.38) for $x \in \mathfrak{p}$ we have
\[ \text{ric}(g)(x, x) = -\frac{1}{2} B(x, x) - \frac{1}{2} \sum_{i=1}^{n+l} \|[x, e_i]|_\mathfrak{p}\|^2 + \frac{1}{2} \sum_{i,j=1}^{n+l} \bar{Q}([e_i, e_j]|_\mathfrak{p}, P(x))^2. \]

Using $[\mathfrak{p}, \mathfrak{p}]_\mathfrak{p} = [\mathfrak{p}, \mathfrak{p}]_\mathfrak{p}$, we arrive at
\[ \text{ric}(g)(x, x) \geq -\frac{1}{2} B(x, x) - \frac{1}{2} \sum_{i=1}^{n} \|[x, e_i]|_\mathfrak{p}\|^2 + \frac{1}{2} \sum_{i,j=1}^{n} \bar{Q}([e_i, e_j]|_\mathfrak{p}, P(x))^2 \]
\[ + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=n+1}^{n+l} \bar{Q}([e_i, e_j]|_\mathfrak{p}, P(x))^2 - \frac{1}{2} \sum_{j=n+1}^{l} \|[x, e_j]|_\mathfrak{p}\|^2. \]

Next, we show that the sum of the fourth and fifth term is nonnegative for an eigenvector $x \in \mathfrak{p}_s$ of $\mathfrak{P}_{ss}$ corresponding to the largest eigenvalue $p$. We have for $i \in \{1, \ldots, n\}$ and $j \in \{n+1, \ldots, n+l\}$
\[ \bar{Q}([e_i, e_j]|_\mathfrak{p}, P(x))^2 - \bar{Q}([x, e_j]|_\mathfrak{p}, P(e_i))^2 = p^2 \cdot Q([e_i, e_j], x)^2 - p_i^2 \cdot Q([x, e_j], e_i)^2 = (p^2 - p_i^2) \cdot Q([e_i, e_j], x)^2 \geq 0, \]
since $Q$ is $\text{Ad}(G)$-invariant and consequently $\text{ad}(v)$ skew symmetric for any $v \in \mathfrak{g}$.

It remains to show that in the above estimate for $\text{ric}(g)(x, x)$ the sum of the first three terms is positive. The above argument shows that we may assume that $B$ is negative definite on $\mathfrak{p}$, hence may assume $P = \mathfrak{P}_{ss}$.

For any $\text{Ad}(H)$-invariant scalar product $g$ on $\mathfrak{p}$, there exists a decomposition $\mathfrak{p} = \mathfrak{p}_1 \oplus \cdots \oplus \mathfrak{p}_r$ of $\mathfrak{p}$ into $\text{Ad}(H)$-irreducible summands, such that $P$ is diagonal with respect to $Q$. That is, we have
\[ g = p_1 \cdot \bar{Q}|_{\mathfrak{p}_1} \perp \cdots \perp p_r \cdot \bar{Q}|_{\mathfrak{p}_r}. \]
with \( p_1, \ldots, p_r > 0 \). Likewise, we have \( P|_{p_i} = p_i \cdot \text{id}|_{p_i} \). For each \( 1 \leq i \leq r \), we set 
\[-B|_{p_i} = b_i \cdot Q|_{p_i} \text{ and } d_i = \text{dim } p_i.\]
Notice, that by assumption we have \( b_i > 0 \) for all \( i \). Moreover, we set
\[
[ijk] = \sum Q([\hat{e}_\alpha, \hat{e}_\beta], \hat{e}_\gamma)^2,
\]
where the sum is taken over \( \{\hat{e}_\alpha\}, \{\hat{e}_\beta\}, \text{ and } \{\hat{e}_\gamma\} \), \( Q \)-orthonormal bases for \( p_1, p_j \), and \( p_k \), respectively. Notice, that \([ijk]\) is invariant under permutations. By [Sa], (see also [WZ]), if \( x \in p_1 \) is an eigenvector of \( P \) with \( Q(x, x) = 1 \) corresponding to the largest eigenvalue \( p = p_1 \) of \( P \), then we have by the above estimate
\[
\text{ric}(g)(x, x) \geq \frac{1}{16d} \cdot \left( 2d_1 b_1 - \sum_{j,k=1}^r [ijk] \frac{2p^2_p - p^2_j}{p_j p_k} \right).
\]
Notice that it is not difficult to deduce this formula from the above formula for the Ricci tensor of \( g \). By [WZ], the identity
\[
d_1 b_1 = 2d_1 c_1 + \sum_{j,k=1}^r [ijk]
\]
holds. For a \( Q \)-orthonormal basis \( \{z_i\} \) of \( \mathfrak{h} \), here \( C_{Q|_{\mathfrak{h}}} = -\sum_j \text{ad } z_j \circ \text{ad } z_i \) denotes the Casimir operator and \( (C_{Q|_{\mathfrak{h}}})|_{p_i} = c_i \cdot \text{id}|_{p_i} \). Recall that \( c_i \geq 0 \) with \( c_i = 0 \) if and only if \([\mathfrak{h}, p_i] = 0\). Notice that the proof of Wang and Ziller carries over to compact, locally homogeneous spaces. We obtain
\[
d_1 b_1 - \sum_{j,k=1}^r [ijk] \frac{2p^2_p - p^2_j}{p_j p_k} = 2d_1 c_1 + \sum_{j,k=1}^r [ijk] \cdot \frac{p^2_j - p^2_k + p_j p_k}{p_j p_k}.
\]
Since \( p \) is the largest eigenvalue of \( P \) we have \( p^2_j - p^2_k + p_j p_k \geq 0 \) for all \( 1 \leq j, k \leq r \). As a consequence \( \text{ric}(g)(x, x) \geq \frac{b_1}{4} > 0 \). This shows the claim under the assumption that \( \text{dim } p_s > 0 \).

If \( p_s = 0 \), then \( p \subset \mathfrak{a} \). Consequently \( \mathfrak{g} \) is abelian, which implies that any locally homogeneous metric on \( G/H \) is flat (cf. [Bes], (7.30)).

As we have seen in the proof of the above theorem, a homogeneous metric \( g \) on \( G/H \) can be considered an endomorphism \( P \) of \( p \) using a background metric \( Q \). It follows from the above proof that if \( g \) is not flat, then there exists \( \bar{x} \in p_s \) with \( P_{ss}(\bar{x}) = p \cdot \bar{x}, g(\bar{x}, \bar{x}) = 1 \) and
\[
\text{ric}(\bar{x}, \bar{x}) \geq \frac{b}{4p}.
\]
Here, \(-b\) denotes the largest negative eigenvalue of the Killing form \( B \), restricted to the semisimple part \( \mathfrak{g}_s \) of \( \mathfrak{g} \). An immediate corollary of this is

**Theorem 2.2.** A homogeneous Ricci flow on a compact locally homogeneous space \( M^n = G/H \) has finite extinction time, if \( G/H \) does not admit flat metrics.

**Proof.** As in the proof of Theorem 2.1, given an \( \text{Ad}(G) \)-invariant scalar product \( Q \) on \( \mathfrak{g} \), we consider the decomposition \( p = p_s \oplus p_a \) of the \( Q \)-orthogonal complement \( p \) of \( \mathfrak{h} \) in \( \mathfrak{g} \), where \( p_a = p \cap J(\mathfrak{g}) \) and \( p_s \) denotes the \( Q \)-orthogonal complement of \( p_a \) in \( p \).

Along a locally homogeneous solution \( (g(t))_{t \in [0, T]} \) to the Ricci flow on \( G/H \) we consider the function
\[
\varphi(t) := \max \{ g(t)(y, y) \mid y \in p_s \text{ and } \|y\|_Q = 1 \}.
\]
Recall that $\varphi(t)$ is nothing but the largest eigenvalue of $P_s(t)$ in (3). By [CLN], page 531, for the Dini derivative
\[ \frac{d^+ \varphi}{dt}(t) := \limsup_{s \to 0, s > 0} \frac{\varphi(t + s) - \varphi(t)}{s} \]
we have
\[ \frac{d^+ \varphi}{dt}(t) = \max \{-2 \text{ric}(g(t))(x, x) \mid g(t)(x, x) = \varphi(t) \text{ and } x \in p_s, \|x\|_Q = 1\}. \]

Using the estimate of Theorem 2.1 we conclude $\text{ric}(g(t))(x, x) \geq \frac{b}{4} > 0$ for any such $x \in p_s$, where $b$ denotes the smallest eigenvalue of $(-B)|_{\mathcal{G} \times \mathcal{G}}$. This shows that $T < +\infty$. \[ \square \]

If a compact, locally homogeneous space $G/H$ admits a flat metric, by [Sp] $G/H$ is globally homogeneous and flat. It follows from the proof of Theorem 2.1 that $G$ does not have any compact factor, thus $G$ is abelian. But then any homogeneous metric on $G/H$ is flat.

3. Immortal solutions of homogeneous Ricci flows

In this section we give the proof of Theorem 4. Before doing so let us note that on the isometry group $E(2)$ of the Euclidean plane there exists a homogeneous immortal solution to the Ricci flow such that $\|R(g(t))\|_{g(t)} \approx \exp(-ct)$ and $\text{scal}(g(t)) \approx \exp(-2ct)$ for a positive constant $c > 0$ (see [IJ]). Hence for immortal solutions the norm of the curvature tensor is not controlled by the absolute value of the scalar curvature, which was true for homogeneous solutions with a Type I singularity. Moreover, the above example provides parabolic rescalings $g_i(t)$, defined on $[0, \infty)$ with $\|R(g_i(0))\| = 1$, which converge on $(0, \infty)$ to the flat metric. Recall also that locally homogeneous metrics maybe incomplete.

**Theorem 3.1.** An immortal locally homogeneous Ricci flow develops a Type III singularity.

**Proof.** Let $(g(t)|_{[0, \infty)})$ denote an immortal locally homogeneous Ricci flow solution. If the initial metric is flat the claim follows. If the initial metric is not flat, then by rescaling we may assume that $\text{scal}(g_0) = -1$. As above we write $s(t) = \text{scal}(g(t))$. By (1) we get $\lim_{t \to \infty} s(t) = 0$.

In a first step we show that $K(t) := \|R(g(t))\|_{g(t)}$ must be bounded. Suppose this is not the case. Then there exists a sequence $\{t_i\}$ of times converging to infinity, such that $i = K(t_i)$ and $K(t) < i$ for all $t \in [0, t_i)$. We may assume that $t_i > \frac{1}{16K(t_0)}$. Next, we choose $\tilde{t}_i \in (0, t_i)$, such that
\[ t_i - \tilde{t}_i = \frac{1}{16K(t_i)}. \]
This is possible, since for $\tilde{t}_i = 0$ we have $t_i > \frac{1}{16K(t_0)}$, whereas for $\tilde{t}_i = t_i$ we have $0 < \frac{1}{16K(t_i)}$. Setting $Q_i := K(\tilde{t}_i)$, we consider the parabolic rescaling
\[ g_i(t) := Q_i \cdot g(\tilde{t}_i + t + \frac{1}{Q_i}). \]
Notice that at time $t = 0$ the norm of the curvature tensor of the metric $g_i(0)$ equals to 1 for all $i$ and that by (5) for given $t_i, \tilde{t}_i$ we have $t_i = \tilde{t}_i + \frac{1}{Q_i}$ if and only if $t = \frac{1}{16}$. As a consequence, by the doubling time property a locally homogeneous limit flow $g_\infty(t)$ will exist on $[0, \frac{1}{16}]$ with $\|R(g_\infty(\frac{1}{16}))\| > 1$ since $K(t_i) > K(\tilde{t}_i)$. 

Of course the scalar curvature of the limit flow is nonpositive. On the other hand side the scalar curvature of the limit metric must vanish at \( t = \frac{1}{16} \) since \( i = K(t_i) \) and \( |\text{scal}(g(t_i))| \leq 1 \). As in Theorem 1.1 we obtain a contradiction.

In the second step we suppose that \( K(t) \) is bounded but does not converge to zero for \( t \to \infty \). Hence there exists a sequence \( \{t_i\} \) of times converging to \(+\infty\) and a constant \( \varepsilon > 0 \), such that \( K(t_i) \geq \varepsilon \). Since \( \lim_{t \to \infty} s(t) = 0 \), we can argue as above to exclude this case as well.

We conclude that \( K(t) \) must converge to zero for \( t \to \infty \). We suppose that the solution \( g(t) \) does not form a Type III singularity. Then there exists a sequence \( \{t_i\} \) of times converging to infinity such that

\[
K(t_i) = \frac{i}{r(i)} \quad \text{and} \quad K(t) < \frac{i}{r(i)} \quad \text{for all} \quad t \in [0, t_i).
\]

We may assume \( t_i > \frac{1}{16 K(0)} \). As above we choose \( t_i \in (0, t_i) \), such that the identity (5) holds. Again we consider the parabolic rescalings \( g_i(t) \) as in (6). By the choice of the sequence \( \{t_i\} \) we have

\[
(7) \quad K(t_i) = r(i) \cdot \frac{i}{t_i + 1} \quad \text{with} \quad r(i) < i.
\]

On the other hand side the doubling time property yields

\[
2 \geq \frac{K(t_i)}{K(t)} = \frac{i}{r(i)} \cdot \frac{i}{t_i + 1} = \frac{i}{r(i)} \cdot (1 + \frac{1}{16 K(t_i)}) = \frac{i}{r(i)} \cdot (1 + \frac{1}{16 K(t_i)}).
\]

We deduce

\[
i > r(i) \geq \frac{i}{2} - \frac{1}{16}.
\]

In the final step we show that \( \|R(g_i(\frac{1}{16}))\| \) is bounded from the below by a positive number. This follows from (5) and (7), since

\[
\|R(g_i(\frac{1}{16}))\| = \frac{i}{r(i)} \cdot \frac{t_i + 1}{t_i + 1} = \frac{i}{r(i)} \cdot (1 - \frac{1}{16 r(i) + 1}).
\]

Let \( g_\infty(t) \) denote a locally homogeneous limit solution for the sequence \( \{g_i(t)\} \), defined on \((0, \frac{1}{16})\). By the above computation we have \( \|R(g_i(\frac{1}{16}))\| \in [\frac{1}{2}, 3] \) for large \( i \). Since \( s(0) = -1 \) we deduce from (1) that \( |s(t)| \cdot (1 + \frac{2}{n} \cdot t) \leq 1 \) for all \( t \geq 0 \).

As a consequence, at time \( t = t_i \) we have

\[
(8) \quad K(t_i) \geq \frac{t_i + 1}{t_i + 1} \cdot (1 + \frac{2}{n} \cdot t_i) \geq \frac{2 i}{n}.
\]

It follows, that the scalar curvature of the limit solution \( g_\infty(t) \) at time \( t = \frac{1}{16} \) vanishes. Again, this is a contradiction.

\[\square\]

4. Collapsing of homogeneous Ricci flows

In this section we will prove Theorem 4.6, from which Theorem 3 and the results on the longtime behavior of the Ricci flow mentioned in the introduction below Theorem 3 follow. Notice that the same results hold for diagonal families of homogeneous metrics, which are preserved by the Ricci tensor.

Let \((g(t))_{t \in [0, T]} \) be a homogeneous Ricci flow on \( M^n \), which develops a Type I singularity. Recall that by [La] we may assume that \( \text{scal}(g(0)) = 1 \). As is well known there exist constants \( c(n) > 0 \) and \( C_{g_0} > 0 \) such that

\[
\frac{c(n)}{T-t} \leq \|R(g(t))\|_{g(t)} \leq \frac{c(n)}{T-t}
\]

\[
\frac{c(n)}{T-t} \leq \|R(g(t))\|_{g(t)} \leq \frac{c(n)}{T-t}
\]
for all \( t \in [0, T) \) (see e.g. \[EMT\]). Moreover, since for such homogeneous Ricci flows the norm of the curvature tensor is controlled by the scalar curvature (see Remark 1.2), there exist constants \( c_{g_0} > 0 \) and \( \bar{C}_{g_0} > 0 \) such that

\[
\frac{c_{g_0}}{T-t} \leq \text{scal}(g(t)) \leq \frac{\bar{C}_{g_0}}{T-t}
\]

for all \( t \in [0, T) \). Let now \( (t_i)_{i \in \mathbb{N}} \) be any sequence in \([0, T)\) with \( \lim_{i \to \infty} t_i = T \). Let \( Q_i := \text{scal}(g(t_i)) \) and

\[
g_i(t) := Q_i \cdot g(t_i + \frac{t}{4T_i}).
\]

Notice that the homogeneous Ricci flow \((g_i(t))\) is defined for \( t \in [-Q_i t_i, (T-t_i) Q_i) \) and that \((T-t_i)Q_i \geq d_{g_0} > 0\), since \( \|R(g_i(0))\|_{g_i(0)} \) is bounded from the above by a constant only depending on \( g_0 \). Notice also that \( \text{scal}(g_i(0)) = 1 \) for all \( i \in \mathbb{N} \).

By [Na], the sequence \((g_i(t))_{i \in \mathbb{N}}\) subconverges to a nonflat homogeneous shrinking soliton \((g_\infty(t))_{t \in (-\infty, T_\infty)}\) on a homogeneous limit space \( M_\infty^n \) with \( T_\infty \geq d_{g_0} \).

The results of [PW] imply that the homogeneous shrinking soliton \( g_\infty(0) \) is up to finite covering the Riemannian product of a compact homogeneous Einstein manifold \((E^n_k, g_\infty)\) with positive scalar curvature and a flat space \((\mathbb{R}^{n-k}, g^2_{\text{flat}})\) endowed with a Gaussian shrinking soliton. Since \( \text{scal}(g_\infty(0)) = 1 \) it follows that there exists a constant \( r_\infty \geq \frac{1}{n} \) such that the eigenvalues of \( \text{Ric}(g_\infty(0)) \) are either equal to \( r_\infty \) or to zero. Here, for a Riemannian metric \( g \) the Ricci-endomorphism \( \text{Ric}(g) \) is defined by

\[
\text{ric}(g)(\cdot, \cdot) = g(\text{Ric}(g)\cdot, \cdot).
\]

**Theorem 4.1.** For a homogeneous Ricci flow with finite extinction time the dimension of the Einstein factor of any limit shrinking soliton does only depend on the initial metric.

**Proof.** There exists \( \bar{T} < T \) such that for all \( t \geq \bar{T} \) the eigenvalues of the Ricci endomorphism of

\[
\bar{g}(t) := \text{scal}(g(t)) \cdot g(t)
\]

come in two blocks: The positive ones are bounded from the below by \( \frac{3}{4n} \) and the small ones are bounded from the above by \( \frac{1}{4n} \). Otherwise, there exists a sequence \((t_i)_{i \in \mathbb{N}}\) of times converging to \( T \), such that at least one eigenvalue of \( \text{Ric}(g_i(0)) \) is contained in \([\frac{1}{4n}, \frac{1}{4n}]\). By passing to a subsequence the same is true for the Ricci endomorphism of a limit soliton \( g_\infty(0) \). Contradiction. \( \square \)

Notice though, that for different initial metrics the dimension of the Einstein factor may vary: For the product Einstein metric on \( S^2 \times S^2 \) the Einstein factor of the limit soliton is of course 4-dimensional, whereas for any other initial metric the dimension of the Einstein factor is 2-dimensional.

We turn now to a class of homogeneous spaces where we can describe how the limit Einstein factor \( E^n_k \) is related to the original homogeneous space \( M^n \). Let \( M^n = G/H \) be a connected compact homogeneous space such that \( G \) and \( H \) are compact Lie groups not necessarily connected. Let \( Q \) denote an \( \text{Ad}(G) \)-invariant scalar product on \( g \) and \( p \) denote the \( Q \)-orthogonal complement to \( h \) in \( g \). Then for any \( G \)-invariant metric \( g \) on \( G/H \) there exists a \( Q \)-orthogonal decomposition \( p = p_1 \oplus \cdots \oplus p_r \) into \( \text{Ad}(H) \)-irreducible summands, such that

\[
g = x_1 \cdot Q|_{p_1} \perp \cdots \perp x_r \cdot Q|_{p_r}
\]
Lemma 4.3. We assume that the moduls $p_i, p_j$ are pairwise inequivalent.

Assumption 4.2. We assume that the moduls $p_1, ..., p_r$ are pairwise inequivalent.

An example would be a homogeneous space where the ranks of $G$ and $H$ agree. As is well known, under assumption 4.2 the Ricci endomorphism also respects the above decomposition of $p$; in general this is not true anymore.

Lemma 4.3. Let $(g(t))_{t \in [0,T)}$ be a homogeneous Ricci flow with finite extinction time and $\text{scal}(g(0)) = 1$. Then, under the assumption 4.2 there exists a nonempty subset $I \subset \{1, ..., r\}$ and a positive constant $r_0 \in [\frac{1}{2}, 1]$, such that the following holds true: For any $\varepsilon > 0$ there exist $T(\varepsilon) < T$, such that for all $t \geq T(\varepsilon)$

$$\|\text{Ric}(\bar{g}(t))|_{p_i} - r_0 \cdot \text{id}_{p_i}\|, \|\text{Ric}(\bar{g}(t))|_{p_{iC}}\| < \varepsilon.$$ 

Moreover, for any $m \in I$, $l \in I^C$ and $t \geq T(\varepsilon)$ we have

$$x_m(t) \leq x_m(T(\varepsilon)) \cdot \left(\frac{T-t}{T-T(\varepsilon)}\right)^{\frac{c_m}{n}}.$$ 

$$x_l(t) \geq x_l(T(\varepsilon)) \cdot \left(\frac{T-t}{T-T(\varepsilon)}\right)^{2C_0}.$$ 

Proof. From Theorem 4.1 it follows that there exists an index set $I \subset \{1, ..., r\}$ and $T_0 < T_\infty$, such that for all $t \geq T_0$ the eigenvalues of $\text{Ric}(\bar{g}(t))$ corresponding to $p_I$ are bounded from the below by $\frac{2}{n}$, whereas all the eigenvalues of $\text{Ric}(\bar{g}(t))$ corresponding to $p_{I^C}$ are bounded from the above by $\frac{1}{n}$. Here $I^C$ denotes the complement of $I$ in $\{1, ..., r\}$. Since for any sequence of times $(t_i)_{i \in \mathbb{N}}$ converging to $T$ there is a limit shrinking soliton (along a subsequence), the eigenvalues of $\text{Ric}(\bar{g}(t))$ must pinch more and more. This shows the first claim.

Let $r_{m}(t)$ denote the eigenvalue of the Ricci endomorphism $\text{Ric}(g(t))$ restricted to $p_m$, $m = 1, ..., r$. Then, under assumption 4.2 the Ricci flow equation is nothing but $x'_m(t) = -2 \cdot x_m(t) \cdot r_{m}(t), m = 1, ..., r$. Since $\text{Ric}(g(t)) = \text{scal}(g(t)) \cdot \text{Ric}(\bar{g}(t))$ we deduce from (9) and the first claim the estimates (11) and (12) by integrating the above differential equation.

In the next step, we described not only the directions, where the Ricci curvature is bounded from the below by $\frac{2}{n}$, but also show that they are related to an intermediate subalgebra.

Definition 4.4. Let $(g(t))_{t \in [0,T)}$ be a homogeneous Ricci flow on a compact homogeneous space $G/H$ with finite extinction time. Let $\{t_a\}$ denote a sequence of times converging to $T$. We assume that at such times $t_a$ the eigenvalues $x_1(t_a), ..., x_r(t_a)$ of $g(t_a)$ can be ordered in the following manner: We have $x_1(t_a) \leq \cdots \leq x_r(t_a)$ and there exists constants $D_1, ..., D_\ell > 0$ and nonempty subsets $I_1 := \{1, ..., i_1\}$, $I_2 := \{i_1+1, ..., i_2\}, ..., I_\ell := \{i_{\ell-1}+1, ..., i_\ell\} = \{r\}$, such that for all $1 \leq s \leq \ell$ and all $k, \bar{k} \in I_s$ we have $x_{k}(t_a) \leq D_s \cdot x_{\bar{k}}(t_a)$ and $\lim_{a \to \infty} \frac{x_{i_s+1}(t_a)}{x_{i_s}(t_a)} = \infty$. For notational reasons we set $i_0 := 0$. Furthermore, we assume that the metrics

$$\tilde{g}_a = \text{scal}(g(t_a)) \cdot g(t_a)$$

converge to a limit soliton metric $g_\infty$. 

Recall that an intermediate subalgebra \( \mathfrak{t} = \mathfrak{h} \oplus \mathfrak{p}_1 \), \( \mathfrak{p}_1 \subset \mathfrak{p} \), is called toral (non-toral), if \( \mathfrak{p}_1 \) is (not) an abelian subalgebra of \( \mathfrak{g} \). For such an intermediate subalgebra \( \mathfrak{t} \) of \( \mathfrak{g} \) we denote by \( K \) the smallest possibly disconnected subgroup of \( G \) containing \( H \) with Lie algebra \( \mathfrak{t} \). Moreover, for a \( G \)-invariant metric \( g \) on \( G/H \) as in (10) we denote by 

\[ g_{K/H} = g|_{\mathfrak{p}_1} \]

the induced metric on \( K/H \).

**Lemma 4.5.** Let \( (g(t))_{t \in [0,T)} \) be a homogeneous Ricci flow on \( G/H \) with finite extinction time and let \( I \subset \{ 1, \ldots, r \} \) be the nonempty subset from Lemma 4.3. Then \( \mathfrak{t}_I := \mathfrak{h} \oplus \mathfrak{p}_I \) is the Lie algebra of a compact subgroup \( K_I \) of \( G \) with \( \dim K_I/H = \dim E_{\mathfrak{k}}^c \). Moreover, there exists \( C > 0 \), such that for all \( t \geq 0 \) and all \( m, \tilde{m} \in I \) we have \( \| \frac{m_I(t)}{x_m(t)} \| \leq C \). Finally, the Ricci curvature of \( \bar{g}(t) \) restricted to \( \mathfrak{p}_I \) is up to lower order terms given by the Ricci curvature of \( \text{scal}(g(t)) \cdot g_{K_I/H}(t) \).

**Proof.** Let \( (t_a)_{a \in \mathbb{N}} \) be any sequence as in Definition 4.4. For \( a \in \mathbb{N} \) we set

\[ \bar{x}(a) := \frac{1}{\text{scal}(g(t_a))}. \]

Recall that \( \bar{g}_a = \frac{1}{x(a)} \cdot g(t_a) \) and that \( \text{scal}(\bar{g}_a) = 1 \) for all \( a \in \mathbb{N} \). Furthermore, by Lemma 4.7 there exists a constant \( \varepsilon(G,H) > 0 \) such that \( \bar{x}(a) \geq \varepsilon(G,H) \cdot x_1(t_a) \) for all \( a \in \mathbb{N} \). In the first step of the proof we assume, that there exists \( C > 0 \), such that

\[ x_1(t_a) \cdot [\tilde{m} j k] \cdot \frac{x_k(t_a)}{x_m(t_a) x_j(t_a)} \geq \bar{\varepsilon} \]

for all \( a \in \mathbb{N} \). By passing to a subsequence eventually and by choosing \( \bar{\varepsilon} \) small enough, we may assume that for indices \( (i, j, k) \) either the estimate (14) holds or the left hand side converges to zero for \( a \to \infty \). If no such indices exist at all the above claim follows since then \( \text{Ric}(g(t_a))|_{\mathfrak{p}_m} \) is of strictly smaller order than \( \frac{1}{x_1(t_a)} \) for all \( m \in \{ 1, \ldots, r \} \setminus I_1 \). This can be deduced easily from the following identity (cf. section 2 and (4)) for the Ricci tensor of a homogeneous metric \( g \) as in (10): We have for \( m \in \{ 1, \ldots, r \} \)

\[ \text{Ric}(g)|_{\mathfrak{p}_m} = \left( \frac{b_m}{2x_m} - \frac{1}{2d_m} \sum_{j,k=1}^{r} [jkm] \cdot \frac{x_k}{x_m x_j} + \frac{1}{4d_m} \sum_{j,k=1}^{r} [jkm] \cdot \frac{x_m}{x_j x_k} \right) \cdot \text{id}_{\mathfrak{p}_m} \]

\[ = \left( 2d_m c_m + \frac{1}{4} \sum_{j=1}^{r} [m j j] \cdot \frac{x_j}{x_j} + \sum_{j < k}^{r} [m j k] \cdot (2 - \frac{x_k}{x_j} - \frac{x_j}{x_k} + \frac{x_m}{x_j x_k}) \right) \cdot \text{id}_{\mathfrak{p}_m} \cdot \frac{x_m}{2d_m x_m}. \]

Notice that \( 2 - \frac{1}{x} - x \leq 0 \) for all \( x > 0 \).
Suppose now that \( \bar{m} \in I_s \), \( 2 \leq s \leq \ell \) is the smallest index satisfying (14). From (15) it follows that for \( i_1 + 1 \leq m < \bar{m} \), we have \( \text{Ric}(\bar{g}_a)|_{p_m} \to 0 \) by the minimality of \( \bar{m} \).

We turn to the computation of \( \text{Ric}(\bar{g}_a)|_{p_m} \). There are two ideas: Firstly we show, that whenever there is a positive term in (15), large in the sense of (14), then there is an even larger negative term, which makes up for it. Secondly, we keep track of the large negative terms, which might show up as positive terms in (15) for \( m > \bar{m} \).

Suppose \( \bar{m} \in I_2 \). Recall that by Lemma 4.7 for \( s \geq 3 \) we have \([\bar{m}jm] = 0 \) for all \( j \in I_1 \) and \( k \in I_1 \cup I_s \). Firstly, we consider the structure constants of type \([\bar{m}jm]\) for \( j \in I_1 \). Notice that two of the corresponding terms in (15) cancel, whereas the remaining term \(-x_1(t_a) \cdot [\bar{m}jm] \cdot \frac{x_{m}(t_a)}{x_{m}(t_a) x_{m}(t_a)} \) converges to zero for \( a \to \infty \). Next, for \( \bar{m} < m \leq i_2 \) and \( j \in I_1 \), the term

\[
(16) \quad x_1(t_a) \cdot [\bar{m}jm] \cdot \left( - \frac{x_{m}(t_a)}{x_{m}(t_a) x_{m}(t_a)} + \frac{x_{\bar{m}}(t_a)}{x_{\bar{m}}(t_a) x_{\bar{m}}(t_a)} \right) \leq 0
\]

is of leading order, if \([\bar{m}jm] > 0 \). Notice that all the remaining positive terms cannot be of leading order.

If \( j, k \in I_2 \), then the terms in (15) corresponding to \([\bar{m}jk] \) are of lower order \( \frac{1}{x_1(t_a)} \). Hence let \( k \in I_s \) with \( s \geq 3 \) and \( \bar{m} \leq m \leq i_2 \). Then the leading term involving a structure constant \([\bar{m}mk] > 0 \) is

\[
(17) \quad -x_1(t_a) \cdot [\bar{m}mk] \cdot \frac{x_{m}(t_a)}{x_{\bar{m}}(t_a) x_{m}(t_a)}
\]

Next, we assume \( 1 \leq j \leq k \leq r \) and \( j, k \in I_s \) for \( s \geq 3 \). Again it follows easily that the leading term

\[
(18) \quad -x_1(t_a) \cdot [\bar{m}jk] \cdot \frac{x_{m}(t_a)}{x_{\bar{m}}(t_a) x_{j}(t_a)}
\]

in (15) involving this structure constant comes with a negative sign. Notice that for \( j, k \in I_s \), \( 3 \leq s \leq \ell \), this term is of lower order.

Since the limit metric \( g_{\infty} \) has nonnegative Ricci curvature, for \( a \to \infty \) we deduce \( \text{Ric}(\bar{g}_a)|_{p_m} \to 0 \). In particular all of the above terms have to converge to zero for \( a \to \infty \). This implies that all the structure constants satisfying (14) have to vanish but the structure constants \([\bar{m}jm] \) with \( j \in I_1 \) and \( \bar{m} < m \leq i_2 \). But if in this case \([\bar{m}jm] > 0 \) we deduce from (16) that \( \lim_{a \to \infty} \frac{x_{m}(t_a)}{x_{\bar{m}}(t_a)} = 1 \). In precisely the same way it follows now that \( \text{Ric}(\bar{g}_a)|_{p_m} \to 0 \) for \( a \to \infty \) and \( \bar{m} < m \leq i_2 \).

Suppose now that \( \bar{m} \in I_3 \) is the smallest index in \( I_3 \) satisfying (14). Choose \( 1 \leq j \leq k \leq r \) for such an \( \bar{m} \). First, we assume that \( j, k \in I_1 \cup I_2 \). Since \([I_1 I_1 I_3] = [I_1 I_2 I_3] = 0 \) by Lemma 4.7 and by (17) terms of type \([I_2 I_2 I_3] \) also cannot satisfy (14), we can assume that \( j \in I_1 \cup I_2 \) and \( k \in I_s \) for some \( s \geq 3 \). For positive structure constants of type \([I_1 \bar{m}m] \), \( m \in I_3 \) and \( \bar{m} \geq \bar{m} \), we conclude as above that \( \frac{x_{m}(t_a)}{x_{\bar{m}}(t_a)} \to 1 \) for \( a \to \infty \), implying that the corresponding term in (16) must converge to zero. Terms of type \([I_2 I_3 I_3] \) are of lower order. Next, we have \([I_1 I_3 I_3] = 0 \) for \( s \geq 4 \). Large terms of type \([I_2 I_3 I_3] \), \( s \geq 4 \), can by (18) not satisfy (14). We are left with structure constants of type \([\bar{m}I_s I_s] \) and \( s, s' \geq 3 \). As above we conclude that such structure constants cannot satisfy (14). This shows \( \text{Ric}(\bar{g}_a)|_{p_m} \to 0 \) for \( a \to \infty \). As above we show that for \( m > \bar{m} \) and \( m \in I_3 \) the same conclusion holds. Inductively one deduces that for all \( m \in I_3 \) with \( s \geq 2 \) one has \( \text{Ric}(\bar{g}_a)|_{p_m} \to 0 \).

Next, we show that \( \text{Ric}(\bar{g}_a)|_{p_{I_1}} \geq \frac{1}{2\bar{m}} \cdot \text{id}|_{p_{I_1}} \) for large \( a \). To this end, suppose that there exists \( \tilde{I} \subset I_1 \) such that for \( a \to \infty \) we have \( \text{Ric}(\bar{g}_a)|_{p_m} \to 0 \) for all \( \bar{m} \in \tilde{I} \).
By Lemma 4.3 we obtain a lower estimate on $x_m(t_a)$ and an upper estimate on $x_1(t)$ for $l \in I \setminus \tilde{I} \neq \emptyset$, yielding a contradiction to the choice of the sequence \{t_a\} in Definition 4.4.

We show now that $K_1/H$ is compact. To this end let $\mathfrak{t}_1^s$ denote the semisimple part of $\mathfrak{t}_1$ and $a = \mathfrak{p}_I, \tilde{I} \subset I_1$, denote the complement of $\mathfrak{z}(\mathfrak{t}_1) \cap \mathfrak{h}$ in $\mathfrak{z}(\mathfrak{t}_1)$. If $a = 0$, then $K_1$ is compact, hence $K_1/H$ as well. Suppose now $a \neq 0$. By assumption we have that $\text{scal}(g(t_a)) \geq \frac{1}{C_{x_i(t_a)}}$ for all $a \in \mathbb{N}$. It follows from the above that for large $a$ and all $m \in I_1$, $r_m(t_a)$ is bounded from below by $\frac{1}{2m \cdot x_i(t_a)}$. On the other hand side, since $a \neq 0$ by (15) we have $r_m(t_a) \leq \bar{C} \cdot \frac{x_m(t_a)}{(x_{i+1}(t_a))^2}$ for $\tilde{m} \in \tilde{I}$.

Contradiction.

In the second part of this proof we will show that there cannot exist $c : \mathbb{N} \to \mathbb{N}$ with $\lim_{n \to \infty} c(a) = +\infty$ such that

\begin{align}
(19) \quad c(a) \cdot x_1(t_a) \leq \bar{x}(a)
\end{align}

for all $a \in \mathbb{N}$. Firstly, we consider indices $\tilde{m} \in I_s$, $s \geq 2$, and $1 \leq j \leq k$ which satisfy (14). For the metric $\tilde{g}_a$ these terms could eventually lead to curvature terms of order $\frac{\tilde{x}(a)}{x_i(t_a)}$. Precisely as above it follows that the above estimates are valid, in particular we know from (16) that

\begin{align}
(20) \quad \left| \frac{x_m(t_a)}{x_m(t_{\tilde{m}})} - \frac{x_m(t_{\tilde{m}})}{x_m(t_a)} \right| \leq \frac{x_1(t_a)}{x_m(t_a)}
\end{align}

for all $m, \tilde{m} \in I_s$, $2 \leq s \leq \ell$, if $|i m \tilde{m}| > 0$ for some $i \in I_1$. Since $|x - \frac{1}{2}| \leq \varepsilon$ implies $|2 - \frac{1}{2} - x| \leq \varepsilon^2$, it follows that

\begin{align}
(21) \quad \left| 2 - \frac{x_m(t_a)}{x_m(t_{\tilde{m}})} - \frac{x_m(t_{\tilde{m}})}{x_m(t_a)} \right| \leq \left( \frac{x_1(t_a)}{x_m(t_{\tilde{m}})} \right)^2.
\end{align}

\[\text{From (15) we deduce for } m \in I_1\]

\begin{align}
(22) \quad \text{Ric}(g)|_{\mathfrak{p}_m} = \text{Ric}(g_1)|_{\mathfrak{p}_m} + \left( \sum_{j,k \in I_1'} \frac{|m k|}{4} \cdot (2 - \frac{x_k}{x_j} - \frac{x_j}{x_k} + \frac{x_k^2}{x_j x_k}) \right) \cdot \frac{\text{id}_{\mathfrak{p}_m}}{dm x_m},
\end{align}

where the Ricci tensor of $(K_1/H, g_1)$ is given by

\[\text{Ric}(g_1)|_{\mathfrak{p}_m} = \left( 2dm cm + \sum_{j,k \in I_1} \frac{|m k|}{4} \cdot (2 - \frac{x_k}{x_j} - \frac{x_j}{x_k} + \frac{x_k^2}{x_j x_k}) \right) \cdot \frac{\text{id}_{\mathfrak{p}_m}}{dm x_m}.
\]

Recall that $g_1 = g_{K_1/H}$ denotes the metric on $K_1/H$ induced by $g$.

Using (21) and (22) we obtain a contradiction if the semisimple part $\mathfrak{t}_1^s$ is not contained in $\mathfrak{h}$, since then on the one hand side at least one of the Ricci curvatures of $(G/H, g(t_a))$ would be of order $\frac{1}{x_i(t_a)}$ whereas $\text{scal}(g(t_a))$ is by (19) of smaller order, contradicting Remark 1.2. It follows that $\mathfrak{t}_1$ is a toral subalgebra. By (19), (21) and (22) we obtain $\text{Ric}(\tilde{g}_a)|_{\mathfrak{t}_1} \to 0$ for $a \to \infty$ provided that $\bar{x}(a) \leq C \cdot x_{i+1}(t_a)$ along a subsequence. But now, by using (12) and (11) we obtain a contradiction to the choice of the sequence \{t_a\} in Definition 4.4.

Next, we show that there cannot be a function $c : \mathbb{N} \to \mathbb{N}$ as above, such that

\begin{align}
(23) \quad c(a) \cdot x_{i+1}(t_a) \leq \bar{x}(a)
\end{align}

for all $a \in \mathbb{N}$. By the above we know that $[I_1 I_1 I_1] = [I_1 I_s I_s] = 0$ for $s \neq s'$. Moreover, by (20) and (21) the sum of all terms in (15) involving eigenvalues $x_m(t_a)$ with $m \in I_1$ is of order $\frac{x_1(t_a)}{x_{i+1}(t_a)} < \frac{\bar{x}(a)}{x_{i+1}(t_a)}$ by (23).
As in (14) we consider now also indices $\vec{m} \in I_s$, $s \leq \ell$, and $1 \leq \ell \leq k \leq r$, such that
\begin{equation}
(24)\quad x_{i_1 + 1}(t_a) \cdot [\vec{m}jkl] \cdot \frac{x_{k}(t_a)}{x_{m}(t_a)} \geq \varepsilon
\end{equation}
for all $a \in \mathbb{N}$. By the above argument we are allowed to disregard all the terms in (15) involving structure constants $[ijk]$ with $i \in I_1$. Using (24), it follows from the proof of Lemma 4.7 that $\mathfrak{e}_2 = \mathfrak{h} \oplus \mathfrak{p}_{I_1} \oplus \mathfrak{p}_{I_2}$ is a subalgebra of $\mathfrak{g}$. Mimicking the arguments after (14) we deduce that $\mathfrak{e}_2$ is a toral subalgebra. As above we conclude $\text{Ric}(\tilde{g}_a)|_{\mathfrak{p}_{I_1} \oplus \mathfrak{p}_{I_2}} \to 0$ provided that $\tilde{x}(a) \leq C \cdot x_{i_1 + 1}(t_a)$ along a subsequence. Again this yields a contradiction. Inductively we obtain that (13) holds true.

Notice that we have shown that for all $m \in I_1$ the estimate (11) and for all $\vec{m} \in \{1, ..., r\} \setminus I_1$ the estimate (12) holds true. This implies $I = I_1$. Moreover, by Lemma 4.3 and by the above there exists a constant $C$ such that for all $m, \vec{m} \in I_1$ we have $\frac{x_m(t_a)}{x_m(t)} \leq C$ for all $t \in [0, T)$. By (16) we deduce that for any structure constants $[ijk]$ with $i \in I_1$ and $j, k \in I_s$, for some $s \geq 2$, one has $\frac{x_i(t)}{x_m(t)} \to 1$ for $t \to T$. It follows from (22) that $\text{Ric}(\tilde{g}_a)|_{\mathfrak{p}_{I_1}}$ equals to $\text{Ric}(\tilde{g}_a)|_{1}$ up to terms of order lower order. This shows the claim. \qed

Let us mention that we have shown that for any sequence $\{t_a\}$ as in Definition 4.4 we have $I = I_1$, that is the subgroup $K_1$ corresponds to the most shrinking directions of $g(t)$. Recall also that for the intermediate subgroup $K_1$ from Lemma 4.5 we denoted by $g_1$ the metric on $K_1/H$ induced by a homogeneous metric $g$ on $G/H$.

**Theorem 4.6.** Let $(g(t))_{t \in [0, T)}$ be a homogeneous Ricci flow on $G/H$ with finite extinction time. Suppose that assumption 4.2 holds. Then there exists a compact, nontoral intermediate subgroup $K_1$ such that $(K_1/H, g_1(t))$ is a totally geodesic submanifold of $(G/H, g(t))$ for all $t \in [0, T)$. Furthermore, the metrics $g_1(t) = \text{scal}(g(t)) \cdot g(t)$ subconverge to an Einstein metric on $K_1/H$ for any sequence $(t_n)_{n \in \mathbb{N}}$ converging to $T$ and the limit Einstein factor $E^K_\infty$ is diffeomorphic to $K_1/H$.

If on $K_1/H$ there exist only finitely many solutions to the homogeneous Einstein equation, then $g_1(t)$ converges for $t \to T$ to an Einstein metric on $K_1/H$.

**Proof.** We first show, that under assumption 4.2 intermediate subalgebras $\mathfrak{k}$ always correspond to totally geodesic submanifolds. Due to formula (7.27) in [Bes], for Killing vector fields $X, Y, N$ one has
\begin{equation}
2g(\nabla_X N, Y) = g([X, N], Y) + g([Y, X], N) + g([N, Y], X).
\end{equation}
By assumption, every $G$-invariant metric on $G/H$ has the special form described in (10). We deduce that for $X, Y \in \mathfrak{k}$ and $N \perp \mathfrak{k}$ we have $g(\nabla_X N, Y) = 0$ since $Q([X, Y], N) = Q([X, N], Y) = 0$.

Next, let $\{t_a\}$ be a sequence of times as in Definition 4.4. By Lemma 4.5, the eigenvalues of $\bar{g}_a = \text{scal}(g(t_a)) \cdot g(t_a)$ restricted to $\mathfrak{p}_{I_1}$ have a uniform positive lower and an upper bound. Moreover, $\text{Ric}(\bar{g}_a)|_{\mathfrak{p}_{I_1}}$ equals to $\text{Ric}(\bar{g}_a)|_{1}$ up to terms of lower order.

We consider now a normalized Ricci flow, which keeps the volume of $g_1(t)$ constant. To this end, let $\text{scal}_1(g) := \text{tr}(\text{Ric}(g)|_{\mathfrak{p}_{I_1}})$ and $k_1 := \dim K_1 - \dim H$. Then the following $K_1/H$-volume normalized Ricci flow
\begin{equation}
(25) \quad \bar{g}'(t) = -2(\text{Ric}(\bar{g}(t)) - \frac{1}{k_1} \cdot \text{scal}_1(\bar{g}(t)) \cdot \bar{g}(t))
\end{equation}
is equivalent to the Ricci flow and leaves the volume of $g_1(t)$ constant. This follows
the same way proving that unnormalized Ricci flow and Ricci flow are equivalent. The
solution $\tilde{g}(t)$ can be obtained (up to parametrization) from a solution $g(t)$ to the Ricci flow by rescaling to keep the volume $\tilde{g}_1(t)$ constant. It follows from Lemma 4.5 that for a solution $\tilde{g}(t)$ the eigenvalues of $\tilde{g}(t)$ restricted to $p_{t_1}$ are bounded uniformly. As a consequence, for any sequence of times $\{t_n\}$ converging to $T$ there exists a subsequence $\{\tilde{t}_n\}$ such that $\tilde{g}_1(\tilde{t}_n)$ converges to a limit metric $\tilde{g}_1^\infty$ on $K_1/H$. Using Lemma 4.5 again, we conclude that $\tilde{g}_1^\infty$ must be an Einstein metric with positive scalar curvature. Since by Lemma 4.5 the scalar curvature $\text{scal}(g(t_n))$ is of order $1/\tilde{x}(t_n)$, the metrics $\tilde{g}_1(t_n)$ on $K_1/H$ subconverge to a limit Einstein metric, too. This shows the second claim.

If there are only finitely many solutions to the Einstein equation on $K_1/H$, then $\tilde{g}_1(t)$ must of course converge to one of those metrics.

Finally, we show that the homogeneous spaces $(K_1/H, \tilde{g}_1(t_n))$ converge to the Einstein manifold $(E_{k_1}^k, g_{k_1}^k)$. We use that $(G/H, \tilde{g}(t_n))$ converges to $(M_{k_1}^k, g_{k_1})$ in pointed $C^\infty$-topology. It follows that the tangent spaces of the totally geodesic submanifolds $(K_1/H, \tilde{g}_1(t_n))$ converge to the tangent space of the Einstein factor $(E_{k_1}^k, g_{k_1}^k)$, since these are precisely the directions where the Ricci curvature is positive. As a consequence $(K_1/H, \tilde{g}_1(t_n))$ converges to $(E_{k_1}^k, g_{k_1}^k)$ in $C^\infty$-topology. By Theorem 1.1 in [BWZ] the limit Einstein factor $E_{k_1}^k$ is diffeomorphic to $K_1/H$. □

Finally, we present the scalar curvature estimates needed above.

Lemma 4.7. Let $(g(t))_{t \in [0,T)}$ be a homogeneous Ricci flow on $G/H$ with finite
extinction time and let $\{t_n\}$ be a sequence of times as in Definition 4.4. Then there exists $s_0 \in \{1, \ldots, \ell\}$, such that $\tilde{t}_{s_0-1} := \mathfrak{h} \oplus \mathfrak{p}_{t_1} \oplus \cdots \oplus \mathfrak{p}_{t_{s_0-1}}$, is the only and
$
\tilde{t}_{s_0} := \tilde{t}_{s_0-1} \oplus \mathfrak{p}_{t_{s_0}}$ a nontoral subalgebra of $\mathfrak{g}$, respectively. Moreover there exists a constant $C(G,H) > 0$, only depending on $G/H$, such that

$$
\text{scal}(g(t_n)) \leq \frac{C(G,H)}{\tilde{x}(t_n)}.
$$

where $\tilde{i} = i_{s_0-1} + 1$

Proof. By [WZ], for a homogeneous metric $g$ as in (10) we have

$$
\text{scal}(g) = \frac{1}{4} \sum_{i=1}^{r} d_h \cdot \frac{d_h}{\tilde{x}_i} - \frac{1}{4} \sum_{i,j,k=1}^{r} [ijk] \cdot \frac{x_{ij}}{\tilde{x}_ix_k}.
$$

Since $\lim_{t \to \infty} \text{scal}(g(t_n)) = +\infty$, we deduce that the terms $[ijk] \cdot \frac{x_{ij}(t_n)}{\tilde{x}_ix_k(t_n)}$ must be bounded for $i \in I_1$, $j \in I_4$ and $k \in I_4$ with $s \neq s'$. Consequently $[I_1I_4I_4] = 0$ for $s \neq s'$. This shows that $\tilde{t}_1 := \mathfrak{h} \oplus \mathfrak{p}_{t_1}$ is a subalgebra. Moreover, it follows from (26) that $\text{scal}(g(t_n)) \leq \frac{C(G,H)}{\tilde{x}(t_n)}$ for a constant $C(G,H) > 0$ depending only on the pair $H \subset G$. If $\tilde{t}_1$ is a nontoral subalgebra, then the above claim follows.

So suppose that $\tilde{t}_1$ is a toral subalgebra. By $[I_1I_1I_1] = 0$ we deduce

$$
\text{scal}(g(t_n)) = \frac{1}{2} \sum_{i \in I_1} d_h \cdot \frac{d_h}{\tilde{x}_i(t_n)} - \frac{1}{4} \sum_{i,j,k \in I_4} [ijk] \cdot \left( \frac{x_{ij}(t_n)}{\tilde{x}_ix_j(t_n)x_k(t_n)} + 2 \frac{x_{ij}(t_n)}{\tilde{x}_ix_j(t_n)x_k(t_n)} + \frac{x_{ij}(t_n)}{\tilde{x}_ix_j(t_n)x_k(t_n)} \right) + \frac{1}{2} \sum_{i \in I_4} d_h \cdot \frac{d_h}{\tilde{x}_i(t_n)} - \frac{1}{4} \sum_{i,j,k \in I_4} [ijk] \cdot \frac{x_{ij}(t_n)}{\tilde{x}_ix_j(t_n)x_k(t_n)}.
$$

Using the identity $d_i b_i = 2 r_i d_i + \sum_{j,k=1}^n [ijk]$ (cf. (4)) and the vanishing of the Casimir operator restricted to $p_{I_1}$, we see that the terms of the first line of the right hand side can be simplified to the following nonpositive term:

$$\frac{1}{2} \sum_{i \in I_1} \frac{1}{\sigma_i(t_a)} \sum_{j,k \in I_1^C} [ijk] \cdot \left( 1 - \frac{x_j(t_a)}{\sigma_k(t_a)} - \frac{x_k(t_a)}{2x_i(t_a)\sigma_k(t_a)} \right) \leq 0$$

Since $\text{scal}(g(t_a)) \to +\infty$ for $a \to \infty$ we obtain by induction the claim. \hfill \square

5. Locally homogeneous Ricci flat metrics are flat

Recall the following result

**Theorem 5.1** (Alekseevski, Kimelfeld [AK]). Let $(M^n, g)$ be a complete, homogeneous Ricci flat manifold. Then $(M^n, g)$ is flat.

Notice that there is a very short proof using the Cheeger-Gromoll splitting theorem (cf. [Bes], p. 191). Later, Spiro observed in [Sp] that the above Theorem is of local nature. For convenience, we provide a proof of Spiro’s result following Spiro’s original approach.

We recall that a Riemannian manifold $(M^n, g)$, not necessarily complete, is called locally homogeneous, if for any two points $p, q \in M^n$ there exists a local isometry mapping $p$ to $q$. Each locally homogeneous space is uniquely described by its so called infinitesimal model. The Nomizu construction associates to each infinitesimal model a Lie algebra $\mathfrak{g}$, a subalgebra $\mathfrak{h}$ and a reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$ (see [Tri]). Notice that $\mathfrak{g}$ is the Lie algebra of the full local isometry group of the Riemannian metric $g$ on $M^n$.

Since there might be also smaller local Lie groups acting locally transitively on $M^n$ the following algebraic definition of a locally homogeneous space has been introduced, for instance in [Sp] or in [L13]:

Let $G$ and $H$ be connected Lie groups with $G$ simply connected and $H \subset G$. Let $\mathfrak{g}$ and $\mathfrak{h}$ denote the Lie algebras of $G$ and $H$, respectively. We call $G/H$ a locally homogeneous space, if the following conditions hold:

(h1) There exists an $\text{Ad}(H)$-invariant complement $\mathfrak{p}$ of $\mathfrak{h}$ in $\mathfrak{g}$.

(h2) There exists an $\text{Ad}(H)$-invariant scalar product $\langle \cdot, \cdot \rangle_p$ on $\mathfrak{p}$.

(h3) The Lie algebra $\mathfrak{h}$ does not contain any nontrivial ideal of $\mathfrak{g}$.

Notice that (h1) is equivalent to $\text{ad}(x)(v) \in \mathfrak{p}$ for all $x \in \mathfrak{h}$ and all $v \in \mathfrak{p}$, where $\text{ad}(x)(v) = [x, v]$ just denotes the Lie bracket of $\mathfrak{g}$. Condition (h2) is equivalent to $\langle \text{ad}(x)(v), w \rangle_p = -\langle v, \text{ad}(x)(w) \rangle_p$ for all $x \in \mathfrak{h}$ and all $v, w \in \mathfrak{p}$. Notice that (h3) ensures that the locally homogeneous space $G/H$ is almost effective.

Having such an algebraic locally homogeneous space one may determine a locally homogeneous metric $g$ on a local quotient $M^n$ as described in [Sp].

The space $G/H$ is regular or globally homogeneous, if $H$ is a closed subgroup of $G$. By [Kow] there exist locally homogeneous spaces which are not globally homogeneous.

**Theorem 5.2** (Spiro [Sp]). Let $(X^n, g)$ be a locally homogeneous Ricci-flat manifold. Then $(X^n, g)$ is flat.

**Proof.** Let $G, H$ be as above and let the homogeneous metric $g$ be induced by an $\text{Ad}(H)$-invariant scalar product $\langle \cdot, \cdot \rangle_p$ on an $\text{Ad}(H)$-invariant complement $\mathfrak{p}$ of $\mathfrak{h}$ in $\mathfrak{g}$. We assume that $H$ is not a closed subgroup of $G$. 

We define a scalar product on $\mathfrak{h}$ as follows:
\[
\langle x, y \rangle_{\mathfrak{h}} := -\text{tr}_p \left( \text{ad}(x)|_p \circ \text{ad}(y)|_p \right).
\]

We claim, that $\langle \cdot, \cdot \rangle_{\mathfrak{h}}$ is $\text{Ad}(H)$-invariant. This is seen as follows: By (h2) we have
\[
\langle x, x \rangle_{\mathfrak{h}} = \sum \| [x, e_i] \|^2_p
\]
for any orthonormal basis $(e_1, \ldots, e_n)$ of $p$. We need to show that for $v \neq 0$ the right hand side is positive. If not, then a short computation shows that the kernel $\mathfrak{k}$ of $\langle \cdot, \cdot \rangle_{\mathfrak{h}}$ is $\text{Ad}(H)$-invariant and $[\mathfrak{k}, p] = 0$. Hence, $\mathfrak{k}$ is a nontrivial ideal of $\mathfrak{g}$, which was excluded by (h3).

The $\text{Ad}(H)$-invariant scalar product $\langle \cdot, \cdot \rangle_{\mathfrak{h}}$ and $\langle \cdot, \cdot \rangle_{\mathfrak{p}}$ on $\mathfrak{h}$ and $\mathfrak{p}$ respectively, induce an $\text{Ad}(\bar{\mathfrak{h}})$-invariant scalar product $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ on $\mathfrak{g}$ by requiring $\langle \mathfrak{h}, \mathfrak{p} \rangle = 0$. Notice that for $x \in \mathfrak{h}$ $\text{ad}(x)$ acts skew symmetrically on $\mathfrak{g}$ with respect to this scalar product.

Let $\bar{H}$ denote the closure of $H$ in $G$ and let $\bar{\mathfrak{h}}$ denote its Lie algebra. Furthermore let $\mathfrak{h}^\perp$ denote the orthogonal complement of $\mathfrak{h}$ in $\bar{\mathfrak{h}}$. Since $\langle \cdot, \cdot \rangle_{\mathfrak{h}}$ is $\text{Ad}(H)$-invariant it is $\text{Ad}(\bar{H})$-invariant as well. Consequently, $\text{ad}(\bar{x})$ acts skew symmetrically on $\mathfrak{g}$ for $\bar{x} \in \mathfrak{h}^\perp$.

We turn now to curvature computations. We denote by $g_G$ the $G$-invariant metric on $G$, which corresponds to $\langle \cdot, \cdot \rangle$ and by $g_H$ the $H$-invariant metric on $H$, which corresponds to $\langle \cdot, \cdot \rangle_{\mathfrak{h}}$.

By Lemma 2.1 of [Mi], for $\bar{x} \in \mathfrak{h}^\perp$ we have $\text{Ric}_{g_G}(\bar{x}, \bar{x}) \geq 0$ and $\text{Ric}_{g_G}(\bar{x}, \bar{x}) = 0$ if and only if $\langle \bar{x}, [v, w] \rangle = 0$ for all $v, w \in \mathfrak{g}$. Using the O'Neill formula for the Riemannian submersion $(H, g_H) \rightarrow (G, g_G) \rightarrow (G/H, g)$ with totally geodesic fibers (cf. [Bes], (9.36c), [Bes] 9.80 – this computation works also for locally homogeneous spaces), we conclude that $\text{Ric}_{g_G}(\bar{x}, \bar{x}) = 0$, since $\text{Ric}_g \equiv 0$ by assumption.

It follows that $\mathfrak{h}^\perp \perp [\mathfrak{g}, \mathfrak{g}]$. Let $\mathfrak{p}$ denote the orthogonal complement of $\mathfrak{h}$ in $\mathfrak{g}$. Then $[\mathfrak{g}, \mathfrak{h} \oplus \mathfrak{p}] \perp \mathfrak{h}^\perp$, that is $\mathfrak{h} \oplus \mathfrak{p}$ is an ideal in $\mathfrak{g}$. As is well-known this ideal corresponds to a closed normal connected subgroup $N$ of $G$, since $G$ is simply connected (cf. [Pr], p.81). Since $N$ is closed and $H \subset N$ we would conclude $H \subset N$. This is of course a contradiction, meaning that $H$ must have been closed.

It should be mentioned that Spiro’s result shows that any locally homogeneous space with nonpositive Ricci curvature comes from a global homogeneous space. Notice also that by [Tsu] the space of globally homogeneous spaces is dense in the space of locally homogeneous spaces.

**References**


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