Prof. Dr. E. Hellmann
S. Huang
SS 2019

Algebraic Geometry II

Exercise Sheet 9

Due Date: 27.06.2019

Exercise 1:

Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be abelian categories and let $G : \mathcal{A} \to \mathcal{B}$ and $F : \mathcal{B} \to \mathcal{C}$ be left exact additive functors.

(i) Let M be an object of $\mathcal{B}$ and let $0 \to M \to K^\bullet$ be an exact complex such that K^i is acyclic for F for all i. Show that there is a canonical isomorphism

$$H^i(F(K^\bullet)) \to R^i F(M).$$

Hint: Proceed by induction on i.

(ii) Assume that G is exact and G maps injective objects to F-acyclic objects. Show that $R^i(FG)(M) \cong R^i F(GM)$.

(iii) Assume that F is exact. Show that $R^i(FG)(M) \cong F(R^i G(M))$.

Exercise 2:

In this exercise we prove Serre's criterion:
Let X be a quasi-compact scheme such that $H^1(X, \mathscr{I}) = 0$ for all quasi-coherent sheaves of ideals $\mathscr{I}$. Then X is affine. To prove this, proceed as follows:

(i) Let $x \in X$ be a closed point and U an affine neighborhood of x. Let $\mathscr{I}_Y$ be the sheaf of ideals defining a scheme structure on the closed complement Y of U. Further let $\mathscr{I}_{Y \cup \{x\}}$ be the sheaf of ideals vanishing on $Y \cup \{x\}$. Use the short exact sequence

$$0 \longrightarrow \mathscr{I}_{Y \cup \{x\}} \longrightarrow \mathscr{I}_Y \longrightarrow \kappa(x) \longrightarrow 0$$

to show that there exists $f \in \Gamma(X, \mathcal{O}_X)$ with $f(x) \neq 0$ such that $X_f = \{y \in X \mid f(y) \neq 0\}$ is affine.
Hint: Show that there exists an f with $X_f \subset U$

(ii) Let $A = \Gamma(X, \mathcal{O}_X)$. Then by (i) there exist $f_1, \ldots, f_n \in A$ such that $X = \bigcup X_{f_i}$. Show that $(f_1, \ldots, f_n) = A$.
Hint: Let

$$\varphi : \bigoplus_{i=1}^{n} \mathcal{O}_X \longrightarrow \mathcal{O}_X$$

be the morphism $(s_i) \mapsto \sum f_i s_i$. Reduce the claim to the claim $H^1(X, \ker \varphi) = 0$. Then use a filtration of $\ker \varphi$ by quasi-coherent sheaves $\mathscr{F}_i$ such that $\mathscr{F}_i / \mathscr{F}_{i-1}$ is a sheaf of ideals on X to prove the vanishing of H^1.

(iii) Deduce that $X = \operatorname{Spec} A$.

Exercise 3:

Let X be a scheme and let $\mathscr{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals defining a closed immersion $Y \hookrightarrow X$. Assume that $\mathscr{I}$ is nilpotent. Show that X is affine if and only if Y is.

Exercise 4:

Let $X = V_+(f) \subset \mathbb{P}^2_k$ be a closed subscheme defined by some $f \in k[T_0, T_1, T_2]$ that is homogenous of degree d. Assume that $(1:0:0) \notin X$, hence X can be covered by $U = D_+(T_1) \cap X$ and $V = D_+(T_2) \cap X$ and let $\mathfrak{U} = \{U, V\}$. Show that

$$\begin{aligned} &\dim_k \check{H}^0(\mathfrak{U}, \mathcal{O}_X) = 1, \\ &\dim_k \check{H}^1(\mathfrak{U}, \mathcal{O}_X) = \tfrac{1}{2}(d-1)(d-2). \end{aligned}$$

Exercise 5:

Let X be a scheme and let $\mathfrak{U} = (U_i)_{i \in I}$ be an open cover of X. Show that there is a canonical isomorphism

$$\check{H}^1(\mathfrak{U}, \mathcal{O}_X^\times) \cong \left\{ \begin{array}{c} \text{isomorphism classes of lines bundles } \mathcal{L} \\ \text{on } X \text{ such that there exists trivializations} \\ \mathcal{L}|_{U_i} \cong \mathcal{O}_{U_i} \text{ for all } i \in I \end{array} \right\}.$$

Exercise 6:

Let

$$\begin{array}{ccc} S_0 & \xrightarrow{u_0} & X \\ \downarrow & & \downarrow{\scriptstyle f} \\ S & \longrightarrow & Y \end{array}$$

be a commutative diagram of schemes, where $S_0 \hookrightarrow S$ is an infinitesimal thickening of first order defined by a sheaf of ideals $\mathscr{I} \subset \mathcal{O}_S$. Let $S = \bigcup_{i \in I} U_i$ be an open covering and for $i \in I$ let $u_i : U_i \to X$ be a deformation of $u_0|_{U_i \cap S_0}$.

(i) Show that $u_i|_{U_i \cap U_j} - u_j|_{U_i \cap U_j}$ determines an element $\partial_{ij} \in \Gamma(U_i \cap U_j, \mathcal{H}om(u_0^* \Omega^1_{X/Y}, \mathscr{I}))$.

(ii) Show that the $\partial = (\partial_{ij})_{i,j \in I}$ is a 1-Čech-cocycle.

(iii) Show that the cohomology class $[\partial] \in \check{H}^1((U_i)_{i \in I}, , \mathcal{H}om(u_0^* \Omega^1_{X/Y}, \mathscr{I}))$ is independent of the choice of the u_i.

(iv) Show that $[\partial] = 0$ if and only if there exists a deformation $u : S \to X$ of u_0.

Homepage: `https://www.uni-muenster.de/Arithm/hellmann/veranstaltungen`