Prof. Dr. E. Hellmann
S. Huang SS 2019

Algebraic Geometry II

Exercise Sheet 7

Due Date: 23.05.2019

Exercise 1:

(i) Let A be a ring and let $B = A[T_1, \ldots, T_n]$. Show that $\Omega^1_{B/A} \cong \bigoplus_{i=1}^n B\, dT_i$ is free of rank n and that for $f \in A[T_1, \ldots, T_n]$ one has

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial T_i} dT_i,$$

where $\partial/\partial T_i : B \to B$ is the formal derivative with respect to the variable T_i (which is a derivation).

(ii) Let A be a ring and $B = A[X, Y]/(f)$ for some $f \in A[X, Y]$. Show that

$$\Omega^1_{B/A} = (B\, dX \oplus B\, dY)/(\tfrac{\partial f}{\partial X} dX + \tfrac{\partial f}{\partial Y} dY).$$

Show that $\Omega^1_{B/A}$ is locally free of rank 1 if and only if the matrix $\nabla f = (\frac{\partial f}{\partial X}, \frac{\partial f}{\partial Y})$ has rank 1 at all points of $\operatorname{Spec} B$.

Exercise 2:

(i) Let k be a field and let $n \geq 1$ be prime to the characteristic of k. Let $\mathbb{G}_m = \operatorname{Spec} k[T, T^{-1}]$. Show that that the morphism $\mathbb{G}_m \to \mathbb{G}_m$ defined by $T \mapsto T^n$ is étale.

(ii) Let k be a field of characteristic p and let $\mathbb{A}^1_k = \operatorname{Spec} k[T]$. Show that the morphism $\mathbb{A}^1_k \to \mathbb{A}^1_k$ defined by $T \mapsto T^p - T$ is étale.

(iii) Let A be a ring and let $f \in A[T]$. Let $B = A[T]/(f)$. Show that $\operatorname{Spec} B \to \operatorname{Spec} A$ is étale if $f' = \frac{df}{dT} \in A[T]$ becomes a unit in B.
(*Hint: First compute* $\Omega^1_{B/A}$.)

Exercise 3:
Let k be an algebraically closed field and let $f : X \to Y$ be a morphism of smooth k-schemes. Show that the following are equivalent:

(a) f is smooth.

(b) $\Omega^1_{X/Y}$ is locally free.

(c) for all $x \in X(k)$ and $y = f(x) \in Y(k)$ the induced map on tangent spaces $T_x X \to T_y Y$ is surjective.

Exercise 4:

Let k be a perfect field and X be a curve over k, i.e. X is an integral k-scheme of finite type which one-dimensional. Show that X is smooth at a closed point $x \in X$ if and only if the local ring $\mathcal{O}_{X,x}$ is a principal ideal domain.

(*Hint: for the difficult direction let* $f \in \mathcal{O}_{X,x}$ *be a generator of the maximal ideal that is defined in a neighborhood* U *of* x. *Show that the morphism* $U \to \mathbb{A}^1_k$ *that is defined by* f *is étale at* x.)

Homepage: `https://www.uni-muenster.de/Arithm/hellmann/veranstaltungen`