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SS 2019

Algebraic Geometry II

Exercise Sheet 6

Due Date: 16.05.2019

Exercise 1+2:

Let X be a k-scheme of finite type and let $k[\epsilon] = k[T]/(T^2)$ be the *ring of dual numbers* which is a first order thickening Spec $k \to$ Spec $k[\epsilon]$.

(i) Let $x \in X(k)$ be a k-valued point. Show that $\Omega^1_{X/k} \otimes k(x) \cong \mathfrak{m}_x/\mathfrak{m}_x^2$ as $k(x)$-vector spaces, where $\mathfrak{m}_x \subset \mathcal{O}_{X,x}$ is the maximal ideal.

(ii) Write $f_x : \mathrm{Spec}\ k \to X$ for the morphism defining x. Show that

$$\begin{aligned}\mathrm{Def}(f_x) :&= \{f_x^{(1)} : \mathrm{Spec}\ k[\epsilon] \to X \text{ morphism of } k \text{ -schemes deforming } f_x\}\\ &= \mathrm{Hom}_{k(x)}(\mathfrak{m}_x/\mathfrak{m}_x^2, k(x))\\ &= (\mathbf{T}_x X)(k),\end{aligned}$$

where $\mathbf{T}_x X = \mathrm{Spec}\,(\mathrm{Sym}(\mathfrak{m}_x/\mathfrak{m}_x^2))$ is the *tangent space* of X at x (viewed as a scheme).

(iii) Show that there is a canonical closed immersion

$$\mathbf{C}_x X := \mathrm{Spec}\ \big(\bigoplus\nolimits_{d\geq 0} \mathfrak{m}_x^d/\mathfrak{m}_x^{d+1}\big) \longrightarrow \mathbf{T}_x X$$

of the *tangent cone* into the tangent space.
Show further that a compatible system of deformations

$$f_x^{(n)} : \mathrm{Spec}\, k[T]/(T^{n+1}) \to X$$

of f_x such that $f_x^{(1)}$ does not factor over Spec k gives rise to a k-valued point

$$f \in \mathrm{Proj}\big(\bigoplus\nolimits_{d\geq 0} \mathfrak{m}_x^d/\mathfrak{m}_x^{d+1}\big)$$

or equivalently to a line in $\mathbf{C}_x X$. Deduce that $\mathbf{C}_x X \to \mathbf{T}_x X$ is an isomorphism if X is smooth at x.

(iv) Assume that X is irreducible of dimension n. Show that $\mathbf{C}_x X$ is n-dimensional.
(*Hint: Show that* $\mathrm{Bl}_{\{x\}}X$ *is* n*-dimensional and deduce that the fiber of* $\mathrm{Bl}_{\{x\}}X$ *over* x *is* $n-1$*-dimensional. Then compare this fiber to* $\mathbf{C}_x X$.)

(v) Compute the tangent space and the tangent cone of X at x in the following cases:

(a) $X = \mathrm{Spec}\ k[T_1, T_2]/(T_1^3 - T_2^2)$ and $x = (0, 0)$.
(b) $X = \mathrm{Spec}\ k[T_1, T_2]/(T_1^2(T_1 + 1) - T_2^2)$ and $x = (0, 0)$.
(c) $X = \mathrm{Spec}\ k[T_1, T_2]/(T_1^2 + T_2^2 - 1)$ and $x = (1, 0)$.

Exercise 3:

Let k be a field. And let $f : \operatorname{Spec} A \to k$ be a morphism. Show that the following are equivalent:

(a) f is étale

(b) f is unramified

(c) A is isomorphic to a direct product of finitely many finite separable field extensions of k.

Exercise 4:

(i) Let A be a noetherian ring an let E be a finitely generated A-module. Write $\mathscr{E}$ for the coherent sheaf on $\operatorname{Spec} A$ defined by E. Show that $\operatorname{Proj}(\operatorname{Sym} E)$ represents the functor

$$(f : X \to \operatorname{Spec} A) \longmapsto \left\{ \begin{matrix} \text{Isomorphism} \\ \text{classes of } (\mathscr{L}, \phi) \end{matrix} \;\middle|\; \begin{matrix} \mathscr{L} \text{ a line bundle on } X \text{ and } \phi : f^*\mathscr{E} \to \mathscr{L} \\ \text{a surjection of } \mathcal{O}_X\text{-modules} \end{matrix} \right\}$$

on the category of A-schemes.

Hint: Choose a surjection $A^{n+1} \twoheadrightarrow E$ *which induces* $\operatorname{Proj}(\operatorname{Sym} E) \hookrightarrow \mathbb{P}^n_A$ *a closed immersion. Show that the morphism* $X \to \mathbb{P}^n_A$ *defined by a surjection* $\mathcal{O}_X^{n+1} \twoheadrightarrow \mathscr{L}$ *agrees with the composition*

$$X \cong \underline{\operatorname{Proj}}_X(\operatorname{Sym} \mathscr{L}) \longrightarrow \underline{\operatorname{Proj}}_X(\operatorname{Sym} \mathcal{O}_X^{n+1}) = \mathbb{P}^n_X \longrightarrow \mathbb{P}^n_A,$$

and deduce that $X \to \mathbb{P}^n_A$ *factors through* $\operatorname{Proj}(\operatorname{Sym} E)$ *if and only if* $\mathcal{O}_X^{n+1} \twoheadrightarrow \mathscr{L}$ *factors through* $\mathcal{O}_X^{n+1} \twoheadrightarrow f^*\mathscr{E}$.

(ii) Let Y be a noetherian scheme and let $\mathscr{E}$ a coherent sheaf on Y. Show that $\underline{\operatorname{Proj}}_Y(\operatorname{Sym} \mathscr{E})$ represents the functor

$$(f : X \to Y) \longmapsto \left\{ \begin{matrix} \text{Isomorphism} \\ \text{classes of } (\mathscr{L}, \phi) \end{matrix} \;\middle|\; \begin{matrix} \mathscr{L} \text{ a line bundle on } X \text{ and } \phi : f^*\mathscr{E} \to \mathscr{L} \\ \text{a surjection of } \mathcal{O}_X\text{-modules} \end{matrix} \right\}$$

on the category of Y-schemes.

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