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Algebraic Geometry II

Exercise Sheet 4

Due Date: 02.05.2019

Exercise 1:

Let $S = \bigoplus_{d\geq 0} S_d$ be a graded ring and let $0 \neq f \in S_d$. Show that f induces a morphism $\phi : \mathcal{O}_X \to \mathcal{O}_X(d)$, where $X = \operatorname{Proj} S$. Furthermore, show that

$$D_+(f) = \{x \in X \mid \phi_x : \mathcal{O}_{X,x} \to \mathcal{O}_X(d)_x \text{ is an isomorphism}\}.$$

Exercise 2:

Let $S = \bigoplus_{d\geq 0} S_d$ be a graded ring such that S is generated by finitely many elements in S_1 as an S_0-algebra. Let $X = \operatorname{Proj} S$ and $U = X \backslash V(S_+) \subset \operatorname{Spec} S$. Finally let $M = \bigoplus_{d\in\mathbb{Z}} M_d$ be a graded S-module.

(i) Let $f \in S_1$. Show that the canonical maps $\operatorname{Spec} S_f \to \operatorname{Spec} S_{(f)}$ glue to give a canonical map $\pi : U \to \operatorname{Proj} S$.

(ii) Let $f \in S_1$. Show that the canonical map $M_{(f)} \otimes_{S_{(f)}} S_f \to M_f$ induced by the inclusion $M_{(f)} \to M_f$ is an isomorphism.

(iii) Let $\mathscr{F}$ denote the quasi-coherent sheaf on $\operatorname{Spec} S$ such that by $\Gamma(\operatorname{Spec} S, \mathscr{F}) = M$ and let $\mathscr{G}$ denote the quasi-coherent sheaf on $\operatorname{Proj} S$ defined by the graded S-module M. Show that there is a canonical isomorphism $\pi^*\mathscr{G} \cong \mathscr{F}|_U$.

(iv) With the notations in (iii), deduce that $\mathscr{G} = 0$ if and only if the coherent sheaf $\mathscr{F}$ is supported on $V(S_+) \subset \operatorname{Spec} S$.

Exercise 3:

Let X be a scheme and let $\mathscr{S} = \bigoplus_{d\geq 0} \mathscr{S}_d$ be a quasi-coherent graded $\mathcal{O}_X$-algebra.

(i) Fix $n > 0$ and define $\mathscr{S}^{(n)} = \bigoplus_{d\geq 0} \mathscr{S}_{dn}$ which is again a quasi-coherent graded $\mathcal{O}_X$-algebra. Show that $\operatorname{Proj}_X \mathscr{S} \cong \operatorname{Proj}_X \mathscr{S}^{(n)}$ as X-schemes.

(ii) Let $\mathscr{L}$ be a line bundle on X. Show that $\mathscr{S}' = \bigoplus_{d\geq 0}(\mathscr{S}_d \otimes \mathscr{L}^{\otimes d})$ is in a natural way a quasi-coherent graded $\mathcal{O}_X$-algebra, and show that $\operatorname{Proj}_X \mathscr{S} \cong \operatorname{Proj}_X \mathscr{S}'$ as X-schemes.

Exercise 4:

Let k be a field and $n \geq 1$ and let $S = k[T_0, \dots, T_n]$ and $\mathbb{P}^n_k = \operatorname{Proj} S$. For $d \in \mathbb{Z}$ we write $S_d \subset S$ for the elements that are homogenous of degree d.

(i) Show that $\Gamma(\mathbb{P}^n_k, \mathcal{O}(d)) = S_d$.

(ii) Assume that $d \geq 0$ and write $N = \dim_k S_d - 1 = \binom{n+d}{d} - 1$. Show that the choice of a k-basis of S_d induces a surjection $\mathcal{O}_{\mathbb{P}^n_k}^{N+1} \to \mathcal{O}(d)$.

(iii) Show that the map $\mathbb{P}^n_k \to \mathbb{P}^N_k$ defined by the surjection from (ii) is a closed embedding. *This embedding is called the d-fold Vernoese embedding.*

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