THE IMAGE OF THE COEFFICIENT SPACE IN THE UNIVERSAL DEFORMATION SPACE OF A FLAT GALOIS REPRESENTATION OF A p-ADIC FIELD

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Abstract. The coefficient space is a kind of resolution of singularities of the universal flat deformation space for a given Galois representation of some local field. It parametrizes (in some sense) the finite flat models for the Galois representation. The aim of this note is to determine the image of the coefficient space in the universal deformation space.

1. Introduction

In the theory of deformations of Galois representations one is often interested in a subfunctor of the universal deformation functor consisting of those deformations that satisfy certain extra conditions, so called *deformation conditions* (cf. [Ma, §23]). If we deal with a representation of the absolute Galois group of a finite extension K of $\mathbb{Q}_p$ in a finite dimensional vector space in characteristic p, there is the deformation condition of being *flat*, which means that there is a finite flat group scheme over the ring of integers of K such that the given Galois representation is isomorphic to the action of the Galois group on the generic fiber. The structure of the ring pro-representing this deformation functor is of interest for modularity lifting theorems (see [Ki1] for example). To get more information about this structure, Kisin constructs some kind of "resolution of singularities" of the spectrum of the flat deformation ring. This resolution is a scheme parametrizing modules with additional structure that define possible extensions of the representation to a finite flat group scheme over the ring of integers. In [PR2] Pappas and Rapoport globalize Kisin's construction and define a so called *coefficient space* parametrizing all Kisin modules that give rise to the given representation.
Following the presentation in [PR2] we want to determine here the image of the coefficient space in the universal deformation space. This question was raised by Pappas and Rapoport in [PR2, 4.c] (see the remark right after Theorem 3.7 for a more preceise comparison of our result with the question of [PR2]). Further we show how to recover Kisin's results from the more abstract setting in [PR2], compare Remark 3.6. The main result of this note is as follows.
Let K be a finite extension of $\mathbb{Q}_p$, where p is an odd prime, and $\bar{\rho}: G_K \longrightarrow \mathrm{GL}_d(\mathbb{F})$ be a continuous flat representation of the absolute Galois group $G_K = \mathrm{Gal}(\bar{K}/K)$ on some d-dimensional vector space over a finite field $\mathbb{F}$ of characteristic p. If $\xi : G_K \longrightarrow \mathrm{GL}_d(A)$ is a deformation of $\bar{\rho}$, we write $C_K(\xi)$ for the coefficient space of (locally free) Kisin modules over Spec A that are related to the flat models for the deformation ξ (see also the definition below).

Theorem 1.1. *Assume that the flat deformation functor of $\bar{\rho}$ is pro-representable by a complete local noetherian ring R^{fl}. We write $\rho : G_K \to \mathrm{GL}_d(R^{\mathrm{fl}})$ for the* *universal flat deformation. Then the morphism* $C_K(\rho) \to \operatorname{Spec} R^{\mathrm{fl}}$ *is topologically surjective.*

Corollary 1.2. *If it exists, the flat deformation ring* R^{fl} *is topologically flat, i.e. the generic fiber* $\operatorname{Spec} R^{\mathrm{fl}}[1/p]$ *is dense in* $\operatorname{Spec} R^{\mathrm{fl}}$.

If the ramification index of the local field K over $\mathbb{Q}_p$ is smaller than $p-1$, then this implies the following result, already contained in [PR2]:

Corollary 1.3. *Denote by* e *the ramification index of* K *over* $\mathbb{Q}_p$. *Assume that the flat deformation functor of* $\bar{\rho}$ *is pro-representable and that* $e < p-1$. *Then* R^{fl} *is the scheme theoretic image of the coefficient space.*

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2. Notations

Let p be an odd prime and K be a finite extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}_K$, uniformizer $\pi \in \mathcal{O}_K$ and residue field $k = \mathcal{O}_K/\pi\mathcal{O}_K$. Denote by K_0 the maximal unramified extension of $\mathbb{Q}_p$ in K and by $W = W(k)$ its ring of integers, the ring of Witt vectors with coefficients in k. We write $E(u) \in W[u]$ for the minimal polynomial of π over K_0.

Fix an algebraic closure $\bar{K}$ of K and denote by $G_K = \operatorname{Gal}(\bar{K}/K)$ the absolute Galois group of K. Further we choose a compatible system π_n of p^n-th roots of the uniformizer π in $\bar{K}$ and denote by K_∞ the subfield $\bigcup K(\pi_n)$ of $\bar{K}$. We write $G_{K_\infty} = \operatorname{Gal}(\bar{K}/K_\infty)$ for its absolute Galois group.

Let $d > 0$ be an integer and $\mathbb{F}$ a finite field of characteristic p. Let $\bar{\rho} : G_K \to \mathrm{GL}_d(\mathbb{F})$ be a continuous representation of G_K and denote by $\bar{\rho}_\infty = \bar{\rho}|_{G_{K_\infty}}$ the restriction of $\bar{\rho}$ to G_{K_∞}.

We consider the deformation functors $\mathcal{D}_{\bar{\rho}}$, $\mathcal{D}_{\bar{\rho}}^{\mathrm{fl}}$ and $\mathcal{D}_{\bar{\rho}_\infty}$ on local Artinian $W(\mathbb{F})$-algebras with residue field $\mathbb{F}$. For a local Artinian ring $(A, \mathfrak{m})$ we have

$$\mathcal{D}_{\bar{\rho}}(A) = \left\{ \begin{array}{c} \text{equivalence classes of } \rho : G_K \to \mathrm{GL}_d(A) \text{ such that} \\ \rho \bmod \mathfrak{m} = \bar{\rho} \end{array} \right\}$$

$$\mathcal{D}_{\bar{\rho}_\infty}(A) = \left\{ \begin{array}{c} \text{equivalence classes of } \rho : G_{K_\infty} \to \mathrm{GL}_d(A) \text{ such that} \\ \rho \bmod \mathfrak{m} = \bar{\rho}_\infty \end{array} \right\},$$

where two lifts ρ_1, ρ_2 are said to be equivalent if they are conjugate under some $g \in \ker(\mathrm{GL}_d(A) \to \mathrm{GL}_d(A/\mathfrak{m}))$. The functor $\mathcal{D}_{\bar{\rho}}^{\mathrm{fl}}$ is the flat deformation functor of Ramakrishna (cf. [Ram]), i.e. the subfunctor of $\mathcal{D}_{\bar{\rho}}$ consisting of all deformations that are (isomorphic to) the generic fiber of some finite flat group scheme over $\operatorname{Spec} \mathcal{O}_K$. Here "isomorphic to" means isomorphic as $\mathbb{Z}_p[G_K]$-modules, as the action of the coefficients in the generic fiber does not need to extend to the group scheme.

If $\mathcal{D}_{\bar{\rho}}$ (resp. $\mathcal{D}_{\bar{\rho}}^{\mathrm{fl}}$) is pro-representable, the pro-representing ring will be denoted by R (resp. R^{fl}).

Remark 2.1. Note that it is not clear whether $\mathcal{D}_{\bar\rho_\infty}$ is representable, even if $\bar\rho_\infty$ is absolutely irreducible, since G_{K_∞} does not satisfy Mazur's p-finiteness condition (cf. [Ma, §1. Definition]). As there is an isomorphism

$$G_{K_\infty} = \mathrm{Gal}(\bar K/K_\infty) \cong \mathrm{Gal}(k((u))^{\mathrm{sep}}/k((u))),$$

each open subgroup of finite index $H \subset G_{K_\infty}$ is isomorphic to the absolute Galois group of some local field in characteristic p,

$$H \cong \mathrm{Gal}(l((t))^{\mathrm{sep}}/l((t))),$$

where l is a finite extension of k and t is an indeterminate. Hence by Artin-Schreier theory (cf. [Se, X §3.a] for example), there is an isomorphism

$$\mathrm{Hom}_{\mathrm{cont}}(H, \mathbb{Z}/p\mathbb{Z}) \cong l((t))/\wp(l((t))),$$

and the latter group is infinite.
If one restricts the attention to G_{K_∞}-representations of E-height $\leq h$, then the deformation functor $\mathcal{D}^{\leq h}_{\bar\rho_\infty}$ is representable if $\mathrm{End}_{\mathbb{F}}(\bar\rho_\infty) = \mathbb{F}$ (see [Kim, Theorem 11.1.2]). The E-height of a p-torsion G_{K_∞}-representation is defined as the minimal h such that the étale ϕ-modules associated to the representation admits an $W[[u]]$-lattice with cokernel of the linearisation of Φ killed by $E(u)^h$ (see [Kim, Definition 5.2.8] for the precise definition).

As the deformation functor will not be pro-representable in the case of G_{K_∞}-representations, it will not be enough to consider the deformation functor, but we really need to work with the deformation groupoid of $\bar\rho_\infty$. That is the fpqc-stack $\mathfrak{D}_{\bar\rho_\infty}$ on local Artinian $W(\mathbb{F})$-algebras with residue field $\mathbb{F}$ whose $(A,\mathfrak{m})$-valued points are given by the groupoid of deformations of $\bar\rho_\infty$ to A, i.e.

$$\mathfrak{D}_{\bar\rho_\infty}(A) = \{\rho : G_K \to \mathrm{GL}_d(A) \text{ such that } \rho \mod \mathfrak{m} = \bar\rho_\infty\}.$$

The morphisms of the groupoid $\mathfrak{D}_{\bar\rho_\infty}$ are all isomorphisms lifting the identity, i.e. a morphism in $\mathfrak{D}_{\bar\rho_\infty}(A)$ is given by g-conjugation, where $g \in \ker(\mathrm{GL}_d(A) \to \mathrm{GL}_d(\mathbb{F}))$. As there can be non-trivial automorphisms we sometimes need to rigidify the situation. This is done by considering framed deformations, i.e. the groupoid

$$\mathfrak{D}^{\square}_{\bar\rho_\infty}(A) = \{\rho : G_K \to \mathrm{GL}_d(A) \text{ such that } \rho \mod \mathfrak{m} = \bar\rho_\infty\},$$

where there are no non-trivial morphisms, i.e. all morphisms in the groupoid $\mathfrak{D}^{\square}_{\bar\rho_\infty}(A)$ are the identities.
Similarly we consider the framed deformation groupoid $\mathfrak{D}^{\square,\mathrm{fl}}_{\bar\rho}$.
Recall that $d > 0$ denotes an integer and consider the following stacks on $\mathbb{Z}_p$-algebras, defined in [PR2]. For a $\mathbb{Z}_p$-algebra R, write R_W for $R \otimes_{\mathbb{Z}_p} W$ and $R_W[[u]]$ for the usual power series ring with coefficients in R_W. Further $R_W((u)) = R_W[[u]](u^{-1})$ denotes the Laurent series ring over R_W. We denote by ϕ the endomorphism of $R_W((u))$ that is the identity on R, the Frobenius on W and that maps u to u^p.
We define an fpqc-stack $\mathcal{R}$ on the category of $\mathbb{Z}_p$-schemes such that for a $\mathbb{Z}_p$-algebra R the groupoid $\mathcal{R}(R)$ is the groupoid of pairs (M, Φ), where M is an $R_W((u))$-module that is fpqc-locally on Spec R free of rank d as an $R_W((u))$-module, and Φ is an isomorphism $\phi^*M \longrightarrow M$.
Further we define a stack $\mathcal{C}$ as follows. The R-valued points are pairs $(\mathfrak{M}, \Phi)$, where $\mathfrak{M}$ is an $R_W[[u]]$-module, fpqc-locally on Spec R free of rank d over $R_W[[u]]$, and

$(\mathfrak{M}[1/u], \Phi) \in \mathcal{R}(R)$. For $m \in \mathbb{Z}$ consider the substacks $\mathcal{C}_m \subset \mathcal{C}$ given by pairs $(\mathfrak{M}, \Phi)$ satisfying

$$u^m\mathfrak{M} \subset \Phi(\phi^*\mathfrak{M}) \subset u^{-m}\mathfrak{M}. \tag{2.1}$$

For $h \in \mathbb{N}$ we write $\mathcal{C}_{h,K}$ for the substack of $\mathcal{C}$ consisting of all $(\mathfrak{M}, \Phi)$ satisfying

$$E(u)^h\mathfrak{M} \subset \Phi(\phi^*\mathfrak{M}) \subset \mathfrak{M}.$$

Here $E(u) \in W[u]$ is the minimal polynomial of the uniformizer $\pi \in \mathcal{O}_K$ over K_0. In the following we will only consider the case $h = 1$ and just write $\mathcal{C}_K$ for $\mathcal{C}_{1,K}$. We will write $\widehat{\mathcal{C}}_K$ resp. $\widehat{\mathcal{R}}$ for the restrictions of the stacks $\mathcal{C}_K$ (resp. $\mathcal{R}$) to the category Nil_p of $\mathbb{Z}_p$-schemes on which p is locally nilpotent. See also [PR2, §2] for the definitions.
The motivations for these definitions are the following equivalences of categories.

Proposition 2.2. *Let A be a local Artin ring with residue field $\mathbb{F}$ a finite field of characteristic p. Then the category of G_{K_∞}-representations on free A-modules of rank d is equivalent to the category of étale ϕ-modules over $(A \otimes_{\mathbb{Z}_p} W)((u))$ that are free of rank d.*

Theorem 2.3. *There is an equivalence between the category of finite flat group schemes $\mathcal{G}$ over* $\mathrm{Spec}\ \mathcal{O}_K$ *and the category of pairs $(\mathfrak{M}, \Phi)$, where $\mathfrak{M}$ is a $W[[u]]$-module of projective dimension* 1 *and $\Phi : \mathfrak{M} \to \mathfrak{M}$ is a ϕ-linear map such that the cokernel of the linearisation of Φ is killed by $E(u)$. Under this equivalence the restriction of the Tate twist of the G_K-representation on $\mathcal{G}(\bar{K})$ to G_{K_∞} corresponds to the étale ϕ-module $(\mathfrak{M}[1/u], \Phi)$.*

The first equivalence is due to Fontaine, see [Fo]. The second one is a theorem of Kisin for $p \geq 3$, see [Ki1, §1] which was independently extended to the case $p = 2$ by Kim [Kim2], Liu [Liu] and Lau [Lau].
Further we will use the following notations: Let $(A, \mathfrak{m})$ be a complete, local noetherian $W(\mathbb{F})$-algebra such that p is nilpotent in $A/\mathfrak{m}$ and $\xi : \mathrm{Spf}\ A \to \widehat{\mathcal{R}}$ be an A-valued point of $\widehat{\mathcal{R}}$. Write ξ_n for the reduction of ξ modulo $\mathfrak{m}^{n+1}$. By [PR2, Corollary 2.6; 3.b] the fiber product

$$\mathrm{Spec}(A/\mathfrak{m}^{n+1}) \times_{\mathcal{R}} \mathcal{C}_K$$

is representable by a projective $A/\mathfrak{m}^{n+1}$-scheme $C_K(\xi_n)$ that is a closed subscheme of some affine Grassmannian over $\mathrm{Spec}(A/\mathfrak{m}^{n+1})$ for all $n \geq 0$. These schemes give rise to a formal scheme $\widehat{C}_K(\xi)$ over $\mathrm{Spf}\ A$. Using the very ample line bundle on the affine Grassmannian this formal scheme is algebraizable. The resulting projective scheme over $\mathrm{Spec}\ A$ will be denoted by $C_K(\xi)$. If A is a complete, local Notherian ring with finite residue field, and if ρ is a deformation of a residual Galois representation $\bar{\rho}$ to A and if we write $\xi : \mathrm{Spf}\ A \to \widehat{\mathcal{R}}$ for the morphism induced by $\rho|_{G_{K_\infty}}$ under the equivalence in Proposition 2.2, then we also write $C_K(\rho) = C_K(\xi)$ for the scheme constructed above.

Remark 2.4. Note that this does not give an arrow $C_K(\xi) \to \mathcal{C}_K$. For example the module $\mathfrak{M} = W[[u]]$ together with the ϕ-linear map Φ given by $\Phi(1) = E(u)$ does not define a $\mathbb{Z}_p$-valued point of $\mathcal{C}_K$ but rather a "formal" point

$$\mathrm{Spf}\ \mathbb{Z}_p \to \widehat{\mathcal{C}}_K.$$

However if B is some $\mathbb{Z}_p$-algebra killed by some power of p, then

$$E(u) \in (B\widehat{\otimes}_{\mathbb{Z}_p} W((u)))^\times,$$

and hence any locally free $B\widehat{\otimes}_{\mathbb{Z}_p} W[[u]]$-modules $\mathfrak{M}$ with semi-linear map Φ satisfying

$$E(u)\mathfrak{M} \subset \Phi(\phi^*\mathfrak{M}) \subset \mathfrak{M}$$

defines a B-valued point of $\mathcal{C}_K$.

3. The image of the coefficient space

In the following we will assume that the representation $\bar{\rho}$ is flat (i.e. is the generic fiber of some finite flat group scheme over Spec $\mathcal{O}_K$) and that $\mathcal{D}_{\bar{\rho}}^{\mathrm{fl}}$ is representable. This is the case if, for example, $\mathrm{End}_{\mathbb{F}}(\bar{\rho}) = \mathbb{F}$ (cf. [Co, Theorem 2.3]). We write ρ for the universal flat deformation.

By Proposition 2.2 we have a map

$$\mathfrak{D}_{\bar{\rho}_\infty} \longrightarrow \widehat{\mathcal{R}} \tag{3.1}$$

of stacks on local Artinian $W(\mathbb{F})$-algebras with residue field $\mathbb{F}$, see also [PR2, 4.a] and [Ki1, 1.2.6, 1.2.7]. Note that this map only exists if we use the deformation groupoid instead of the deformation functor, as the deformation functor is not pro-representable.

For some local Artinian ring A and some $\xi \in \mathfrak{D}_{\bar{\rho}_\infty}(A)$ we write $M(\xi) \in \widehat{\mathcal{R}}(A)$ for the corresponding Φ-module. More precisely, this map identifies $\mathfrak{D}_{\bar{\rho}_\infty}$ with $\widehat{\mathcal{R}}_{[\bar{\rho}_\infty]}$ (cf. [PR2, 4.a]). The latter fibered category is given by all deformations in $\widehat{\mathcal{R}}$ of the Φ-module $M(\bar{\rho}_\infty)$. Especially we find that the map in (3.1) is formally smooth and hence so it the morphism $\mathfrak{D}_{\bar{\rho}_\infty}^{\square} \to \widehat{\mathcal{R}}$.

Lemma 3.1. *Write for the moment $\widehat{C}_K$ for the 2-fiber product of $\widehat{\mathcal{C}}_K$ and $\mathfrak{D}_{\bar{\rho}}^{\mathrm{fl}}$ over $\widehat{\mathcal{R}}$. Composing the canonical projection $\widehat{C}_K \to \mathfrak{D}_{\bar{\rho}}^{\mathrm{fl}}$ with the morphism $\mathfrak{D}_{\bar{\rho}}^{\mathrm{fl}} \to \mathfrak{D}_{\bar{\rho}_\infty}$ obtained by restriction to G_{K_∞}, we obtain a 2-cartesian diagram of stacks on local Artinian $W(\mathbb{F})$-algebras with residue field $\mathbb{F}$:*

$$\begin{array}{ccc} \widehat{C}_K & \longrightarrow & \widehat{\mathcal{C}}_K \\ \downarrow & & \downarrow \\ \mathfrak{D}_{\bar{\rho}_\infty} & \longrightarrow & \widehat{\mathcal{R}}. \end{array}$$

The same conclusion also holds true if $\mathfrak{D}_{\bar{\rho}}^{\mathrm{fl}}$ and $\mathfrak{D}_{\bar{\rho}_\infty}$ are replaced by $\mathfrak{D}_{\bar{\rho}}^{\square,\mathrm{fl}}$ and $\mathfrak{D}_{\bar{\rho}_\infty}^{\square}$, respectively.

Proof. Let A be a local Artinian $W(\mathbb{F})$-algebra such that $p^nA = 0$ for some $n > 0$. We have to show that there is a natural equivalence of categories

$$\widehat{C}_K(A) \to \left\{ \begin{array}{c} ((\mathfrak{M},\Phi),\xi,\alpha) \text{ with } (\mathfrak{M},\Phi) \in \mathcal{C}_K(A)\ ,\ \xi \in \mathfrak{D}_{\bar{\rho}_\infty}(A) \\ \text{and an isomorphism } \alpha : (\mathfrak{M}[\tfrac{1}{u}],\Phi) \longrightarrow M(\xi) \end{array} \right\}. \tag{3.2}$$

where

$$\widehat{C}_K(A) = \left\{ \begin{array}{c} ((\mathfrak{M},\Phi),\rho',\beta) \text{ with } (\mathfrak{M},\Phi) \in \mathcal{C}_K(A) \\ \text{a flat lift } \rho' \in \mathfrak{D}_{\bar{\rho}}^{\mathrm{fl}}(A) \text{ of } \bar{\rho} \\ \text{and an isomorphism } \beta : (\mathfrak{M}[\tfrac{1}{u}],\Phi) \longrightarrow M(\rho'|_{G_{K_\infty}}) \end{array} \right\}.$$

First it is clear that $\mathfrak{D}^{\mathrm{fl}}_{\bar\rho} \to \mathfrak{D}_{\bar\rho_\infty}$ induces the natural map in 3.2. This map is fully faithful, as the restriction to G_{K_∞} is fully faithful on the category of flat p-power torsion representations by [Br, Theorem 3.4.3]. We have to show that it is essentially surjective.
Let $x = ((\mathfrak{M}, \Phi), \xi, \alpha)$ be an A-valued point of the right hand side.
Then $(\mathfrak{M}, \Phi) \in \mathcal{C}_K(A)$ and by Theorem 2.3 there is an associated flat representation $\widetilde{\xi}$ of G_K such that

$$\beta : (\mathfrak{M}[1/u], \Phi) \xrightarrow{\ \cong\ } M(\widetilde{\xi}|_{G_{K_\infty}}).$$

This shows that $y = ((\mathfrak{M}, \Phi), \widetilde{\xi}, \beta)$ defines a unique point in $\widehat{C}_K(A)$. It follows from the construction that this point maps to x. $\square$

Recall that R^{fl} pro-represents the flat deformation functor with universal representation ρ. We denote by $R^{\square,\mathrm{fl}}$ and $\rho^\square$ the corresponding objects for framed deformations.

Proposition 3.2. *Let $C_K(\rho^\square)$ denote the projective $R^{\square,\mathrm{fl}}$-scheme obtained from $\widehat{C}_K(\rho^\square)$ by algebraization.*

(i) *The generic fiber $C_K(\rho^\square) \otimes_{W(\mathbb{F})} W(\mathbb{F})[1/p]$ is reduced, normal and Cohen-Macaulay.*
(ii) *The reduced subscheme underlying the special fiber $C_K(\rho^\square) \otimes_{W(\mathbb{F})} \mathbb{F}$ is normal and with at most rational singularities.*
(iii) *The scheme $C_K(\rho^\square)$ is topologically flat, i.e. its generic fiber is dense.*

Proof. This is similar to [Ki1, Proposition 2.4.6].
Denote by $y : \operatorname{Spec} \mathbb{F} \to \mathcal{R}$ the $\mathbb{F}$-valued point defined by $\bar\rho_\infty$. Let x be a closed point of $C_K(\rho^\square)$. Extending scalars if necessary, we may assume that x is defined over $\mathbb{F}$. Denote by $(M_0, \Phi_0) \in \mathcal{R}(\mathbb{F})$ the Φ-module defined by y and by $(\mathfrak{M}_0, \Phi_0) \in \mathcal{C}_K(\mathbb{F})$ the Φ-module defined by x. We want to compare the structure of the local ring $\mathcal{O}_{C_K(\rho^\square),x}$ (resp. its completion) to the structure of a local model M_K defined in [PR2, 3.a]. By loc. cit. Theorem 0.1. there is a "local model"-diagram of stacks on the category Nil_p

$$\begin{tikzcd} & \widehat{\widetilde{\mathcal{C}}}_K \arrow[dl, "\pi"'] \arrow[dr, "\phi"] & \\ \widehat{\mathcal{C}}_K & & \widehat{M}_K, \end{tikzcd} \tag{3.3}$$

with π and ϕ formally smooth. Here the B-valued points of the stack $\widetilde{\mathcal{C}}_K$ are the Φ-modules $(\mathfrak{M}, \Phi) \in \mathcal{C}_K(B)$ together with an isomorphism $\mathfrak{M} \to (B\widehat{\otimes}_{\mathbb{Z}_p} W[[u]])^d$, for a $\mathbb{Z}_p$-algebra B.
We consider the following groupoids on local Artinian $W(\mathbb{F})$-algebras: Denote by $\mathfrak{D}_x$ and $\mathfrak{D}_y$ the groupoids

$$\mathfrak{D}_x(B) = \left\{ \begin{array}{c} (\mathfrak{M}, \Phi) \in \mathcal{C}_K(B) \text{ equipped with an isomorphism} \\ (\mathfrak{M} \otimes_B (B/\mathfrak{m}_B), \Phi \otimes \mathrm{id}) \cong (\mathfrak{M}_0 \otimes_{\mathbb{F}} (B/\mathfrak{m}_B), \Phi_0 \otimes \mathrm{id}) \end{array} \right\},$$

$$\mathfrak{D}_y(B) = \left\{ \begin{array}{c} (M, \Phi) \in \mathcal{R}(B) \text{ equipped with an isomorphism} \\ (M \otimes_B (B/\mathfrak{m}_B), \Phi \otimes \mathrm{id}) \cong (M_0 \otimes_{\mathbb{F}} (B/\mathfrak{m}_B), \Phi_0 \otimes \mathrm{id}) \end{array} \right\}.$$

Fixing a basis of $\mathfrak{M}_0$ we may view x as an $\mathbb{F}$-valued point of $\widetilde{\mathcal{C}}_K$. Denote by $\widetilde{\mathfrak{D}}_x$ the groupoid of deformations of x in $\widetilde{\mathcal{C}}_K$.

Under the morphism ϕ in (3.3), the point x maps to a point $\bar{x}$ of M_K. This point defines an $\mathbb{F}\otimes_{\mathbb{Z}_p}\mathcal{O}_K$-submodule $L\subset(\mathbb{F}\otimes_{\mathbb{Z}_p}\mathcal{O}_K)^d$. Let $\mathfrak{D}_{\bar{x}}$ be the groupoid of deformations of $\bar{x}$, i.e.

$$\mathfrak{D}_{\bar{x}}(B)=\{B\otimes_{\mathbb{Z}_p}\mathcal{O}_K-\text{submodules } \mathcal{L}\subset(B\otimes_{\mathbb{Z}_p}\mathcal{O}_K)^d\mid \mathcal{L}\otimes_B(B/\mathfrak{m}_B)\cong L\otimes_{\mathbb{F}}(B/\mathfrak{m}_B)\}.$$

This groupoid is pro-represented by the completion of the local ring $\mathcal{O}_{M_K,\bar{x}}$. Now we have the following commutative diagram

$$\begin{tikzcd}
 & & \widetilde{\mathfrak{D}}_x \arrow[dl, "\pi"'] \arrow[dr, "\phi"] & \\
\operatorname{Spf}\widehat{\mathcal{O}}_{C_K(\rho^\square),x} \arrow[r, "\xi'"] \arrow[d] & \mathfrak{D}_x \arrow[d] & & \mathfrak{D}_{\bar{x}} \\
\mathfrak{D}^\square_{\bar{\rho}_\infty} \arrow[r, "\xi"] & \mathfrak{D}_y, & &
\end{tikzcd}$$

where the lower left square is cartesian by Lemma 3.1. As remarked above ξ is formally smooth and hence so is ξ'. As $\mathfrak{D}_{\bar{x}}$ is pro-represented by the complete local ring at some closed point of the local model M_K and as ξ', π and ϕ are formally smooth, the assertion of the Proposition is true if it is true for M_K. But if follows from the definitions (using the notation of [PR2, 3.c]) that

$$M_K\otimes_{\mathbb{Z}_p}\mathbb{Q}_p=\coprod_{\mu_i}M^{\mathrm{loc}}_{\mu_i,K}\otimes_{\mathbb{Z}_p}\mathbb{Q}_p, \tag{3.4}$$

for some cocharacters

$$\mu_i:\mathbb{G}_{m,\bar{\mathbb{Q}}_p}\longrightarrow(\operatorname{Res}_{K/\mathbb{Q}_p}\mathrm{GL}_d)_{\bar{\mathbb{Q}}_p}=\prod_{\psi:K\to\bar{\mathbb{Q}}_p}\mathrm{GL}_{d,\bar{\mathbb{Q}}_p},$$

such that

$$\bar{\mathbb{Q}}_p^d=\bigoplus_{n\in\{0,1\}}V^\psi_{n,i},$$

where $V^\psi_{n,i}=\{v\in\bar{\mathbb{Q}}_p^d\mid(\mathrm{pr}_\psi\circ\mu_i)(a)v=a^nv\text{ for all }a\in\bar{\mathbb{Q}}_p^\times\}$ and each of the $M^{\mathrm{loc}}_{\mu_i,K}$ is a local model in the sense of [PR1] (compare [PR2, Remark 3.3]). Hence, by [PR1, Theorem 5.4], the generic fiber of the local model $M_K\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$ is normal, reduced and Cohen-Macaulay. The special fiber decomposes as follows:

$$M_K\otimes_{\mathbb{Z}_p}\mathbb{F}_p=\coprod_{\nu}M^{\mathrm{loc}}_{\mu_{\max}(\nu),K}\otimes_{\mathbb{Z}_p}\mathbb{F}_p, \tag{3.5}$$

where ν runs over cocharacters

$$\mathbb{G}_{m,\bar{\mathbb{Q}}_p}\longrightarrow(\operatorname{Res}_{K/\mathbb{Q}_p}\mathbb{G}_m)_{\bar{\mathbb{Q}}_p},$$

and where $\mu_{\max}(\nu)$ is the maximal dominant cocharacter $\mathbb{G}_m\to\operatorname{Res}_{K/\mathbb{Q}_p}\mathrm{GL}_d$ (for the dominance order) such that the composition

$$\mathbb{G}_{m,\bar{\mathbb{Q}}_p}\longrightarrow(\operatorname{Res}_{K/\mathbb{Q}_p}\mathrm{GL}_d)_{\bar{\mathbb{Q}}_p}\xrightarrow{\det}(\operatorname{Res}_{K/\mathbb{Q}_p}\mathbb{G}_m)_{\bar{\mathbb{Q}}_p}$$

equals ν. The decomposition in (3.5) runs over all cocharacters ν such that the local model $M^{\mathrm{loc}}_{\mu_{\max}(\nu),K}$ is non-empty. Further we have $\mu_i=\mu_{\max}(\nu_i)$ for some cocharacter ν_i. It follows that every connected component of the special fiber is the special fiber of a local model appearing in the decomposition of the generic fiber (3.4). Now the claim again follows from [PR1, Theorem 5.4]. $\square$

Remark 3.3. We need to formulate the result on the local structure of the special fiber as a result about the underlying reduced scheme as the local models $M^{\mathrm{loc}}_{\mu,K}$ are in general not defined over $\mathbb{Z}_p$ but over a ramified extension and hence there are nilpotent elements in the special fiber $M_K \otimes_{\mathbb{Z}_p} \mathbb{F}_p$. However, the local structure of the special fiber is not needed in the sequel. We make only use of the fact that the gneric fiber is dense.

Proposition 3.4. *The map $C_K(\rho^\square) \to \operatorname{Spec} R^{\square,\mathrm{fl}}$ becomes an isomorphism in the generic fiber over $W(\mathbb{F})$, i.e.*

$$C_K(\rho^\square) \otimes_{W(\mathbb{F})} \operatorname{Frac}(W(\mathbb{F})) \xrightarrow{\cong} \operatorname{Spec}(R^{\square,\mathrm{fl}}[\tfrac{1}{p}]).$$

Proof. Using the result on the local structure of $C_K(\rho^\square)$, the proof is the same as in [Ki1, Proposition 2.4.8]. The main point is to check that the map is a bijection on points. $\square$

Corollary 3.5. *The map $C_K(\rho) \to \operatorname{Spec} R^{\mathrm{fl}}$ becomes an isomorphism in the generic fiber. Further the generic fiber of $C_K(\rho)$ is dense in $C_K(\rho)$.*

Proof. This is an obvious consequence of the statements on $C_K(\rho^\square)$. $\square$

Remark 3.6. As a consequence of the above we recover Kisin's result on the comparison of the connected components of the generic fiber of $\operatorname{Spec} R^{\mathrm{fl}}$ with the connected components of a scheme in characteristic p from the more abstract set up in [PR2]. More precisely, let

$$\mu : \mathbb{G}_{m,\bar{\mathbb{Q}}_p} \longrightarrow (\operatorname{Res}_{K/\mathbb{Q}_p}\mathrm{GL}_d)_{\bar{\mathbb{Q}}_p}$$

be a miniscule cocharacter which is dominant with respect to the Borel subgroup that is the Weil restriction of the upper triangular matrices. We write $\operatorname{Spec} R^{\mathrm{fl},\mu}$ for the flat closure of the union of the connected components of $\operatorname{Spec} R^{\mathrm{fl}}[1/p]$ where the Hodge-Tate weights of the universal flat representations are given by μ (compare [He, 5] for the definition). We define a (dominant) cocharacter

$$\nu : \mathbb{G}_{m,\bar{\mathbb{F}}_p} \longrightarrow (\operatorname{Res}_{k/\mathbb{F}_p}\mathrm{GL}_d)_{\bar{\mathbb{F}}_p}$$

as follows. Let

$$\mu_\psi : t \longmapsto \operatorname{diag}(t^{a_\psi}, t^{b_\psi})$$

denote the component of μ corresponding to an embedding $\psi : K \hookrightarrow \bar{\mathbb{Q}}_p$. Then the component of ν corresponding to $\bar{\psi} : k \hookrightarrow \bar{\mathbb{F}}_p$ is given by

$$\nu_{\bar{\psi}} : t \longmapsto \operatorname{diag}(t^{\sum_{\psi \bmod \pi = \bar{\psi}} a_\psi}, t^{\sum_{\psi \bmod \pi = \bar{\psi}} b_\psi}).$$

If one defines a closed subscheme $C_\nu(\bar{\rho}) \subset C_K(\bar{\rho})$ as the variety of all lattices $\mathfrak{M}$ such that the elementary divisors of $E(u)\mathfrak{M} \subset \mathfrak{M}$ are given by ν, then the proofs of Proposition 3.2 and [Ki1, Corollary 2.4.10] show that there is a bijection[1]

$$\pi_0(\operatorname{Spec} R^{\mathrm{fl},\mu}[1/p]) = \pi_0(C_\nu(\bar{\rho})).$$

[1]Actually rewriting Kisin's arguments in this context would yield a bijection $\pi_0(\operatorname{Spec} R^{\square,\mathrm{fl},\mu}[1/p]) = \pi_0(C_\nu(\bar{\rho}^\square))$. However, this bijection is of course equivalent to the one above.

In fact this bijection is Kisin's motivation to study the morphism $C_K(\rho) \to \operatorname{Spec} R^{\mathrm{fl}}$. In the 2-dimensional case the connected components of the variety $C_\nu(\bar\rho)$ can be computed in many cases.

Theorem 3.7. *Suppose that the universal flat deformation ring R^{fl} of $\bar\rho$ exists and denote by ρ the universal flat deformation. Then the morphism $C_K(\rho) \to \operatorname{Spec} R^{\mathrm{fl}}$ is topologically surjective.*

We will prove this theorem in section 4 below. The theorem gives a partial answer to a question raised in [PR2, 4.c]. In loc. cit. Pappas and Rapoport make the following construction. If A is a complete local Noetherian ring with finite residue field $\mathbb{F}$ and if ρ is a deformation of a residual G_K-representation $\bar\rho$ over $\mathbb{F}$ to A, they define a quotient $A^{\mathrm{fl}} \to A^K$ as the scheme theoretic image of the morphism

$$C_K(\rho) \longrightarrow \operatorname{Spec} A^{\mathrm{fl}}.$$

They show that the map $A^{\mathrm{fl}} \to A^K$ is an isomorphism if $[K : K_0] < p-1$, and ask whether it is an isomorphism in general. Our main result gives a partial answer to this question in the sense that, if $A = R^{\mathrm{fl}}$ is the universal flat deformation ring, then the induced map $\operatorname{Spec} R^K \to \operatorname{Spec} R^{\mathrm{fl}}$ is an isomorphism on the level of points, i.e. the reduced rings

$$(R^{\mathrm{fl}})^{\mathrm{red}} \longrightarrow (R^K)^{\mathrm{red}}$$

are equal.
We will conclude this section with some consequences of Theorem 3.7.

Corollary 3.8. *Assume that R^{fl} exists, then $\operatorname{Spec} R^{\mathrm{fl}}$ is topologically flat.*

Proof. This follows from Theorem 3.7 and the corresponding result on $C_K(\rho)$. □

Proposition 3.9. *Assume that the universal deformation ring R of $\bar\rho$ exists with universal deformation ρ^{univ}. Then $C_K(\rho^{\mathrm{univ}}) \to \operatorname{Spec} R$ factors over $\operatorname{Spec} R^{\mathrm{fl}}$ and is (canonically) isomorphic to $C_K(\rho)$.*

Proof. For $n \geq 0$ denote by $\rho_n : \operatorname{Spec}(R^{\mathrm{fl}}/\mathfrak{m}_{R^{\mathrm{fl}}}^{n+1}) \to \mathcal{R}$ the reduction of $\rho|_{G_{K_\infty}}$ modulo $\mathfrak{m}_{R^{\mathrm{fl}}}^{n+1}$, and similarly $(\rho^{\mathrm{univ}})_n$. We consider the following diagram with all rectangles cartesian.

$$\begin{tikzcd}
C_K(\rho_n) \arrow[r] \arrow[d] & C_K(\rho_n^{\mathrm{univ}}) \arrow[r] \arrow[d] & \mathcal{C}_K \arrow[d] \\
\operatorname{Spec}(R^{\mathrm{fl}}/\mathfrak{m}_{R^{\mathrm{fl}}}^{n+1}) \arrow[r] \arrow[rr, bend right, "\rho_n"'] & \operatorname{Spec}(R/\mathfrak{m}_R^{n+1}) \arrow[r, "(\rho^{\mathrm{univ}})_n"] & \mathcal{R}.
\end{tikzcd}$$

By [PR2, Proposition 4.3] the morphism $C_K(\rho_n^{\mathrm{univ}}) \to \operatorname{Spec}(R/\mathfrak{m}_R^{n+1})$ factors over $\operatorname{Spec}(R^{\mathrm{fl}}/\mathfrak{m}_{R^{\mathrm{fl}}}^{n+1})$ and hence $\widehat{C}_K(\rho_n) \to \widehat{C}_K(\rho_n^{\mathrm{univ}})$ is an isomorphism.
As $\operatorname{Spf} R^{\mathrm{fl}} \to \operatorname{Spf} R$ is a closed immersion the formal scheme $\widehat{C}_K(\rho)$ is a projective formal $\operatorname{Spf} R$-scheme and applying formal GAGA (see [EGA3, 5.4]) over $\operatorname{Spf} R$ we find that also the algebraizations $C_K(\rho)$ and $C_K(\rho^{\mathrm{univ}})$ are isomorphic over $\operatorname{Spec} R$. □

Proposition 3.10. *Assume that $e = [K : K_0] < p-1$. Then the morphism*

$$C_K(\rho) \longrightarrow \operatorname{Spec} R^{\mathrm{fl}}$$

is an isomorphism.

Proof. It is enough to show that $\widehat{C}_K(\rho) \to \mathrm{Spf}\ R^{\mathrm{fl}}$ is an isomorphism. We show that both objects pro-represent the same functor, i.e. $\widehat{C}_K(\rho)$ pro-represents the deformation functor $\mathcal{D}^{\mathrm{fl}}_{\bar\rho}$. This is already contained in [PR2, Remark 4.4, Proposition 4.5]. We repeat the argument here. Let A be a local Artinian ring and $\xi \in \mathcal{D}^{\mathrm{fl}}_{\bar\rho}(A)$ a flat deformation of $\bar\rho$. By a result of Raynaud (cf. [Ra, Proposition 3.3.2]) there is a unique flat model for this deformation. Denote by $(\mathfrak{M}, \Phi)$ the $W[[u]]$-module associated with this group scheme by Kisin's classification. This is a $W[[u]]$-submodule of the étale ϕ-module (M, Φ) over $A_W((u))$ corresponding to the (twist of) the restriction of ξ to G_{K_∞}. As the module $\mathfrak{M}$ is the unique $W[[u]]$-submodule of M satsfying

$$E(u)\mathfrak{M} \subset \Phi(\phi^*\mathfrak{M}) \subset \mathfrak{M}$$

it has to equal its $A_W[[u]]$-span inside M. As $\mathfrak{M}$ is unique one easily sees that $\mathfrak{M}\otimes_A\mathbb{F}$ equals the image of $\mathfrak{M}$ in $M \otimes_A \mathbb{F}$ and hence is free of rank d over $\mathbb{F}_W[[u]]$. It follows from [PR2, Lemma 4.2] that $\mathfrak{M}$ is free over $A_W[[u]]$. This defines the unique point in $\mathcal{C}_K(A)$ above ξ. We have shown that the functor morphism $\widehat{C}_K(\rho) \to \mathcal{D}^{\mathrm{fl}}_{\bar\rho}$ is bijective on A-valued points. The claim follows. □

4. Proof of Theorem 3.7

In this section we prove the main result, Theorem 3.7.

Let $e = [K : K_0]$ denote the ramification index of K over $\mathbb{Q}_p$. Then the degree of the Eisenstein polynomial $E(u)$ is e and its reduction modulo p is $u^e \in k[u]$.

For the rest of the section we denote by $\mathcal{O}_F = l[[\varpi]]$ a complete discrete valuation ring in characteristic p with finite residue field l containing k. We will use the notation $A_n = \mathcal{O}_F/(\varpi^{n+1}) \otimes_{\mathbb{F}_p} k$. For a ring R and a free $R((u))$-module $R((u))^d$, a finitely generated projective $R[[u]]$-submodule that generates $R((u))^d$ will be called a lattice in $R((u))^d$.

Finally, we will write $\mathcal{O}_F\{\{u\}\}$ for the ϖ-adic completion of $\mathcal{O}_F((u))$ and similarly $(\mathcal{O}_F \otimes_{\mathbb{F}_p} k)\{\{u\}\}$ for the ϖ-adic completion of $(\mathcal{O}_F \otimes_{\mathbb{F}_p} k)((u))$.

Lemma 4.1. *Let $(M, \Phi) \in \mathcal{R}(\mathcal{O}_F/(\varpi^{n+1}))$ and $\mathfrak{M} \subset M$ a finitely generated $A_n[[u]]$-submodule such that $\mathfrak{M}[1/u] = M$ and*

$$u^e\mathfrak{M} \subset \Phi(\phi^*\mathfrak{M}) \subset \mathfrak{M}.$$

Then the l-dimension of the u-torsion part of the finitely generated $l[[u]]$-module $\mathfrak{M}/\varpi\mathfrak{M}$ is bounded by

$$\dim_l(\mathfrak{M}/\varpi\mathfrak{M})^{\mathrm{tors}} \leq [k : \mathbb{F}_p]d\tfrac{e}{p-1}.$$

Proof. We can describe the u-torsion as follows.

There is a filtration $\mathcal{F}^0 \supset \mathcal{F}^1 \supset \cdots \supset \mathcal{F}^{n+1} = 0$ of $\mathfrak{M}/\varpi\mathfrak{M}$ such that

$$\mathcal{F}^i/\mathcal{F}^{i+1} = (\mathfrak{M} \cap \varpi^i M)/(\varpi\mathfrak{M} \cap \varpi^i M + \mathfrak{M} \cap \varpi^{i+1} M).$$

Here $\mathcal{F}^0/\mathcal{F}^1$ is the free part in the quotient $\mathfrak{M}/\varpi\mathfrak{M}$ and $\mathcal{F}^i/\mathcal{F}^{i+1}$ is the image of the contribution of the elements in $\mathfrak{M} \cap (\varpi^i M \backslash \varpi^{i+1} M)$ to the u-torsion. Further for $i \in 1, \ldots, n-1$ we have

$$\begin{aligned} &\dim_l(\mathfrak{M} \cap \varpi^i M)/(\varpi^i\mathfrak{M} + \mathfrak{M} \cap \varpi^{i+1} M) \\ &+ \dim_l(\mathfrak{M} \cap \varpi^{i+1} M)/(\varpi\mathfrak{M} \cap \varpi^{i+1} M + \mathfrak{M} \cap \varpi^{i+2} M) \\ &= \dim_l(\mathfrak{M} \cap \varpi^{i+1} M)/(\varpi^{i+1}\mathfrak{M} + \mathfrak{M} \cap \varpi^{i+2} M). \end{aligned}$$

This can be seen using the interpretation of $\dim_l(\mathfrak{M}\cap\varpi^iM)/(\varpi^i\mathfrak{M}+\mathfrak{M}\cap\varpi^{i+1}M)$ as the sum of all elementary divisors of the lattice $(\mathfrak{M}\cap\varpi^iM)/(\mathfrak{M}\cap\varpi^{i+1}M)$ with respect to $\varpi^i\mathfrak{M}/(\mathfrak{M}\cap\varpi^{i+1}M)$ as $l[[u]]$-lattices in $\varpi^iM/\varpi^{i+1}M$ and the fact that the multiplication by ϖ induces isomorphisms from $\varpi^iM/\varpi^{i+1}M$ to $\varpi^{i+1}M/\varpi^{i+2}M$ for $i\leq n-1$. Now we find that

$$\dim_l(\mathfrak{M}/\varpi\mathfrak{M})^{\rm tors}=\dim_l(\mathfrak{M}\cap\varpi^nM)/\varpi^n\mathfrak{M}.$$

The Lemma now follows from the following claim:

$$u^{\lfloor\frac{e}{p-1}\rfloor}(\mathfrak{M}\cap\varpi^nM)\subset\varpi^n\mathfrak{M}.$$

We denote by j the minimal integer such that $u^j(\mathfrak{M}\cap\varpi^nM)\subset\varpi^n\mathfrak{M}$.
Then pj is the minimal integer r such that $u^r\Phi(\phi^*(\mathfrak{M}\cap\varpi^nM))\subset\varpi^n\Phi(\phi^*\mathfrak{M})$. But we have

$$\varpi^n\Phi(\phi^*\mathfrak{M})\supset u^e\varpi^n\mathfrak{M}\supset u^{e+j}(\mathfrak{M}\cap\varpi^nM)\supset u^{e+j}\Phi(\phi^*(\mathfrak{M}\cap\varpi^nM)).$$

Hence $pj\leq e+j$ and the claim follows. □

Lemma 4.2. *Let $(M,\Phi)\in\mathcal{R}(\mathcal{O}_F/(\varpi^{n+1}))$. Then there are at most finitely many finitely generated $A_n[[u]]$-submodules $\mathfrak{M}\subset M$ such that $\mathfrak{M}[1/u]=M$ and*

$$u^e\mathfrak{M}\subset\Phi(\phi^*\mathfrak{M})\subset\mathfrak{M}.$$

Proof. The module M is a $nd[k:\mathbb{F}_p]$-dimensional $l((u))$ vector space. Every finitely generated $A_n[[u]]$ submodule $\mathfrak{M}\subset M$ with $\mathfrak{M}[1/u]=M$ is an $l[[u]]$-lattice in M. Hence the argument of [Ki1, Proposition 2.1.7] shows that there exists a lattice $\mathfrak{M}_0\subset M$ and integers $i_1,i_2\in\mathbb{Z}$ such that all $\mathfrak{M}\subset M$ satisfying the properties of the Lemma satisfy

$$u^{i_1}\mathfrak{M}_0\subset\mathfrak{M}\subset u^{i_2}\mathfrak{M}_0.$$

These are only finitely many lattices. □

We will use the following notation: If $\mathrm{Spf}\,\mathcal{O}_F\to\widehat{\mathcal{R}}$ is a formal point defined by $(M_n,\Phi_n):\mathrm{Spec}(\mathcal{O}_F/\varpi^{n+1})\to\mathcal{R}$, we denote this point by $(\widehat{M},\widehat{\Phi})$ and view this as the ϕ-module

$$\widehat{M}=\varprojlim M_n$$

over $(\mathcal{O}_F\otimes_{\mathbb{F}_p}k)\{\{u\}\}$.

Proposition 4.3. *Let $(\widehat{M},\widehat{\Phi})$ be a point* $\mathrm{Spf}\ \mathcal{O}_F\to\widehat{\mathcal{R}}$ *and denote by (M_n,Φ_n) the reduction modulo ϖ^{n+1}, i.e. the ϕ-module defined by* $\mathrm{Spec}\ \mathcal{O}_F/\varpi^{n+1}\to\mathcal{R}$. *Assume that there exist finitely generated $A_n[[u]]$ submodules $\mathfrak{M}_n\subset M_n$ such that $\mathfrak{M}_n[1/u]=M_n$ and*

$$u^e\mathfrak{M}_n\subset\Phi_n(\phi^*\mathfrak{M}_n)\subset\mathfrak{M}_n.$$

Then there exists $(M,\Phi)\in\mathcal{R}(\mathcal{O}_F)$ such that

$$(M/\varpi^{n+1}M,\Phi\mod\varpi^{n+1})=(M_n,\Phi_n).$$

Proof. We denote by $\mathcal{Z}_n$ the set of all finitely generated $A_n[[u]]$-submodules $\mathfrak{N}\subset M_n$ such that $\mathfrak{N}[1/u]=M_n$ and

$$u^e\mathfrak{N}\subset\Phi_n(\phi^*\mathfrak{N})\subset\mathfrak{N}.$$

By assumption these sets are non empty and by Lemma 4.2 they are finite. Further if $\mathfrak{N}\in M_n$ and $m<n$, then the image of $\mathfrak{N}$ under the map

$$M_n\longrightarrow M_m$$

defines an element of $\mathcal{Z}_m$, denoted by $f_{nm}(\mathfrak{N})$. As the sets $\mathcal{Z}_i$ are non empty and finite we can inductively construct a sequence $\widetilde{\mathfrak{M}}_n \in \mathcal{Z}_n$ such that $f_{nm}(\widetilde{\mathfrak{M}}_n) = \widetilde{\mathfrak{M}}_m$ for $m \leq n$. We denote this sequence again by $\mathfrak{M}_n$ instead of $\widetilde{\mathfrak{M}}_n$.
By Lemma 4.1 there are only finitely many possibilities for the isomorphism class of the u-torsion in $\mathfrak{M}_n/\varpi\mathfrak{M}_n$. Hence there exists a strictly increasing sequence $n_i \in \mathbb{N}$ such that

$$\mathfrak{M}_{n_i}/\varpi\mathfrak{M}_{n_i} \xrightarrow{\cong} \mathfrak{M}_{n_j}/\varpi\mathfrak{M}_{n_j} \tag{4.1}$$

for $j \leq i \in \mathbb{N}$. Now there is an isomorphism

$$\mathfrak{M}_{n_i}/\varpi^{n_j+1}\mathfrak{M}_{n_i} \cong \mathfrak{M}_{n_j} \oplus (\mathfrak{M}_{n_i}/\varpi^{n_j+1}\mathfrak{M}_{n_i})^{\mathrm{tors}},$$

where the last summand is the u-torsion part of the left hand side. We find that

$$(\mathfrak{M}_{n_i}/\varpi\mathfrak{M}_{n_i})^{\mathrm{tors}} = (\mathfrak{M}_{n_j}/\varpi\mathfrak{M}_{n_j})^{\mathrm{tors}} \oplus ((\mathfrak{M}_{n_i}/\varpi^{n_j+1}\mathfrak{M}_{n_i})^{\mathrm{tors}})/\varpi.$$

Using (4.1) and Nakayama's Lemma it follows that

$$\mathfrak{M}_{n_i}/\varpi^{n_j+1}\mathfrak{M}_{n_i} \xrightarrow{\cong} \mathfrak{M}_{n_j}$$

for all $j \leq i$. Especially there is an $r \in \mathbb{N}$ (independent of i) and generators $b_1^{(i)}, \dots, b_r^{(i)}$ of $\mathfrak{M}_{n_i}$ as an $A_{n_i}[[u]]$-module such that the $b_\nu^{(i)}$ reduce to $b_\nu^{(j)}$ modulo ϖ^{n_j+1} for all $j \leq i$. Choosing a compatible expression of $\Phi_n(b_j^{(i)})$ in terms of the $b_j^{(i)}$ we can define commutative diagrams for $j \leq i$:

$$\begin{tikzcd}
(A_{n_i}[[u]]^r, \widetilde{\Phi}_{n_i}) \arrow[r] \arrow[d] & (\mathfrak{M}_{n_i}, \Phi_{n_i}) \arrow[d] \\
(A_{n_j}[[u]]^r, \widetilde{\Phi}_{n_j}) \arrow[r] & (\mathfrak{M}_{n_j}, \Phi_{n_j}),
\end{tikzcd}$$

where all arrows are surjective. In the limit we get morphisms

$$\begin{tikzcd}
((\mathcal{O}_F \otimes_{\mathbb{F}_p} k)[[u]]^r, \widetilde{\Phi}) \arrow[d] \arrow[dr] & \\
\widetilde{M} = \varprojlim (A_{n_i}((u))^r, \widetilde{\Phi}_{n_i}) \arrow[r] & (\widehat{M}, \widehat{\Phi}).
\end{tikzcd} \tag{4.2}$$

Here the lower arrow is surjective by the Mittag-Leffler criterion: The modules in question have finite length. Note that we do not claim that the linearisation of $\widetilde{\Phi}$ is an isomorphism after inverting u.
Now the image of the vertical arrow defines (after inverting u) a free $(\mathcal{O}_F \otimes_{\mathbb{F}} k)((u))$-submodule $\widetilde{N}$ of the $(\mathcal{O}_F \otimes_{\mathbb{F}_p} k)\{\{u\}\}$-module $\widetilde{M}$ such that

$$\widetilde{N} \otimes_{\mathcal{O}_F((u))} \mathcal{O}_F\{\{u\}\} \xrightarrow{\cong} \widetilde{M}.$$

The image of $\widetilde{N}$ under $\widetilde{M} \to \widehat{M}$ defines a finitely generated $(\mathcal{O}_F \otimes_{\mathbb{F}_p} k)((u))$ submodule N such that

$$N \otimes_{\mathcal{O}_F((u))} \mathcal{O}_F\{\{u\}\} \xrightarrow{\cong} \widehat{M}.$$

The surjectivity follows from the discussion above, and injectivity can be shown as follows: Let $(\mathfrak{N}_n)_{n\in\mathbb{N}}$ be a system of free $A_n[[u]]$-submodules of M_n such that

$$\mathfrak{N}_n/\varpi^n\mathfrak{N}_n = \mathfrak{N}_{n-1}$$

for all $n \geq 1$, and such that $\mathfrak{M}_{n_i} \subset \mathfrak{N}_{n_i}$ for all i. Then $\widetilde{\mathfrak{M}} \subset \mathfrak{N} := \lim_{\leftarrow} \mathfrak{N}_i$, where $\widetilde{\mathfrak{M}}$ is the image of the diagonal arrow in (4.2). Now $\mathcal{O}_F\{\{u\}\}$ is flat over $\mathcal{O}_F[[u]]$, as it is the ϖ-adic completion of the Noetherian and flat $\mathcal{O}_F[[u]]$-algebra $\mathcal{O}_F((u))$. It follows that

$$N \otimes_{\mathcal{O}_F((u))} \mathcal{O}_F\{\{u\}\} = \widetilde{\mathfrak{M}} \otimes_{\mathcal{O}_F[[u]]} \mathcal{O}_F\{\{u\}\} \hookrightarrow \mathfrak{N} \otimes_{\mathcal{O}_F[[u]]} \mathcal{O}_F\{\{u\}\} = \widehat{M}.$$

Further N is $\widehat{\Phi}$-stable by construction. We claim that N is free.
As $k \subset l$ we have isomorphisms

$$(\mathcal{O}_F \otimes_{\mathbb{F}_p} k)((u)) \longrightarrow \prod_{\psi: k \hookrightarrow l} \mathcal{O}_F((u)).$$

And hence $\widehat{M} = \widehat{M}^{(\psi_0)} \times \widehat{M}^{(\phi\psi_0)} \times \cdots \times \widehat{M}^{(\phi^{f-1}\psi_0)}$ where ψ_0 is a fixed embedding, ϕ is the absolute Frobenius on k and $f = [k : \mathbb{F}_p]$. The endomorphism $\widehat{\Phi}$ maps $\widehat{M}^{(\phi^i\psi_0)}$ to $\widehat{M}^{(\phi^{i+1}\psi_0)}$. As N is $\widehat{\Phi}$-stable we find that $N = N^{(\psi_0)} \times \cdots \times N^{(\phi^{f-1}\psi_0)}$, where $N^{(\phi^i\psi_0)}$ is a finitely generated $\mathcal{O}_F((u))$-submodule of $\widehat{M}^{(\phi^i\psi_0)}$ that generates $\widehat{M}^{(\phi^i\psi_0)}$ over $\mathcal{O}_F\{\{u\}\}$ and hence is free of rank d, as $\mathcal{O}_F((u))$ is a principal ideal domain.
Now $(N, \widehat{\Phi})$ is the object claimed in the Proposition: It follows from the construction that $(N, \widehat{\Phi})$ reduces to (M_n, Φ_n) modulo ϖ^{n+1} and hence it follows from Nakayama's lemma that the linearisation of $\widehat{\Phi}$ is invertible on N. $\square$

Before we continue we want to remind the reader that not every $A_n[[u]]$-submodule $\mathfrak{M}_n \subset M_n$ satisfying $u^e\mathfrak{M}_n \subset \Phi_n(\phi^*\mathfrak{M}_n) \subset \mathfrak{M}_n$ defines an $\mathcal{O}_F/\varpi^{n+1}$-valued point of $\mathcal{C}_K$. This is only the case if $\mathfrak{M}_n$ is a free $A_n[[u]]$-module.

Proposition 4.4. *Let $(M, \Phi) \in \mathcal{R}(\mathcal{O}_F)$ and denote by $(M_n, \Phi_n) \in \mathcal{R}(\mathcal{O}_F/\varpi^{n+1})$ the reduction modulo ϖ^{n+1}. Assume that there exist finitely generated $A_n[[u]]$-submodules $\mathfrak{M}_n \subset M_n$ such that $\mathfrak{M}_n[1/u] = M_n$ and*

$$u^e\mathfrak{M}_n \subset \Phi_n(\phi^*\mathfrak{M}_n) \subset \mathfrak{M}_n.$$

Then there exists the diagonal arrow in the diagram

$$\begin{tikzcd} & \mathcal{C}_K \arrow[d] \\ \operatorname{Spec} \mathcal{O}_F \arrow[r] \arrow[ur] & \mathcal{R}. \end{tikzcd}$$

Proof. Consider the free $(F \otimes_{\mathbb{F}_p} k)((u))$-module $M \otimes_{\mathcal{O}_F((u))} F((u))$. We choose an $(F \otimes_{\mathbb{F}_p} k)[[u]]$-lattice $\widetilde{\mathfrak{N}} \subset M \otimes_{\mathcal{O}_F((u))} F((u))$. As the linearisation of Φ is an isomorphism, there exist $r \in \mathbb{N}$ such that

$$u^r\widetilde{\mathfrak{N}} \subset \Phi(\phi^*\widetilde{\mathfrak{N}}) \subset u^{-r}\widetilde{\mathfrak{N}},$$

i.e. $\widetilde{\mathfrak{N}}$ is an F-valued point of the stack $\mathcal{C}_r$ defined in (2.1).
By [PR2, Corollary 2.6] and the valuative criterion of properness, the diagonal

arrow in the diagram below exists,

$$\begin{tikzcd} \mathrm{Spec}\ F \arrow[r] \arrow[d] & \mathcal{C}_r \arrow[d] \\ \mathrm{Spec}\ \mathcal{O}_F \arrow[r] \arrow[ur] & \mathcal{R}. \end{tikzcd}$$

This means that $\widetilde{\mathfrak{N}}$ extends to an $(\mathcal{O}_F \otimes_{\mathbb{F}_p} k)[[u]]$-lattice $\mathfrak{N}$ such that

$$u^r\mathfrak{N} \subset \Phi(\phi^*\mathfrak{N}) \subset u^{-r}\mathfrak{N}.$$

We denote by $\mathfrak{N}_n$ the reduction of $\mathfrak{N}$ modulo ϖ^{n+1}.
By assumption there are finitely generated $A_n[[u]]$-submodules $\mathfrak{M}_n \subset M_n$ such that $\mathfrak{M}_n[1/u] = M_n$ and

$$u^e\mathfrak{M}_n \subset \Phi_n(\phi^*\mathfrak{M}_n) \subset \mathfrak{M}_n.$$

By the same argument as in the proof of Proposition 4.3 we can assume that $\mathfrak{M}_n$ maps onto $\mathfrak{M}_{n-1}$ under the projection $M_n \to M_{n-1}$ for all n. Now the argument of [Ki1, Proposition 2.1.7] shows that there is an integer s only depending on r and e such that

$$u^s\mathfrak{N}_n \subset \mathfrak{M}_n \subset u^{-s}\mathfrak{N}_n.$$

If we write $\mathfrak{M}$ for $\varprojlim \mathfrak{M}_n$, then this shows

$$u^s\mathfrak{N} \subset \mathfrak{M} \subset u^{-s}\mathfrak{N}.$$

Hence $\mathfrak{M}$ is finitely generated over $\mathcal{O}_F[[u]]$ and contains an $(\mathcal{O}_F \otimes_{\mathbb{F}_p} k)((u))$-basis of M. Further it still satisfies $u^e\mathfrak{M} \subset \Phi(\phi^*\mathfrak{M}) \subset \mathfrak{M}$ and $\mathfrak{M} \otimes_{\mathcal{O}_F[[u]]} F[[u]]$ is free over $(F \otimes_{\mathbb{F}_p} k)[[u]]$: As $(F \otimes_{\mathbb{F}_p} k)[[u]]$ is a product of principal ideal domains and $\mathfrak{M} \otimes_{\mathcal{O}_F[[u]]} F[[u]]$ has no u-torsion it has to be a product of free $F[[u]]$-modules, all of which have the same rank, as Φ permutes these factors and is an isomorphism after inverting u. Hence we obtain the following commutative diagram

$$\begin{tikzcd} \mathrm{Spec}\ F \arrow[r] \arrow[d] & \mathcal{C}_K^1 \arrow[d] \\ \mathrm{Spec}\ \mathcal{O}_F \arrow[r] & \mathcal{R}, \end{tikzcd}$$

where $\mathcal{C}_K^1 = \mathcal{C}_K \times_{\mathrm{Spec}\ \mathbb{Z}_p} \mathrm{Spec}\,\mathbb{Z}/p\mathbb{Z}$ is the reduction of $\mathcal{C}_K$ modulo p (compare [PR2, 3.b.]). By loc. cit. the stack $\mathcal{C}_K^1$ is a closed substack of $\mathcal{C}_e^1 = \mathcal{C}_e \times_{\mathrm{Spec}\ \mathbb{Z}_p} \mathrm{Spec}\,\mathbb{Z}/p\mathbb{Z}$. Using the valuative criterion of properness again we obtain the desired arrow. □

Proof of Theorem 3.7. By Corollary 3.5 the morphism $C_K(\rho) \to \mathrm{Spec}\,R^{\mathrm{fl}}$ is an isomorphism in the generic fiber over $W(\mathbb{F})$. Especially it is surjective.
We write $\overline{C}_K(\rho) = C_K(\rho) \otimes_{W(\mathbb{F})} \mathbb{F}$ for the special fiber of $C_K(\rho)$ and $\overline{R}^{\mathrm{fl}}$ for $R^{\mathrm{fl}}/pR^{\mathrm{fl}}$. Let η be a point of $\overline{R}^{\mathrm{fl}}$ that is not the unique closed point x_0. We mark the specialization $\eta \rightsquigarrow x_0$ by a morphism

$$\mathrm{Spec}\ \mathcal{O}_F \longrightarrow \mathrm{Spec}\ \overline{R}^{\mathrm{fl}},$$

where Spec $\mathcal{O}_F$ is a complete discrete valuation ring and where the morphism maps the generic point of Spec $\mathcal{O}_F$ to η and the special point to x_0. As $C_K(\rho) \to \mathrm{Spec}\,R^{\mathrm{fl}}$ is proper, the image is closed and, by a Zariski density argument, it suffices to consider the case where η is a height 1 prime ideal. Hence we may assume that

$\mathcal{O}_F$ has residue field $\mathbb{F}$ (recall that R^{fl} is a quotient of a power series ring in finitely many variables over $W(\mathbb{F})$). Let us write $\mathbb{F}'$ for the composition field of $\mathbb{F}$ and k and Art' for the category of local Artinian $W(\mathbb{F}')$-modules with residue field $\mathbb{F}'$. Further we write $\overline{R}' = \overline{R}^{\mathrm{fl}} \otimes_{\mathbb{F}} \mathbb{F}'$ and $\mathcal{O}_{F'} = \mathcal{O}_F \otimes_{\mathbb{F}} \mathbb{F}'$. Then $\overline{R}'$ is a complete local Noetherian ring with residue field $\mathbb{F}'$ and $\mathcal{O}_{F'}$ is a complete discrete valuation ring with residue field $\mathbb{F}'$. Let ρ' denote the representation

$$\rho' : G_K \to \mathrm{GL}_d(\overline{R}')$$

obtained by composing $(\rho \mod p)$ with the inclusion $\overline{R}^{\mathrm{fl}} \to \overline{R}'$ and write $\bar{\rho}'_\infty$ for the restriction of $\bar{\rho}' = \rho' \otimes_{\overline{R}'} \mathbb{F}' = \bar{\rho} \otimes_{\mathbb{F}} \mathbb{F}'$ to G_{K_∞}. It is obvious that the representation $\bar{\rho}'$ is flat, as $\bar{\rho}$ is flat, and that the representations on $\mathrm{GL}_d(\mathcal{O}_{F'}/\varpi^{n+1})$ induced by $\mathrm{Spec}\ \mathcal{O}_{F'} \to \mathrm{Spec}\ \overline{R}'$ are flat as well, as they are scalar extensions of flat representations with coefficents in $\mathcal{O}_F/\varpi^{n+1}$. Now

$$\mathrm{Spf}\ \mathcal{O}_{F'} \longrightarrow \mathrm{Spf}\ \overline{R}' \longrightarrow \mathfrak{D}_{\bar{\rho}'_\infty} \tag{4.3}$$

is a morphism of stacks on Art' and induces modules $(M_n, \Phi_n) \in \mathcal{R}(\mathcal{O}_{F'}/\varpi^{n+1})$. By Kisin's classification of finite flat group schemes (Theorem 2.3), there exist finitely generated $k[[u]]$-submodules $\mathfrak{M}_n \subset M_n$ such that $\mathfrak{M}_n[1/u] = M_n$ and

$$u^e \mathfrak{M}_n \subset \Phi_n(\phi^* \mathfrak{M}_n) \subset \mathfrak{M}_n.$$

Replacing $\mathfrak{M}_n$ by the $(\mathcal{O}_{F'}/\varpi^{n+1} \otimes_{\mathbb{F}_p} k)[[u]]$-modules that it generates, we may assume that $\mathfrak{M}_n$ is stable under the action of $\mathcal{O}_{F'}/\varpi^{n+1}$. By Proposition 4.3 the arrow $\mathrm{Spf}\ \mathcal{O}_{F'} \to \widehat{\mathcal{R}}$ is algebraizable to a morphism $\mathrm{Spec}\ \mathcal{O}_{F'} \to \mathcal{R}$ and by Proposition 4.4 we obtain a commutative diagram:

$$\begin{tikzcd} & \mathcal{C}_K \arrow[d] \\ \mathrm{Spec}\ \mathcal{O}_{F'} \arrow[ur] \arrow[r] & \mathcal{R}. \end{tikzcd}$$

We have to show that the arrow $\mathrm{Spf}\ \mathcal{O}_{F'} \to \mathfrak{D}_{\bar{\rho}'_\infty}$ induced by restricting the lower vertical arrow in this diagram to Art' factors over $\overline{R}'$ and that this morphism coincides with the arrow in (4.3).

By Theorem 2.3 we obtain from the morphisms $\mathrm{Spec}\ \mathcal{O}_{F'}/\varpi^{n+1} \to \mathcal{C}_K$ flat G_K-representations such that the restriction to G_{K_∞} induces the objects (M_n, Φ_n) under the morphism (3.1). By [Br, Theorem 3.4.3] the restriction to G_{K_∞} is fully faithful on the category of flat p-torsion G_K-representations and hence the Galois representation obtained from these group schemes coincides the one obtained from the morphism $\mathrm{Spf}\ \mathcal{O}_{F'} \to \mathrm{Spf}\ \overline{R}'$. We have shown that the image of the morphism

$$C_K(\rho') \longrightarrow \mathrm{Spec}\ \overline{R}'$$

contains the image of $\mathrm{Spec}\ \mathcal{O}_{F'} \to \mathrm{Spec}\ \overline{R}'$ which is enough to proof the claim. $\square$

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