ON THE STRUCTURE OF SOME MODULI SPACES OF FINITE FLAT GROUP SCHEMES

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1. Introduction and Notations

Let p be an odd prime and k a finite field of characteristic p. Let $W = W(k)$ be the ring of Witt vectors with coefficients in k and $K_0 = W[\frac{1}{p}]$ its fraction field. We consider a finite, totally ramified extension K/K_0 and denote by $e = [K : K_0]$ the degree of the extension. Let us fix a uniformizer $\pi \in \mathcal{O}_K$ with minimal polynomial $E(u) \in W[u]$ over K_0. Further we fix an algebraic closure $\bar{K}$ of K.
Let $\mathbb{F}$ be a finite field of characteristic p and $\rho : G_K \to GL(V_{\mathbb{F}})$ a continuous representation of the absolute Galois group $G_K = \mathrm{Gal}(\bar{K}/K)$ of K in a finite dimensional $\mathbb{F}$-vector space $V_{\mathbb{F}}$ whose dimension will be denoted by d.
This datum is equivalent to a finite commutative group scheme $\widetilde{\mathcal{G}} \to \mathrm{Spec}\ K$ with an operation of $\mathbb{F}$: The $\bar{K}$-valued points become an $\mathbb{F}$-vector space with a natural action of G_K and we want $\widetilde{\mathcal{G}}(\bar{K})$ and $V_{\mathbb{F}}$ to be isomorphic as $\mathbb{F}[G_K]$-modules.
If $\mathbb{F}'$ is a finite extension of $\mathbb{F}$, the representation ρ induces a representation ρ' on $V_{\mathbb{F}'} = V_{\mathbb{F}} \otimes_{\mathbb{F}} \mathbb{F}'$.
By the construction in Kisin's article [Ki], there is a projective $\mathbb{F}$-scheme $\mathcal{GR}_{V_{\mathbb{F}},0}$ whose $\mathbb{F}'$-valued points parametrize the isomorphism classes of finite flat models of $V_{\mathbb{F}'}$, i.e. finite flat group schemes $\mathcal{G} \to \mathrm{Spec}\ \mathcal{O}_K$ with an operation of $\mathbb{F}'$ such that the generic fiber of $\mathcal{G}$ is the G_K-representation on $V_{\mathbb{F}'}$ in the above sense.
Our aim is to analyze the structure of (some stratification of) $\mathcal{GR}_{V_{\mathbb{F}},0}$ in the case $d = 2$ and $k = \mathbb{F}_p$.
First we recall some constructions from [Ki], see also [PR2]. We assume $k = \mathbb{F}_p$ to simplify the situation.
For each n let $\pi_n \in \bar{K}$ be a p^n-th root of the uniformizer π such that $\pi_n^p = \pi_{n-1}$ for all n. Define $K_\infty = \bigcup_{n\geq 1} K(\pi_n)$ and denote by $G_{K\infty} = \mathrm{Gal}(\bar{K}/K_\infty)$ the absolute Galois group of K_∞.
For each algebraic extension $\mathbb{F}'$ of $\mathbb{F}$ we denote by $\phi : \mathbb{F}'((u)) \to \mathbb{F}'((u))$ the homomorphism which takes u to its p-th power and which is the identity on the coefficients:

$$\phi(\sum_i a_i u^i) = \sum a_i u^{pi}.$$

Denote by $\mathrm{Mod}^{\phi}_{/\mathbb{F}'((u))}$ the category of finite dimensional $\mathbb{F}'((u))$-modules M together with a ϕ-linear map $\Phi : M \to M$ such that the linearization $\mathrm{id} \otimes \Phi : \phi^* M \to M$ is an isomorphism. The morphism are $\mathbb{F}'((u))$-linear maps commuting with Φ.
By ([Ki] 1.2.6, Lemma 1.2.7), there is an equivalence of abelian categories

$$\mathrm{Mod}^{\phi}_{/\mathbb{F}'((u))} \longleftrightarrow \left\{ \begin{array}{c} \text{continuous } G_{K\infty}\text{-representations} \\ \text{on finite dimensional } \mathbb{F}'\text{-vector spaces} \end{array} \right\}$$

which preserves the dimensions and is compatible with finite base change $\mathbb{F}''/\mathbb{F}'$. This is a version with coefficients of the equivalence of categories of Fontaine (cf. [Fo], A3).
Denote by $(M_{\mathbb{F}}, \Phi)$ the d-dimensional $\mathbb{F}((u))$-vector space with semi-linear endomorphism Φ, associated to the restriction of the Tate-twist $V_{\mathbb{F}}(-1)$ to G_{K_∞} under the above equivalence. By the descriptions in [Ki], the finite flat models $\mathcal{G} \to \operatorname{Spec} \mathcal{O}_K$ of $V_{\mathbb{F}}$ correspond to $\mathbb{F}[\![u]\!]$-lattices $\mathfrak{M} \subset M_{\mathbb{F}}$ satisfying $u^e\mathfrak{M} \subset \langle\Phi(\mathfrak{M})\rangle \subset \mathfrak{M}$. Here $\langle\Phi(\mathfrak{M})\rangle = (\mathrm{id} \otimes \Phi)\phi^*\mathfrak{M}$ is the $\mathbb{F}[\![u]\!]$-lattice in $M_{\mathbb{F}}$ generated by $\Phi(\mathfrak{M})$.
Under this description the multiplicative group schemes correspond to the lattices $\mathfrak{M}$ such that $\langle\Phi(\mathfrak{M})\rangle = \mathfrak{M}$ and the étale group schemes correspond to the lattices with $u^e\mathfrak{M} = \langle\Phi(\mathfrak{M})\rangle$. These lattices will be called multiplicative resp. étale.
This construction is compatible with base change in the following sense. Suppose $\mathfrak{M}_{\mathbb{F}} \subset M_{\mathbb{F}}$ is a $\mathbb{F}[\![u]\!]$-lattice corresponding to a finite flat model $\mathcal{G}$ of $V_{\mathbb{F}}$. If $\mathbb{F}'$ is a finite extension of $\mathbb{F}$ with $n = [\mathbb{F}' : \mathbb{F}]$, then the $\mathbb{F}'[\![u]\!]$ lattice

$$\mathfrak{M}_{\mathbb{F}'} = \mathfrak{M}\widehat{\otimes}_{\mathbb{F}}\mathbb{F}' \subset M_{\mathbb{F}'} = M_{\mathbb{F}}\widehat{\otimes}_{\mathbb{F}}\mathbb{F}'$$

corresponds to the finite flat model $\mathcal{G}' = \mathcal{G} \boxtimes_{\mathbb{F}} \mathbb{F}'$ of $V_{\mathbb{F}'}$. Here the exterior tensor product $\mathcal{G} \boxtimes_{\mathbb{F}} \mathbb{F}'$ is the following group scheme: Choose a $\mathbb{F}$-basis $e_1 \dots e_n$ of $\mathbb{F}'$. Then $\mathcal{G} \boxtimes_{\mathbb{F}} \mathbb{F}' = \prod_{i=1}^n \mathcal{G}$ and $z \in \mathbb{F}'$ operates via the matrix $A \in GL_n(\mathbb{F})$ describing the multiplication by z on $\mathbb{F}'$ in the fixed $\mathbb{F}$-basis.
The scheme $\mathcal{GR}_{V_{\mathbb{F}},0}$ is constructed as a closed subscheme of the affine Grassmannian $\mathrm{Grass}\ \mathrm{M}_{\mathbb{F}}$ for $GL(M_{\mathbb{F}})$ and its closed points are given by

$$\mathcal{GR}_{V_{\mathbb{F}},0}(\mathbb{F}') = \{\mathbb{F}'[\![u]\!]\text{-lattices } \mathfrak{M} \subset M_{\mathbb{F}'} \mid u^e\mathfrak{M} \subset \langle\Phi(\mathfrak{M})\rangle \subset \mathfrak{M}\} \tag{1.1}$$

for every finite extension $\mathbb{F}'$ of $\mathbb{F}$.
In the following we will forget about the Galois representation and finite flat group schemes and will consider lattices. We will drop the condition $p \neq 2$. All results hold for arbitrary p, except those using the interpretation of the closed points as finite flat group schemes. We will always assume that there exists a finite flat model for $V_{\mathbb{F}}$ at least after extending scalars.
For each $\mathbb{Q}_p$-algebra embedding $\psi : K \to \bar{K}_0$ we now fix an integer $v_\psi \in \{0, \dots, d\}$. Denote by $\mathbf{v} = (v_\psi)_\psi$ the collection of the v_ψ and by $\mathbf{r} = \check{\mathbf{v}}$ the dual partition, i.e. $r_i = \sharp\{\psi \mid v_\psi \geq i\}$.
Kisin constructs closed, reduced subschemes

$$\mathcal{GR}^{\mathbf{v},\mathrm{loc}}_{V_{\mathbb{F}},0} \subset \mathcal{GR}_{V_{\mathbb{F}},0}$$

whose $\mathbb{F}'$-valued points are given by

$$\mathcal{GR}^{\mathbf{v},\mathrm{loc}}_{V_{\mathbb{F}},0}(\mathbb{F}') = \{\mathfrak{M} \in \mathcal{GR}_{V_{\mathbb{F}},0}(\mathbb{F}') \mid J(u|_{\langle\Phi(\mathfrak{M})\rangle/u^e\,\mathfrak{M}}) \leq \mathbf{r}\} \tag{1.2}$$

for a finite extension $\mathbb{F}'$ of $\mathbb{F}$ (cf. [Ki], Prop. 2.4.6). Here $J(u|_{\langle\Phi(\mathfrak{M})\rangle/u^e\,\mathfrak{M}})$ denotes the Jordan type of the nilpotent endomorphism on $\langle\Phi(\mathfrak{M})\rangle/u^e\,\mathfrak{M}$ induced by the multiplication with u. Recall that for $d = 2$

$$(a_1, b_1) \leq (a_2, b_2) \Leftrightarrow \begin{cases} a_1 \leq a_2, \\ a_1 + b_1 = a_2 + b_2 \end{cases} \tag{1.3}$$

for pairs $(a_i, b_i) \in \mathbb{Z}^2$ with $a_i \geq b_i$. The local structure of $\mathcal{GR}^{\mathbf{v},\mathrm{loc}}_{V_{\mathbb{F}},0}$ is linked to the structure of the local models studied in [PR1]. These schemes are named *" closed Kisin varieties"* in [PR2].

Kisin conjectures in ([Ki] 2.4.16) that, if $\mathrm{End}_{\mathbb{F}[G_K]}(V_{\mathbb{F}}) = \mathbb{F}$, the connected components of $\mathcal{GR}^{\mathbf{v},\mathrm{loc}}_{V_{\mathbb{F}},0}$ are given by the open and closed subschemes on which both the rank of the maximal multiplicative subobject and the rank of the maximal étale quotient are fixed. In ([Ki], 2.5) he proves this conjecture in the case $d = 2$, $k = \mathbb{F}_p$ and $v_\psi = 1$ for all ψ. For $d = 2$ and $v_\psi = 1$ for all ψ this result is generalized by Imai to the case of arbitrary k (see [Im]). In this paper we want to analyze the situation in the case $k = \mathbb{F}_p$, $d = 2$ but arbitrary $\mathbf{v}$. It turns out that the conjecture is not true in general. Our main results are as follows.

For $(a, b) \in \mathbb{Z}^2$ with $a \geq b$, we introduce a locally closed subscheme of the affine Grassmannian

$$\mathcal{G}_{V_{\mathbb{F}}}(a, b) \subset \mathrm{Grass}\, M_{\mathbb{F}},$$

with closed points the lattices $\mathfrak{M}$ such that the elementary divisors of $\langle \Phi(\mathfrak{M}) \rangle$ with respect to $\mathfrak{M}$ are given by (a, b).

Theorem 1.1. *Assume that $(M_{\mathbb{F}'}, \Phi) = (M_{\mathbb{F}} \widehat{\otimes}_{\mathbb{F}} \mathbb{F}', \Phi)$ is simple for all finite extensions $\mathbb{F}'$ of $\mathbb{F}$.*
(i) If $\mathcal{G}_{V_{\mathbb{F}}}(a, b) \neq \emptyset$, there exists a finite extension $\mathbb{F}'$ of $\mathbb{F}$ such that

$$\mathcal{G}_{V_{\mathbb{F}'}}(a, b) = \mathcal{G}_{V_{\mathbb{F}}}(a, b) \otimes_{\mathbb{F}} \mathbb{F}' \cong \mathbb{A}^n_{\mathbb{F}'}$$

for $n = \lfloor \frac{a-b}{p+1} \rfloor$.
(ii) The scheme $\mathcal{GR}^{\mathbf{v},\mathrm{loc}}_{V_{\mathbb{F}},0}$ is geometrically connected and irreducible. There exists a finite extension $\mathbb{F}'$ of $\mathbb{F}$ such that $\mathcal{GR}^{\mathbf{v},\mathrm{loc}}_{V_{\mathbb{F}},0} \otimes_{\mathbb{F}} \mathbb{F}'$ is isomorphic to a Schubert variety in the affine Grassmannian for $GL(M_{\mathbb{F}'})$.
The dimension of $\mathcal{GR}^{\mathbf{v},\mathrm{loc}}_{V_{\mathbb{F}},0}$ is either $\lfloor \frac{r_1 - r_2}{p+1} \rfloor$ or $\lfloor \frac{r_1 - r_2}{p+1} \rfloor - 1$. Here $r_i = \sharp\{\psi \mid v_\psi \geq i\}$.

In the treatment of the reducible case we consider the set $\mathcal{S}(\mathbf{v})$ of isomorphism classes $[M']$ of one dimensional objects in $\mathrm{Mod}^{\phi}_{/\bar{\mathbb{F}}((u))}$ which admit an $\bar{\mathbb{F}}[\![u]\!]$-lattice $\mathfrak{M}_{[M']} \subset M'$ such that $\langle \Phi(\mathfrak{M}_{[M']}) \rangle = u^{e-r_1} \mathfrak{M}_{[M']}$. We will define subschemes

$$X^{\mathbf{v}}_{[M']} \subset \mathcal{GR}^{\mathbf{v},\mathrm{loc}}_{V_{\mathbb{F}},0} \otimes_{\mathbb{F}} \bar{\mathbb{F}}.$$

A lattice defines a closed point of $X^{\mathbf{v}}_{[M']}$ if it admits a Φ-stable subobject isomorphic to $\mathfrak{M}_{[M']}$. A lattice $\mathfrak{M}$ is called $\mathbf{v}$-*ordinary* iff it defines a closed point of $X^{\mathbf{v}}_{[M']}$ for some $[M'] \in \mathcal{S}(\mathbf{v})$. The subscheme of non-$\mathbf{v}$-ordinary points will be denoted by $X^{\mathbf{v}}_0$. We will prove the following Theorem.

Theorem 1.2. *Assume that $(M_{\mathbb{F}'}, \Phi) = (M_{\mathbb{F}} \widehat{\otimes}_{\mathbb{F}} \mathbb{F}', \Phi)$ is reducible for some finite extension $\mathbb{F}'$ of $\mathbb{F}$.*
(i) The subschemes $X^{\mathbf{v}}_0$ and $X^{\mathbf{v}}_{[M']}$ are open and closed in $\mathcal{GR}^{\mathbf{v},\mathrm{loc}}_{V_{\mathbb{F}},0} \otimes_{\mathbb{F}} \bar{\mathbb{F}}$ for all isomorphism classes $[M'] \in \mathcal{S}(\mathbf{v})$.
(ii) The scheme $X^{\mathbf{v}}_0$ is connected.
(iii) For each $[M'] \in \mathcal{S}(\mathbf{v})$ the scheme $X^{\mathbf{v}}_{[M']}$ is connected. If it is non empty, it is either a single point, or isomorphic to $\mathbb{P}^1_{\bar{\mathbb{F}}}$.
(iv) There are at most two isomorphism classes $[M'] \in \mathcal{S}(\mathbf{v})$ such that $X^{\mathbf{v}}_{[M']} \neq \emptyset$.

The structure of the subscheme $X^{\mathbf{v}}_0$ of non-$\mathbf{v}$-ordinary lattices is much more complicated than in the absolutely simple case. In general $X^{\mathbf{v}}_0$ has many irreducible components of varying dimensions. The main result concerning the irreducible components of $X^{\mathbf{v}}_0$ is the following theorem.

Theorem 1.3. *If $(M_{\mathbb{F}}, \Phi)$ is not isomorphic to the direct sum of two isomorphic one-dimensional ϕ-modules, then the irreducible components of $X_0^{\mathbf{v}}$ are Schubert varieties. Especially they are normal.*

Theorem 1.2 proves a modified version of Kisin's conjecture in the case $k = \mathbb{F}_p$ and $d = 2$, as follows.
For an integer s denote by

$$\mathcal{GR}^{\mathbf{v},\mathrm{loc},s}_{V_{\mathbb{F}},0} \subset \mathcal{GR}^{\mathbf{v},\mathrm{loc}}_{V_{\mathbb{F}},0}$$

the open and closed subscheme where the rank of the maximal Φ-stable subobject $\mathfrak{M}_1$, satisfying $\langle \Phi(\mathfrak{M}_1)\rangle = u^{e-r_1}\mathfrak{M}_1$, is equal to s.

Corollary 1.4. *Assume $p \neq 2$ and let $\rho : G_K \to V_{\mathbb{F}}$ be any two dimensional continuous representation of G_K. Assume that $\mathrm{End}_{\mathbb{F}'[G_K]}(V_{\mathbb{F}'})$ is a simple algebra for all finite extensions $\mathbb{F}'$ of $\mathbb{F}$. Then $\mathcal{GR}^{\mathbf{v},\mathrm{loc},s}_{V_{\mathbb{F}},0}$ is geometrically connected for all s. Furthermore*
(i) If $s = 1$ and $\mathrm{End}_{\mathbb{F}'[G_K]}(V_{\mathbb{F}'}) = \mathbb{F}'$ for all finite extensions $\mathbb{F}'$ of $\mathbb{F}$, then $\mathcal{GR}^{\mathbf{v},\mathrm{loc},s}_{V_{\mathbb{F}},0}$ is either empty or a single point.
If $s = 1$ and $\mathrm{End}_{\mathbb{F}'[G_K]}(V_{\mathbb{F}'}) = M_2(\mathbb{F}')$ for some finite extension $\mathbb{F}'$ of $\mathbb{F}$, then $\mathcal{GR}^{\mathbf{v},\mathrm{loc},s}_{V_{\mathbb{F}},0}$ is either empty or becomes isomorphic to $\mathbb{P}^1_{\mathbb{F}'}$ after extending the scalars to $\mathbb{F}'$.
(ii) If $s = 2$, then $\mathcal{GR}^{\mathbf{v},\mathrm{loc},s}_{V_{\mathbb{F}},0}$ is either empty or a single point.

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2. Some notations in the building

The method of this paper is to determine all lattices in the building of $GL_2\left(\bar{\mathbb{F}}((u))\right)$ that correspond to closed points of $\mathcal{GR}^{\mathbf{v},\mathrm{loc}}_{V_{\mathbb{F}},0}$. As we know that the scheme we study is a closed reduced subscheme of the affine Grassmannian, we can get information on the structure of $\mathcal{GR}^{\mathbf{v},\mathrm{loc}}_{V_{\mathbb{F}},0}$ by looking at its closed points.
For the rest of this paper, we fix the following notations: Let $(M_{\mathbb{F}}, \Phi)$ be the object in $\mathrm{Mod}^{\phi}_{/\mathbb{F}((u))}$ corresponding to the 2-dimensional Galois representation ρ on $V_{\mathbb{F}}$. Let $\mathbf{v} = (v_\psi)_\psi$ be a collection of integers $v_\psi \in \{0, 1, 2\}$ for every $\psi : K \to \bar{K}_0$. Define

$$d' = \sum_\psi v_\psi. \tag{2.1}$$

Denote by $\mathbf{r} = \check{\mathbf{v}}$ the dual partition, i.e.

$$r_1 = \sharp\{\psi \mid v_\psi \geq 1\}$$
$$r_2 = \sharp\{\psi \mid v_\psi \geq 2\}.$$

Denote by $\mathcal{B}$ the Bruhat-Tits building for $GL_2(\mathbb{F}((u)))$. For any finite extension $\mathbb{F}'$ of $\mathbb{F}$ the building for $GL_2(\mathbb{F}'((u)))$ will be denoted by $\mathcal{B}_{\mathbb{F}'}$. We write

$$\bar{\mathcal{B}} = \bigcup_{\mathbb{F}'/\mathbb{F}} \mathcal{B}_{\mathbb{F}'}$$

for the building for $GL_2\left(\bar{\mathbb{F}}((u))\right)$.
We choose an $\mathbb{F}((u))$-basis e_1, e_2 of $M_{\mathbb{F}}$. Denote by $\mathfrak{M}_0 = \langle e_1, e_2\rangle$ the standard lattice in the standard apartment $\mathcal{A}_0$ determined by e_1, e_2. In this apartment we choose the following coordinates:
Let $(m,n)_0$ denote the lattice $\langle u^m e_1, u^n e_2\rangle$. Further, we consider another set of coordinates given by $[x,y]_0 = (\frac{x+y}{2}, \frac{y-x}{2})_0$ for $x, y \in \mathbb{Z},\ x \equiv y \mod 2$; i.e. $(m,n)_0 = [m-n, m+n]_0$.
Let $q \in \mathbb{F}((u))^\times$ and set $k = v_u(q) \in \mathbb{Z}$, where v_u is the valuation on $\mathbb{F}((u))$ with $v_u(u) = 1$. The basis $e_1, qe_1 + e_2$ of $M_{\mathbb{F}}$ defines another apartment $\mathcal{A}_q$ which is branching off from the standard apartment at the line defined by $x = k$. Using the Iwasawa decomposition we find

$$\mathcal{B} = \bigcup_{q \in \mathbb{F}((u))} \mathcal{A}_q.$$

For arbitrary $q \in \mathbb{F}((u))$ we choose coordinates in the apartments $\mathcal{A}_q$, similar to the case of $\mathcal{A}_0$. Define

$$(m,n)_q = [m-n, m+n]_q := \langle u^m e_1, u^n(qe_1 + e_2)\rangle \in \mathcal{A}_q.$$

Remark 2.1. *(i)* The systems of coordinates in the various apartments are compatible in the following sense: For any $x, y \in \mathbb{Z}$, $x \equiv y \mod 2$ and $q, q' \in \mathbb{F}((u))$ we have $[x,y]_q = [x,y]_{q'}$ iff $x \leq v_u(q - q')$ which implies

$$[x,y]_q = [x,y]_{q'} \Leftrightarrow [x,y]_q \in \mathcal{A}_q \cap \mathcal{A}_{q'} \Leftrightarrow [x,y]_{q'} \in \mathcal{A}_q \cap \mathcal{A}_{q'}.$$

(ii) We will make use of these coordinates for arbitrary points in the building (not only points corresponding to lattices). We see that $[x,y]_q$ defines a lattice if and only if $x, y \in \mathbb{Z}$ and $x \equiv y \mod 2$.
(iii) We extend the above notations in the obvious way to the buildings $\bar{\mathcal{B}}$ and $\mathcal{B}_{\mathbb{F}'}$ for arbitrary finite extensions $\mathbb{F}'$ of $\mathbb{F}$.
(iv) Two points $[x,y]_q, [x',y']_q \in \mathcal{A}_q$ define the same point in the building for $PGL_2\left(\mathbb{F}((u))\right)$ if and only if $x = x'$. Thus the projection from $\mathcal{B}$ onto the building for $PGL_2\left(\mathbb{F}((u))\right)$ is given by the projection onto the x-coordinate for every apartment $\mathcal{A}_q \subset \mathcal{B}$.

Definition 2.2. Let $\mathfrak{M}$ and $\mathfrak{M}'$ be lattices in $M_{\mathbb{F}}$. Let a, b be the elementary divisors of $\mathfrak{M}'$ with respect to $\mathfrak{M}$, i.e. there exists a basis e_1', e_2' of $\mathfrak{M}$ such that $\mathfrak{M}' = \langle u^a e_1', u^b e_2'\rangle$. Define

$$d_1(\mathfrak{M}, \mathfrak{M}') = |a-b|$$
$$d_2(\mathfrak{M}, \mathfrak{M}') = a+b.$$

Remark 2.3. These quantities have the following meaning in the building:
If $\mathfrak{M} = [x,y]_q$ and $\mathfrak{M}' = [x',y']_{q'}$, then $d_2(\mathfrak{M}, \mathfrak{M}') = y' - y$. Further $d_1(\mathfrak{M}, \mathfrak{M}')$ is the distance between $\mathfrak{M}$ and $\mathfrak{M}'$ in the building for $PGL_2(\mathbb{F}((u)))$. Here, the distance between two lattices joined by an edge is equal to 1. This can be seen as follows: Assume that $\mathfrak{M} = \langle e_1, e_2\rangle$ is the standard lattice and $\mathfrak{M}' = A\mathfrak{M} = [x,y]_q$, with

$$A = (a_{ij})_{ij} = \begin{pmatrix} u^m & u^n q \\ 0 & u^n \end{pmatrix}.$$

If $a \geq b$ are the elementary divisors of $\mathfrak{M}'$ with respect to $\mathfrak{M}$, then, by the theory of elementary divisors,

$$d_1(\mathfrak{M}, \mathfrak{M}') = a + b - 2b = m + n - 2 \min_{i,j} v_u(a_{ij}).$$

FIGURE 1. The distance between two lattices in the building for $PGL_2(\mathbb{F}((u)))$ in the cases $x \leq v_u(q)$ and $x \geq v_u(q) \geq 0$.

If $x = m - n \leq v_u(q)$ or $v_u(q) \geq 0$, then $\min_{i,j} v_u(a_{ij}) = \min\{m, n\}$ and hence

$$d_1(\mathfrak{M}, \mathfrak{M}') = a - b = |m - n| = |x|.$$

If $x > v_u(q)$ and $v_u(q) < 0$, then $\min_{i,j} v_u(a_{ij}) = n + v_u(q)$. In this case we find

$$d_1(\mathfrak{M}, \mathfrak{M}') = a - b = m - n - 2v_u(q) = (x - v_u(q)) + (0 - v_u(q)).$$

Compare also Fig. 1 and Fig. 2.

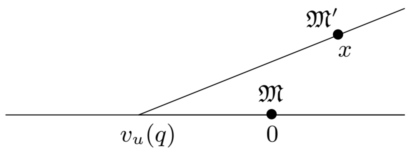

FIGURE 2. The distance between two lattices in the building for $PGL_2(\mathbb{F}((u)))$ in the case $x > v_u(q)$ and $v_u(q) < 0$.

We see that the distance $d_1(\mathfrak{M}, \mathfrak{M}')$ only depends on x, x' (and on $v_u(q - q')$), while $d_2(\mathfrak{M}, \mathfrak{M}')$ only depends on y and y'.
Using this remark, we can extend the distances d_1 and d_2 in an obvious way to the whole building $\mathcal{B}$ (and to $\bar{\mathcal{B}}$, $\mathcal{B}_{\mathbb{F}'}$). For example

$$d_1([x, y]_q, [0, 0]_0) = \begin{cases} x & \text{if } x \geq 0,\ v_u(q) \geq 0 \\ -x & \text{if } x < 0,\ x < v_u(q) \\ x - 2v_u(q) & \text{if } v_u(q) < x,\ v_u(q) < 0. \end{cases}$$

Lemma 2.4. *Define d' as in* (2.1). *The closed points $z \in \mathcal{GR}^{\mathbf{v},\mathrm{loc}}_{V_{\bar{\mathbb{F}}},0}(\bar{\mathbb{F}})$ correspond to the lattices $\mathfrak{M} \subset M_{\bar{\mathbb{F}}}$ which satisfy*

$$d_1(\mathfrak{M}, \langle \Phi(\mathfrak{M}) \rangle) \leq r_1 - r_2$$
$$d_2(\mathfrak{M}, \langle \Phi(\mathfrak{M}) \rangle) = 2e - d'.$$

Proof. If $\mathfrak{M} \subset M_{\bar{\mathbb{F}}}$ is any lattice and if a, b are the elementary divisors of $\langle \Phi(\mathfrak{M}) \rangle$ with respect to $\mathfrak{M}$, then the above conditions read

$$a - b \leq r_1 - r_2$$
$$a + b = 2e - d' = 2e - (r_1 + r_2).$$

This implies $u^e \mathfrak{M} \subset \langle \Phi(\mathfrak{M}) \rangle \subset \mathfrak{M}$. The Jordan type of u on $\langle \Phi(\mathfrak{M}) \rangle / u^e \mathfrak{M}$ is given by

$$J(u|_{\langle \Phi(\mathfrak{M}) \rangle / u^e \mathfrak{M}}) = (e - a, e - b).$$

Assuming $a \geq b$ we find:

$$J(u|_{\langle\Phi(\mathfrak{M})\rangle/u^e\,\mathfrak{M}}) \leq \mathbf{r} \Leftrightarrow \begin{cases} b \geq e - r_1 \\ a + b = 2e - d' = 2e - (r_1 + r_2). \end{cases}$$

The lemma follows easily from this. □

Definition 2.5. A lattice $\mathfrak{M}$ is called $\mathbf{v}$-*admissible* if it satisfies

$$d_1(\mathfrak{M}, \langle\Phi(\mathfrak{M})\rangle) \leq r_1 - r_2 \quad \text{and} \quad d_2(\mathfrak{M}, \langle\Phi(\mathfrak{M})\rangle) = 2e - d'.$$

Let $\mathfrak{M}$ be a lattice in $M_{\mathbb{F}}$ and $A \in GL_2(\mathbb{F}((u)))$ be a matrix. We will use the notation $\mathfrak{M} \sim A$ if $\mathfrak{M}$ admits a $\mathbb{F}[\![u]\!]$-basis b_1, b_2 satisfying $\Phi(b_i) = Ab_i$. Similarly we will use the notation $M_{\mathbb{F}} \sim A$ (use a $\mathbb{F}((u))$-basis of $M_{\mathbb{F}}$).

Lemma 2.6. *For $i = 1, 2$, let $z_i \in \mathcal{GR}^{\mathbf{v},\mathrm{loc}}_{V_{\mathbb{F}},0}(\bar{\mathbb{F}})$ be closed points corresponding to lattices $\mathfrak{M}_i = [x_i, y_i]_{q_i} \in \bar{\mathcal{B}}$. Then $y_1 = y_2$.*

Proof. Choose $A, B \in GL_2\left(\bar{\mathbb{F}}((u))\right)$ such that $\mathfrak{M}_2 = A\mathfrak{M}_1$ and $\mathfrak{M}_1 \sim B$. Then $\mathfrak{M}_2 \sim \phi(A)BA^{-1}$. Using the theory of elementary divisors it follows that

$$v_u(\det B) = d_2(\mathfrak{M}_i, \langle\Phi(\mathfrak{M}_i)\rangle) = (p-1)v_u(\det A) + v_u(\det B)$$

and hence $v_u(\det A) = 0$ which yields the claim. □

Definition 2.7. For each $m \in \mathbb{Z}$ define the following subset of $\bar{\mathcal{B}}$:

$$\bar{\mathcal{B}}(m) := \bigcup_{q \in \bar{\mathbb{F}}((u))} \{[x, y]_q \in \mathcal{A}_q \mid y = m\}.$$

Viewing $\mathcal{GR}^{\mathbf{v},\mathrm{loc}}_{V_{\mathbb{F}},0}(\bar{\mathbb{F}})$ as a subset of $\bar{\mathcal{B}}$, Lemma 2.6 implies:

$$\mathcal{GR}^{\mathbf{v},\mathrm{loc}}_{V_{\mathbb{F}},0}(\bar{\mathbb{F}}) \subset \bar{\mathcal{B}}(m)$$

for some $m = m(\mathbf{v}) \in \mathbb{Z}$. The subset $\bar{\mathcal{B}}(m)$ is a tree which is (as a topological space) isomorphic to the building for $PGL_2(\bar{\mathbb{F}}((u)))$.
The difference is that not every vertex represents a lattice: A vertex $[x, m]_q \in \bar{\mathcal{B}}(m)$ represents a lattice $\mathfrak{M} \subset M_{\bar{\mathbb{F}}}$ iff $x \equiv m \mod 2$.

Remark 2.8. By construction we have

$$\mathcal{GR}_{V_{\mathbb{F}},0} \subset \mathrm{Grass}\, M_{\mathbb{F}},$$

where $\mathrm{Grass}\, \mathrm{M}_{\mathbb{F}}$ denotes the affine Grassmannian for GL_2. Since the determinant condition in (1.2) fixes the dimension

$$\dim \langle\Phi(\mathfrak{M})\rangle/u^e\,\mathfrak{M} = \sum_{\psi} v_{\psi} = d',$$

the closed subscheme $\mathcal{GR}^{\mathbf{v},\mathrm{loc}}_{V_{\mathbb{F}},0}$ lies in a connected component of this Grassmannian: If $\mathfrak{M} = A\mathfrak{M}_0$ defines a closed point (where $\mathfrak{M}_0$ is the standard lattice and A is a matrix), then the valuation of $\det A$ is determined by the dimension of $\langle\Phi(\mathfrak{M})\rangle/u^e\,\mathfrak{M}$.

Definition 2.9. For a given collection $\mathbf{v}$ denote by $\mathcal{GR}^{\mathbf{v}}_{V_{\mathbb{F}},0}$ the closed subscheme of $\mathcal{GR}_{V_{\mathbb{F}},0}$ consisting of all lattices $\mathfrak{M}$ such that $\dim\langle\Phi(\mathfrak{M})\rangle/u^e\,\mathfrak{M} = \sum_{\psi} v_{\psi}$.

Proposition 2.10. *If any two of the v_{ψ} differ at most by 1, then*

$$\mathcal{GR}^{\mathbf{v}}_{V_{\mathbb{F}},0} = \mathcal{GR}^{\mathbf{v},\mathrm{loc}}_{V_{\mathbb{F}},0}.$$

Proof. Let $\mathfrak{M}$ be a lattice and $a \geq b$ the elementary divisors of $\langle \Phi(\mathfrak{M}) \rangle$ with respect to $\mathfrak{M}$. Then $\mathfrak{M}$ defines a closed point of $\mathcal{GR}^{\mathbf{v}}_{V_{\mathbb{F}},0}$ if and only if $0 \leq b \leq a \leq e$ and $a + b = 2e - d'$ with $d' = \sum_\psi v_\psi$. This is equivalent to

$$\begin{aligned} a + b &= 2e - d' \\ a - b &\leq \max\{d', 2e - d'\}. \end{aligned}$$

Indeed, if $a+b = 2e-d'$, then $a-b \leq d'$ is equivalent to $a \leq e$, while $a-b \leq 2e-d'$ is equivalent to $b \geq 0$.
Now if any of the v_ψ differ at most by 1 we must have $r_1 = e$ (if all $v_\psi \geq 1$) or $r_2 = 0$ (if all $v_\psi \leq 1$). In both cases we find $r_1 - r_2 = \max\{d', 2e - d'\}$.
(See also [Ki], Prop. 2.4.6 (4) and [PR1], Thm. B *(iii)*). □

3. The absolutely simple case

In this section we will analyze the structure of $\mathcal{GR}^{\mathbf{v},\mathrm{loc}}_{V_{\mathbb{F}},0}$ in the case where $(M_{\mathbb{F}}, \Phi)$ is absolutely simple, i.e. for every (finite) extension $\mathbb{F}'/\mathbb{F}$ there is no proper Φ-stable subobject of $(M_{\mathbb{F}'}, \Phi)$.

Lemma 3.1. *If $(M_{\mathbb{F}}, \Phi)$ is absolutely simple, there exists a finite extension $\mathbb{F}'$ of $\mathbb{F}$, a basis e_1, e_2 of $M_{\mathbb{F}'}$ and $a \in \mathbb{F}'^{\times}$, $s \in \mathbb{Z}$ satisfying*

$$0 \leq s < p^2 - 1 \text{ and } s \not\equiv 0 \mod (p+1)$$

such that

$$M_{\mathbb{F}'} \sim \begin{pmatrix} 0 & au^s \\ 1 & 0 \end{pmatrix}.$$

Proof. This follows from ([Ca], Cor. 8), except that we need to check that $s \not\equiv 0$ mod $(p+1)$. If $p+1|s$, then there would be a proper Φ-stable subspace of $M_{\mathbb{F}''}$ for a quadratic extension $\mathbb{F}''$ of $\mathbb{F}'$, namely $\langle \sqrt{a} u^{s/p+1} e_1 + e_2 \rangle \subset M_{\mathbb{F}'} \widehat{\otimes}_{\mathbb{F}'} \mathbb{F}'[\sqrt{a}]$.
The constructions in [Ca] give a basis after extending scalars to the algebraic closure $\bar{\mathbb{F}}$ of $\mathbb{F}$, but of course this also gives a basis after finite field extension, as there are only finitely many equations to solve. See also ([Im], Lemma 1.2). □

For the rest of this section we fix the basis e_1, e_2 of Lemma 3.1 as the standard basis of $M_{\bar{\mathbb{F}}}$ and use the coordinates introduced in section 2. Furthermore we fix the point

$$P_{\mathrm{irred}} := [\tfrac{s}{p+1}, -\tfrac{s}{p-1}]_0 \in \mathcal{A}_0 \subset \bar{\mathcal{B}}. \tag{3.1}$$

Proposition 3.2. *(i) The map Φ extends to a map $\bar{\mathcal{B}} \to \bar{\mathcal{B}}$ also denoted by Φ.*
(ii) Let $[x, y]_0 \in \mathcal{A}_0$ be any point in the standard apartment. Then

$$\Phi([x, y]_0) = [-px + s, py + s]_0.$$

(iii) For any $q \in \bar{\mathbb{F}}((u))^\times$ with $k = v_u(q)$ and $[x, y] \in \mathcal{A}_q \backslash \mathcal{A}_0$, i.e. $x > k$, the map Φ is given by

$$\Phi([x, y]_q) = [px - 2pk + s, py + s]_{q'} \in \mathcal{A}_{q'}$$

for some $q' \in \bar{\mathbb{F}}((u))^\times$ with $v_u(q') = -pk + s \neq k$.
(iv) The point P_{irred}, as defined in (3.1), satisfies $\Phi(P_{\mathrm{irred}}) = P_{\mathrm{irred}}$.
(v) If $Q \in \mathcal{B}$ is an arbitrary point, then

$$\begin{aligned} d_1(Q, \Phi(Q)) &= (p+1) d_1(Q, P_{\mathrm{irred}}) \\ d_2(Q, \Phi(Q)) &= (p-1) d_2(Q, P_{\mathrm{irred}}). \end{aligned}$$

Proof. *(i)* We can use the expressions in *(ii)* and *(iii)* to extend Φ.
(ii) We have

$$\Phi(u^m e_1) = u^{pm}\Phi(e_1) = u^{pm}e_2$$
$$\Phi(u^n e_2) = u^{pn}\Phi(e_2) = au^{pn+s}e_1$$

and hence $\Phi((m,n)_0) = (pn+s, pm)_0$. The statement follows from this.
(iii) We put $v_u(q) = k$ and $\phi(q) = \alpha u^{pk}$ for some $\alpha \in \bar{\mathbb{F}}[\![u]\!]^\times$. If $\mathfrak{M} = (m,n)_q$, then

$$\langle\Phi(\mathfrak{M})\rangle = \langle u^{pm}e_2\ ,\ u^{pn}\phi(q)e_2 + au^{pn+s}e_1\rangle.$$

As $\mathfrak{M} = [m-n, m+n]_q \notin \mathcal{A}_0$ we have $m > n+k$. Hence

$$\begin{aligned}\langle\Phi(\mathfrak{M})\rangle &= \langle u^{pm}e_2 - \alpha^{-1}u^{p(m-n-k)}(u^{pn}\phi(q)e_2 + au^{pn+s}e_1), u^{pn}\phi(q)e_2 + au^{pn+s}e_1\rangle \\ &= \langle u^{p(m-k)+s}e_1, u^{p(n+k)}(q'e_1 + e_2)\rangle\end{aligned}$$

with $q' = \alpha^{-1}au^{-pk+s}$. And thus $\Phi((m,n)_q) = (p(m-k)+s, p(n+k))_{q'}$ with $v_u(q') = -pk+s \neq k$, as $k \not\equiv 0 \mod (p+1)$.
(iv) Obvious.
(v) If $\mathfrak{M} = [x,y]_q$, then the statement on d_2 follows immediately from *(ii)* and *(iii)*:

$$d_2(\mathfrak{M}, \langle\Phi(\mathfrak{M})\rangle) = (p-1)y + s = (p-1)d_2([x,y]_q, [\tfrac{s}{p+1}, -\tfrac{s}{p-1}]_0).$$

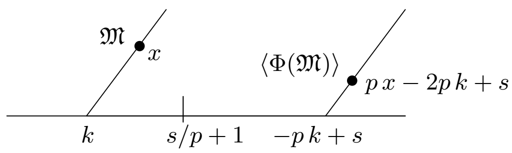

FIGURE 3. The images of $\mathfrak{M}$ and $\Phi(\mathfrak{M})$ in the building for $PGL_2(\bar{\mathbb{F}}((u)))$ in the case $\mathfrak{M} \notin \mathcal{A}_0$.

For the statement on d_1 first assume that $\mathfrak{M} = [x,y]_0 \in \mathcal{A}_0$. Then *(ii)* implies

$$d_1(\mathfrak{M}, \langle\Phi(\mathfrak{M})\rangle) = |(p+1)x - s| = (p+1)d_1([x,y]_0, [\tfrac{s}{p+1}, -\tfrac{s}{p-1}]_0).$$

If $\mathfrak{M} = [x,y]_q \in \mathcal{A}_q \backslash \mathcal{A}_0$, then $x > k$ and $px - 2pk + s > -pk + s$ which implies $\langle\Phi(\mathfrak{M})\rangle \notin \mathcal{A}_0$. Now *(iii)* implies

$$\begin{aligned}d_1(\mathfrak{M}, \langle\Phi(\mathfrak{M})\rangle) &= (x-k) + |-pk+s-k| + (px - 2pk + s - (-pk+s)) \\ &= (p+1)(x-k) + (p+1)\left|k - \tfrac{s}{p+1}\right| \\ &= \begin{cases}(p+1)(x - \frac{s}{p+1}) & \text{if } k > \frac{s}{p+1} \\ (p+1)(x - k + \frac{s}{p+1} - k) & \text{if } k < \frac{s}{p+1},\end{cases}\end{aligned}$$

using $k \neq -pk + s$. In both cases the claim follows. (See also Fig. 3) $\square$

Remark 3.3. The Proposition shows that the absolutely simple case is exactly the case discussed in ([PR2], 6.d, A1). The fixed point in the building is the point P_{irred} and its projection onto the building for $PGL_2(\bar{\mathbb{F}}((u)))$ lies on the edge between two vertices. The set of lattices $\mathfrak{M}$ with $d_1(\mathfrak{M}, \langle\Phi(\mathfrak{M})\rangle) \leq r_1 - r_2$ is identified with a ball around this fixed point.

Let $\mathbf{v}$ be a collection of integers as in the introduction. By Lemma 2.4 and Proposition 3.2*(v)* we find $\mathcal{GR}^{\mathbf{v},\mathrm{loc}}_{V_{\mathbb{F}},0}(\bar{\mathbb{F}}) \subset \bar{\mathcal{B}}(m(\mathbf{v}))$, where

$$m(\mathbf{v}) = (2e - d' - s)/(p-1). \tag{3.2}$$

Corollary 3.4. *The scheme* $\mathcal{GR}^{\mathbf{v},\mathrm{loc}}_{V_{\mathbb{F}},0}$ *is empty if* $2e - d' \not\equiv s \mod (p-1)$.

Proof. This follows from Lemma 2.4 and Proposition 3.2. □

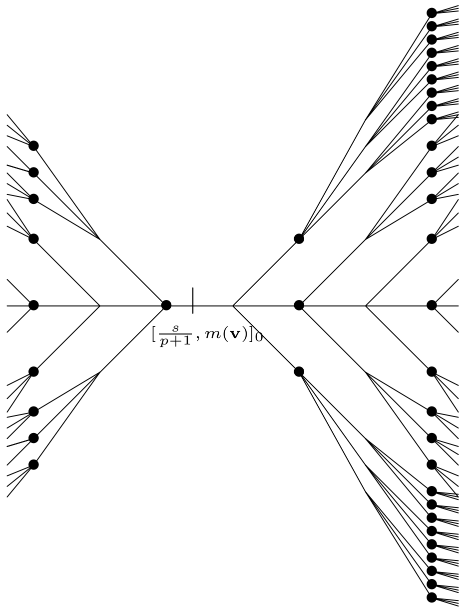

FIGURE 4. This picture illustrates the subset of $\mathbf{v}$-admissible lattices in the case $p = 3$ and $\mathbb{F} = \mathbb{F}_3$. This subset is given by all lattices $\mathfrak{M} \in \bar{\mathcal{B}}(m(\mathbf{v}))$ satisfying $d_1(\mathfrak{M}, P_{\mathrm{irred}}) \leq (r_1 - r_2)/(p+1)$. The fat points correspond to $\mathbf{v}$-admissible lattices.

Now we want to define locally closed subschemes of $\operatorname{Grass} M_{\mathbb{F}}$ on which the elementary divisors of $\langle \Phi(\mathfrak{M}) \rangle$ with respect to $\mathfrak{M}$ are fixed. Define a function

$$E : \operatorname{Grass} M_{\mathbb{F}} \to \mathbb{Z}^2.$$

For an extension field L of $\mathbb{F}$ and an L-valued point $z \in (\operatorname{Grass} M_{\mathbb{F}})(L)$ consider the $\mathbb{F}[\![u]\!]\widehat{\otimes}_{\mathbb{F}} L$-lattice $\mathfrak{M}_z$ in $M_{\mathbb{F}} \widehat{\otimes}_{\mathbb{F}} L$ corresponding to z. Then $E(z) = (j_1, j_2)$, where $j_1 \geq j_2$ are the elementary divisors of $\langle \Phi(\mathfrak{M}_z) \rangle$ with respect to $\mathfrak{M}_z$. Recall that there is a partial order on the pairs $(a,b) \in \mathbb{Z}^2$ given by (1.3).

Lemma 3.5. *The function E is lower semi-continuous with respect to the Zariski topology on* $\operatorname{Grass} M_{\mathbb{F}}$.

Proof. Let $\eta \rightsquigarrow z$ be a specialization and let $\mathfrak{M}_\eta$ and $\mathfrak{M}_z$ be the lattices corresponding to the points η and z. Denote by $E(\eta) = (a(\eta), b(\eta))$ and $E(z) = (a(z), b(z))$ the elementary divisors of $\langle \Phi(\mathfrak{M}_\eta) \rangle$ with respect to $\mathfrak{M}_\eta$ (resp. the elementary divisors of $\langle \Phi(\mathfrak{M}_z) \rangle$ with respect to $\mathfrak{M}_z$). We mark the specialization by a morphism $f : \operatorname{Spec} R \to \mathcal{GR}^{\mathbf{v},\mathrm{loc}}_{V_{\mathbb{F}},0}$, where R is a discrete valuation ring with uniformizer t. The morphism f defines a $R[\![u]\!]$-lattice $\mathfrak{M}_R$ in $M_{\mathbb{F}}\widehat{\otimes}_{\mathbb{F}}R$. After choosing a basis we find a matrix $C=(c_{ij})_{ij}\in GL_2(R((u)))\cap M_2(R[\![u]\!])$ such that $\mathfrak{M}_R\sim C$. Denote by $\bar{c}_{ij}$ the reduction $\mod t$ of the matrix coefficients. Using the theory of elementary divisors we find

$$b(\eta)=\min_{i,j} v_u(c_{ij})\leq \min_{i,j} v_u(\bar{c}_{ij})=b(z)$$

and hence $E(\eta)\geq E(z)$ which yields the claim. $\square$

Definition 3.6. Let $(a,b)\in\mathbb{Z}^2$ such that $a\geq b$. The *"Kisin variety"* associated to (a,b) is

$$\mathcal{G}_{V_{\mathbb{F}}}(a,b)=E^{-1}(a,b)\subset \operatorname{Grass} M_{\mathbb{F}}.$$

By Lemma 3.5, this is a locally closed subset and it will be considered as a subscheme with the reduced scheme structure (See also [PR2]).

Now we want to analyze the structure of $\mathcal{G}_{V_{\mathbb{F}}}(a,b)$ and $\mathcal{GR}^{\mathbf{v},\mathrm{loc}}_{V_{\mathbb{F}},0}$. We will make use of the following fact.

Lemma 3.7. *Let b_1,b_2 be any basis of $M_{\bar{\mathbb{F}}}$. There exists a morphism*

$$\chi:\mathbb{A}^1_{\bar{\mathbb{F}}}\to \operatorname{Grass} M_{\bar{\mathbb{F}}}$$

such that $\chi(z)=\langle b_1, zu^{-1}b_1+b_2\rangle$ for every closed point $z\in\mathbb{A}^1_{\bar{\mathbb{F}}}$. The morphism χ extends in a unique way to a morphism

$$\bar{\chi}:\mathbb{P}^1_{\bar{\mathbb{F}}}\to \operatorname{Grass} M_{\bar{\mathbb{F}}}.$$

The image of the point at infinity is given by $\bar{\chi}(\infty)=\langle u^{-1}b_1, ub_2\rangle$.

Proof. Consider the family

$$\langle b_1, Tu^{-1}b_1+b_2\rangle_{\bar{\mathbb{F}}[T][\![u]\!]}\subset M_{\bar{\mathbb{F}}}\widehat{\otimes}_{\bar{\mathbb{F}}}\bar{\mathbb{F}}[T]$$

of lattices on $\mathbb{A}^1=\operatorname{Spec}\bar{\mathbb{F}}[T]$. This family defines the morphism χ.
Let X be the closed subscheme of $\operatorname{Grass} M_{\bar{\mathbb{F}}}$ consisting of all lattices $\mathfrak{M}$ that satisfy $u\langle b_1,b_2\rangle\subset\mathfrak{M}\subset u^{-1}\langle b_1,b_2\rangle$ and that lie in the same connected component of $\operatorname{Grass} M_{\bar{\mathbb{F}}}$ as $\langle b_1,b_2\rangle$. The scheme X is identified with a closed subscheme of the (ordinary) Grassmann variety $\operatorname{Grass}_{\bar{\mathbb{F}}}(4,2)$ of 2-dimensional subspaces in $\bar{\mathbb{F}}^4$. The morphism χ factors as follows:

$$\begin{tikzcd} \mathbb{A}^1_{\bar{\mathbb{F}}} \arrow[rr, "\chi"] \arrow[dr, dotted, "\chi'"] & & \operatorname{Grass} M_{\bar{\mathbb{F}}} & \\ & X \arrow[ur, hook] \arrow[r] & \operatorname{Grass}_{\bar{\mathbb{F}}}(4,2) \arrow[r, "\iota"] & \mathbb{P}^5_{\bar{\mathbb{F}}} \end{tikzcd}$$

where ι is the Plücker embedding. As X is projective, the valuative criterion shows that χ extends in a unique way to $\mathbb{P}^1$.
We view $\operatorname{Grass}_{\bar{\mathbb{F}}}(4,2)$ as the quotient $GL_{2,\bar{\mathbb{F}}}\backslash V$, where V is the scheme of 2×4 matrices of rank 2 and $GL_{2,\bar{\mathbb{F}}}$ acts on V by left multiplication. Now, the computations using Plücker coordinates gives

$$\chi'(z)=\begin{pmatrix}0&1&0&0\\ z&0&0&1\end{pmatrix}\ ,\ \iota(\chi'(z))=(-z:0:0:0:1:0)$$

for all closed points $z\in\mathbb{A}^1(\bar{\mathbb{F}})$. Hence the extension to $\mathbb{P}^1$ is

$$(z_1:z_2)\mapsto(-z_1:0:0:0:z_2:0).$$

The image of the point at infinity is

$$\bar{\chi}((1:0)) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \mapsto (-1:0:0:0:0:0).$$

This is the lattice $\langle u^{-1}b_1, ub_2\rangle$. $\square$

Remark 3.8. In the building, the $\mathbb{F}$-valued points of the image of the morphism $\bar{\chi}$ can be illustrated in the following way (if the morphism is defined over $\mathbb{F}$):

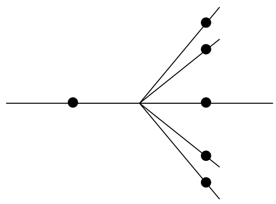

FIGURE 5. The morphism $\bar{\chi}$ in the building for $p=5$ and $\mathbb{F}=\mathbb{F}_5$.

Similarly, we can define morphisms $\bar{\chi}_1, \bar{\chi}_2 : \mathbb{P}^1_{\bar{\mathbb{F}}} \to \operatorname{Grass} M_{\bar{\mathbb{F}}}$ such that

$$\begin{aligned} \operatorname{im}(\bar{\chi}_1) &= \{\langle u^{n-1}b_1, u^{-(n-1)}(zu^{-1}b_1+b_2)\rangle \mid z \in \bar{\mathbb{F}}\} \cup \{\langle u^{-n}b_1, u^n b_2\rangle\} \\ \operatorname{im}(\bar{\chi}_2) &= \{\langle u^{n}b_1, u^{-n}(zb_1+b_2)\rangle \mid z \in \bar{\mathbb{F}}\} \cup \{\langle u^{-n}b_1, u^n b_2\rangle\} \end{aligned}$$

Theorem 3.9. *Assume that $(M_{\mathbb{F}}, \Phi)$ is absolutely simple. Fix a finite extension $\mathbb{F}'$ of $\mathbb{F}$ such that the normal form for Φ of Lemma* 3.1 *is defined over $\mathbb{F}'$.*
(a) For any $(a,b) \in \mathbb{Z}^2$ with $a \geq b$:

$$\mathcal{G}_{V_{\mathbb{F}}}(a,b) \neq \emptyset \Leftrightarrow a+b \equiv s \mod (p-1)\ ,\ \begin{cases} & pa+b \equiv s \mod (p^2-1) \\ or & pa+b \equiv ps \mod (p^2-1). \end{cases}$$

This condition being satisfied, there exists an isomorphism

$$\mathcal{G}_{V_{\mathbb{F}}}(a,b) \otimes_{\mathbb{F}} \mathbb{F}' \cong \mathbb{A}^n_{\mathbb{F}'}$$

with $n = \lfloor \frac{a-b}{p+1} \rfloor$. Further

$$\overline{\mathcal{G}_{V_{\mathbb{F}}}(a,b)} = \bigcup_{(a',b') \leq (a,b)} \mathcal{G}_{V_{\mathbb{F}}}(a',b').$$

(b) The scheme $\mathcal{GR}^{\mathbf{v},\mathrm{loc}}_{V_{\mathbb{F}},0}$ is geometrically connected and irreducible. After extending the scalars to $\mathbb{F}'$ it becomes isomorphic to a Schubert variety in the affine Grassmannian for $M_{\mathbb{F}'} = M_{\mathbb{F}} \widehat{\otimes}_{\mathbb{F}} \mathbb{F}'$ with dimension given by

$$\dim \mathcal{GR}^{\mathbf{v},\mathrm{loc}}_{V_{\mathbb{F}},0} = \left\lfloor \tfrac{r_1-r_2}{p+1} - (-1)^{\epsilon} \tfrac{s}{p+1} \right\rfloor + \left\lfloor (-1)^{\epsilon} \tfrac{s}{p+1} \right\rfloor,$$

with $\epsilon = \lfloor \frac{r_1-r_2}{p+1} \rfloor + \lfloor \frac{s}{p+1} \rfloor + \frac{2e-d'-s}{p-1}$.
[Here as in the rest of the paper, $\lfloor x \rfloor$ denotes the integral part of a real number x.]

Proof. (a) Assume $\mathfrak{M} \in \mathcal{G}_{V_{\mathbb{F}}}(a,b) \neq \emptyset$. Without loss of generality, we may assume $\mathfrak{M} = [x,y]_0 \in \mathcal{A}_0$: if $\mathfrak{M}$ is an arbitrary lattice, then there exists a lattice $\mathfrak{M}' \in \mathcal{A}_0$ such that $d_i(\mathfrak{M}, P_{\mathrm{irred}}) = d_i(\mathfrak{M}', P_{\mathrm{irred}})$ for $i=1,2$ (compare Fig. 4, for example). By Lemma 3.2*(v)* and Definition 2.2, the condition for $\mathfrak{M} = [x,y]_0 \in \mathcal{G}_{V_{\mathbb{F}}}(a,b)$ is

$$(p+1)d_1(\mathfrak{M}, P_{\mathrm{irred}}) = a-b \qquad (p-1)d_2(\mathfrak{M}, P_{\mathrm{irred}}) = a+b.$$

By an explicit computation of these distances, this is equivalent to

$$\left|x - \tfrac{s}{p+1}\right| = \tfrac{a-b}{p+1} \qquad y + \tfrac{s}{p-1} = \tfrac{a+b}{p-1}.$$

The second equation gives $s \equiv a+b \mod (p-1)$ and the sum of both equations gives $s \equiv pa+b \mod (p^2-1)$ if $(p+1)x > s$ and $ps \equiv pa+b \mod (p^2-1)$ if $(p+1)x < s$ (using the fact that $x+y$ and $x-y$ are even).
Conversely, suppose $s \equiv a+b \mod (p-1)$ and $s \equiv pa+b \mod (p^2-1)$ and define

$$x = \tfrac{a-b+s}{p+1} \qquad y = \tfrac{a+b-s}{p-1}.$$

Then we have $y \in \mathbb{Z}$ and $x+y \in 2\mathbb{Z}$. Thus $[x,y]_0$ defines a lattice $\mathfrak{M} \in \mathcal{G}_{V_{\mathbb{F}}}(a,b)$. If $ps \equiv pa+b \mod (p^2-1)$ we use

$$x = \tfrac{s-(a-b)}{p+1} \qquad y = \tfrac{a+b-s}{p-1}.$$

Now fix the sum $a+b$ and denote by y the integer solving the equation

$$(p-1)y + s = a+b.$$

Let us assume that $x_0 := \lfloor \tfrac{s}{p+1} \rfloor \equiv y \mod 2$ (the case $x_0 \not\equiv y \mod 2$ admits a similar treatment). In this case $[x_0,y]_0$ defines a lattice $\mathfrak{M}_0$ and we denote by X the connected component of $\operatorname{Grass} M_{\mathbb{F}'}$ containing $\mathfrak{M}_0$, i.e. $X(\bar{\mathbb{F}}) = \{\mathfrak{M} \in \bar{\mathcal{B}}(y)\}$.
For each $m \geq 0$, there is a morphism $f_m : \mathbb{A}^{2m+1}_{\mathbb{F}'} \to X$ given by the family of lattices

$$\left\langle u^{(x_0+y)/2}u^{m+1}e_1 \ , \ u^{(y-x_0)/2}u^{-(m+1)}\left((\sum_{i=1}^{2m+1} T_i u^{i+x_0})e_1 + e_2 \right) \right\rangle$$
$$\subset M_{\mathbb{F}'}\widehat{\otimes}_{\mathbb{F}'}(\mathbb{F}'[T_1,\dots,T_{2m+1}])$$

on $\mathbb{A}^{2m+1}_{\mathbb{F}'} = \operatorname{Spec}\mathbb{F}'[T_1,\dots,T_{2m+1}]$. Let $V_m \cong \mathbb{A}^{2m+1}_{\mathbb{F}'}$ be its image. We have

$$\begin{aligned} &\bar{\mathcal{B}}(y) \supset V_m(\bar{\mathbb{F}}) = \\ &\{\langle u^{(x_0+y)/2}u^{m+1}e_1 \, , \, u^{(y-x_0)/2}u^{-(m+1)}(qe_1+e_2)\rangle \mid q = \sum_{i=1}^{2m+1} a_i u^{i+x_0}\}, \end{aligned} \tag{3.3}$$

with $a_1 \dots a_{2m+1} \in \bar{\mathbb{F}}$. Similarly, define for $m \geq 0$ a morphism $g_m : \mathbb{A}^{2m}_{\mathbb{F}'} \to X$ given by the family of lattices

$$\left\langle u^{(x_0+y)/2}u^{-m}\left(e_1 + (\sum_{i=0}^{2m-1} T_i u^{i-x_0})e_2 \right) \ , \ u^{(y-x_0)/2}u^m e_2 \right\rangle$$
$$\subset M_{\mathbb{F}'}\widehat{\otimes}_{\mathbb{F}'}(\mathbb{F}'[T_0,\dots,T_{2m-1}])$$

and let $U_m \cong \mathbb{A}^{2m}_{\mathbb{F}'}$ be its image. We have

$$\begin{aligned} &\bar{\mathcal{B}}(y) \supset U_m(\bar{\mathbb{F}}) = \\ &\{\langle u^{(x_0+y)/2}u^{-m}(e_1+qe_2) \, , \, u^{(y-x_0)/2}u^m e_2\rangle \mid q = \sum_{i=0}^{2m-1} a_i u^{i-x_0}\}, \end{aligned} \tag{3.4}$$

with $a_0 \dots a_{2m-1} \in \bar{\mathbb{F}}$. It is easy to see that every lattice $\mathfrak{M} \in \bar{\mathcal{B}}(y)$ is either of the form (3.3) or of the form (3.4) for some $m \geq 0$. Thus

$$X = (\bigcup_{m\geq 0} V_m) \cup (\bigcup_{m \geq 0} U_m).$$

We claim

$$\mathfrak{M} \in V_m(\bar{\mathbb{F}}) \Longrightarrow d_1(\mathfrak{M}, P_{\text{irred}}) = 2m + 2 - \xi \tag{3.5}$$

$$\mathfrak{M} \in U_m(\bar{\mathbb{F}}) \Longrightarrow d_1(\mathfrak{M}, P_{\text{irred}}) = 2m + \xi, \tag{3.6}$$

where $\xi = \frac{s}{p+1} - x_0$ denotes the fractional part of $\frac{s}{p+1}$.
Indeed, if $\mathfrak{M} \in V_m(\bar{\mathbb{F}})$, then $\mathfrak{M} = [x_0 + 2m + 2, y]_q$ for some $q \in \bar{\mathbb{F}}((u))$ with $v_u(q) > x_0$ and hence

$$d_1(\mathfrak{M}, P_{\text{irred}}) = x_0 + 2m + 2 - \tfrac{s}{p+1} = 2m + 2 - \xi.$$

The statement on U_m follows by a more complicated computation or by a symmetry argument: The choice of apartments $\mathcal{A}_q$ and coordinates $[-,-]_q$ depends on the order of e_1 and e_2. Interchanging e_1 and e_2 yields expressions for the lattices $\mathfrak{M} \in U_m$ similar to the above expressions for V_m (if $\mathfrak{M} \in U_m$ is a lattice, then $\mathfrak{M} = [-x_0 + 2m, y]_q$ for some q) while it maps the point P_{irred} to $[-\frac{s}{p+1}, -\frac{s}{p-1}]_0$ and hence the claim follows by the same computation.
Now equation (3.5) and (3.6) together with Proposition 3.2 *(v)* imply

$$\begin{aligned} V_m(\bar{\mathbb{F}}) &\subset \mathcal{G}_{V_{\mathbb{F}}}(a_{\text{odd}}(m), b_{\text{odd}}(m))(\bar{\mathbb{F}}) \\ U_m(\bar{\mathbb{F}}) &\subset \mathcal{G}_{V_{\mathbb{F}}}(a_{\text{even}}(m), b_{\text{even}}(m))(\bar{\mathbb{F}}) \end{aligned} \tag{3.7}$$

for some $(a_{\text{odd}}(m), b_{\text{odd}}(m))$, $(a_{\text{even}}(m), b_{\text{even}}(m)) \in \mathbb{Z}^2$ with

$$\begin{aligned} a_{\text{odd}}(m) + b_{\text{odd}}(m) &= a_{\text{even}}(m') + b_{\text{even}}(m') = (p-1)y + s \\ a_{\text{odd}}(m) - b_{\text{odd}}(m) &= (p+1)(2m + 2 - \xi) \\ a_{\text{even}}(m) - b_{\text{even}}(m) &= (p+1)(2m + \xi) \end{aligned} \tag{3.8}$$

and $0 < \xi < 1$ implies that all these pairs are pairwise distinct when m runs over all positive integers.
As U_m and V_m cover X, the inclusions in (3.7) are actually equalities. Furthermore $V_m = \mathcal{G}_{V_{\mathbb{F}}}(a_{\text{odd}}(m), b_{\text{odd}}(m))$ as schemes, as both are reduced locally closed subschemes of Grass $M_{\mathbb{F}'}$ with the same underlying point set. Finally (3.8) yields

$$\dim V_m = 2m + 1 = \lfloor 2m + 2 - \xi \rfloor = \lfloor \tfrac{a_{\text{odd}}(m) - b_{\text{odd}}(m)}{p+1} \rfloor.$$

The conclusion for U_m is similar.
To finish the proof of *(a)*, it remains to show that $U_m \subset \overline{V_m}$ and $V_{m-1} \subset \overline{U_m}$. We will prove the first assertion: the second is proved in the same way.
Let $z_1 \in U_m$ be an arbitrary point corresponding to a lattice

$$\mathfrak{M}_1 = \langle u^{(x_0+y)/2} u^{-m}(e_1 + qe_2)\,,\, u^{(y-x_0)/2} u^m e_2 \rangle$$

with $q = \sum_{i=0}^{2m-1} a_i u^{i-x_0}$ and let $z_2 \in V_m$ be the point corresponding to

$$\mathfrak{M}_2 = \langle u^{(x_0+y)/2} u^{m+1} e_1\,,\, u^{(y-x_0)/2} u^{-(m+1)} e_2 \rangle.$$

There exists a basis b_1 and b_2 of $M_{\bar{\mathbb{F}}}$ such that

$$\begin{aligned} \langle b_1, b_2 \rangle &= \mathfrak{M}_0 = [x_0, y]_0, \\ \langle u^{-m} b_1, u^m b_2 \rangle &= \mathfrak{M}_1, \\ \langle u^{m+1} b_1, u^{-(m+1)} b_2 \rangle &= \mathfrak{M}_2. \end{aligned}$$

Explicitly, we may choose

$$b_1 = u^{(x_0+y)/2}(e_1 + qe_2)\ ,\ b_2 = u^{(y-x_0)/2} e_2.$$

Applying Lemma 3.7 (resp. Remark 3.8) with the basis $ub_1, u^{-1}b_2$, we obtain a morphism $\chi : \mathbb{A}^1_{\bar{\mathbb{F}}} \to \operatorname{Grass} M_{\bar{\mathbb{F}}}$ that is given by $\chi(z) = \langle u^{m+1}b_1, u^{-(m+1)}(zub_1 + b_2)\rangle$ on closed points and we easily find $\operatorname{im}\chi \subset V_m \otimes_{\mathbb{F}'} \bar{\mathbb{F}}$. As $\overline{V_m} \otimes_{\mathbb{F}'} \bar{\mathbb{F}}$ is projective, the morphism χ extends to a morphism from $\mathbb{P}^1$ to $\overline{V_m} \otimes_{\mathbb{F}'} \bar{\mathbb{F}}$ and the point at infinity is mapped to z_1 (Fig. 6 illustrates the image of the morphism $\bar{\chi}$ in the building. The fat points are the lattices in the image of $\bar{\chi}$). Hence $z_1 \in \overline{V_m}(\bar{\mathbb{F}})$ and the claim follows.

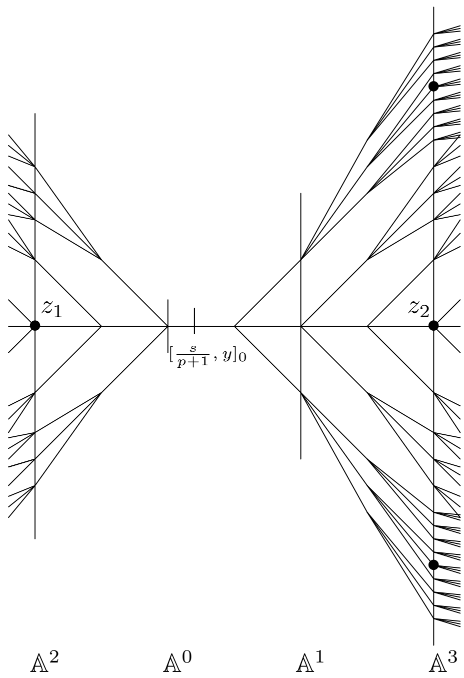

FIGURE 6. The stratification with affine spaces in the building. Fat points mark the image of an exemplary morphism $\bar{\chi}$.

(b) For a given collection $\mathbf{v}$ we have

$$\mathcal{GR}^{\mathbf{v},\mathrm{loc}}_{V_{\mathbb{F}},0} = \bigcup_{\substack{a+b=2e-d' \\ e-r_1 \le b \le a \le e-r_2}} \mathcal{G}_{V_{\mathbb{F}}}(a,b), \tag{3.9}$$

where d' is the integer defined in (2.1). Hence the scheme is geometrically irreducible, because the restriction of the order "$\le$" on the pairs

$$\{(a,b) \in \mathbb{Z}^2 \mid a+b = m(\mathbf{v})\},$$

where $m(\mathbf{v})$ is given by (3.2), is a total order. Of course this also implies connectedness.

The dimension of $\mathcal{GR}^{\mathbf{v},\mathrm{loc}}_{V_{\mathbb{F}},0}$ is given by the dimension of the maximal affine space in (3.9). We assume that ϵ is even, i.e. $\lfloor \frac{r_1-r_2}{p+1} \rfloor + x_0 \equiv m(\mathbf{v}) \mod 2$. The computations in the other case are similar.

In this case the affine subspace of maximal dimension consists of all lattices

$\mathfrak{M} \in \bar{\mathcal{B}}(m(\mathbf{v}))$ with

$$d_1(\mathfrak{M}, P_{\text{irred}}) = d_1([x_0 - \lfloor \tfrac{r_1-r_2}{p+1} \rfloor, m(\mathbf{v})]_0, P_{\text{irred}}).$$

(if the latter distance is $\leq \frac{r_1-r_2}{p+1}$) or of the lattices with

$$d_1(\mathfrak{M}, P_{\text{irred}}) = d_1([x_0 + \lfloor \tfrac{r_1-r_2}{p+1} \rfloor, m(\mathbf{v})]_0, P_{\text{irred}})$$

(if $d_1([x_0 - \lfloor \frac{r_1-r_2}{p+1} \rfloor, m(\mathbf{v})]_0, P_{\text{irred}}) > \frac{r_1-r_2}{p+1}$). Hence its dimension is either $n := \lfloor \frac{r_1-r_2}{p+1} \rfloor$ (in the first case) or $n-1$ (in the second case). This yields the claim on the dimension:

$$\begin{aligned} \dim \mathcal{GR}^{\mathbf{v},\text{loc}}_{V_{\mathbb{F}},0} &= \lfloor \tfrac{r_1-r_2}{p+1} - \tfrac{s}{p+1} \rfloor + \lfloor \tfrac{s}{p+1} \rfloor \\ &= \begin{cases} n & \text{if } \frac{s}{p+1} - \lfloor \frac{s}{p+1} \rfloor \leq \frac{r_1-r_2}{p+1} - \lfloor \frac{r_1-r_2}{p+1} \rfloor \\ n-1 & \text{if } \frac{s}{p+1} - \lfloor \frac{s}{p+1} \rfloor > \frac{r_1-r_2}{p+1} - \lfloor \frac{r_1-r_2}{p+1} \rfloor. \end{cases} \end{aligned}$$

We further see that the set of $\mathbf{v}$-admissible lattices is exactly the set of lattices in $\bar{\mathcal{B}}(m(\mathbf{v}))$ with

$$\begin{array}{ll} d_1(\mathfrak{M}, [x_0, m(\mathbf{v})]_0) \leq n & \text{if } x_0 + \lfloor \frac{r_1-r_2}{p+1} \rfloor - \frac{s}{p+1} \leq \frac{r_1-r_2}{p+1} \\ d_1(\mathfrak{M}, [x_0+1, m(\mathbf{v})]_0) \leq n-1 & \text{otherwise} \end{array} \tag{3.10}$$

and hence this is the set of lattices whose elementary divisors (a,b) with respect to a lattice $\mathfrak{N}$ satisfy $(a,b) \leq (a_{\max}, b_{\max})$ for some given integers $a_{\max}, b_{\max}$. For $\mathfrak{N}$ we choose one of the lattices

$$[x_0, m(\mathbf{v})]_0 \ , \ [x_0, m(\mathbf{v})-1]_0 \quad \text{or} \quad [x_0+1, m(\mathbf{v})]_0 \ , \ [x_0+1, m(\mathbf{v})-1]_0$$

depending on the cases as listed in (3.10) and on $x_0 - m(\mathbf{v}) \mod 2$. Since we know that $\mathcal{GR}^{\mathbf{v},\text{loc}}_{V_{\mathbb{F}},0}$ is reduced, we find that it is isomorphic to a Schubert variety in the affine Grassmannian after extending the scalars to $\mathbb{F}'$.
If ϵ is odd, then the maximal affine subspace consists of all lattices $\mathfrak{M} \in \bar{\mathcal{B}}(m(\mathbf{v}))$ with

$$d_1(\mathfrak{M}, P_{\text{irred}}) = d_1([x_0 + 1 + \lfloor \tfrac{r_1-r_2}{p+1} \rfloor, m(\mathbf{v})]_0, P_{\text{irred}})$$

(if the latter distance is $\leq \frac{r_1-r_2}{p+1}$) or of the lattices with

$$d_1(\mathfrak{M}, P_{\text{irred}}) = d_1([x_0 + 1 - \lfloor \tfrac{r_1-r_2}{p+1} \rfloor, m(\mathbf{v})]_0, P_{\text{irred}})$$

(if $d_1([x_0 + 1 + \lfloor \frac{r_1-r_2}{p+1} \rfloor, m(\mathbf{v})]_0, P_{\text{irred}}) > \frac{r_1-r_2}{p+1}$). We find

$$\begin{aligned} \dim \mathcal{GR}^{\mathbf{v},\text{loc}}_{V_{\mathbb{F}},0} &= \lfloor \tfrac{r_1-r_2}{p+1} + \tfrac{s}{p+1} \rfloor + \lfloor -\tfrac{s}{p+1} \rfloor \\ &= \begin{cases} n & \text{if } 1 - (\frac{s}{p+1} - \lfloor \frac{s}{p+1} \rfloor) \leq \frac{r_1-r_2}{p+1} - \lfloor \frac{r_1-r_2}{p+1} \rfloor \\ n-1 & \text{if } 1 - (\frac{s}{p+1} - \lfloor \frac{s}{p+1} \rfloor) > \frac{r_1-r_2}{p+1} - \lfloor \frac{r_1-r_2}{p+1} \rfloor \end{cases} \end{aligned}$$

and the conclusion for the isomorphism with a Schubert variety is similar. □

As a consequence of the theorem, we may determine the cases when $\mathcal{GR}^{\mathbf{v},\text{loc}}_{V_{\mathbb{F}},0}$ is a single point.

Corollary 3.10. *Denote by* $\xi = \frac{s}{p+1} - \lfloor \frac{s}{p+1} \rfloor$ *the fractional part of* $\frac{s}{p+1}$.

$$\mathcal{GR}^{\mathbf{v},\text{loc}}_{V_{\mathbb{F}},0} = \{*\} \Leftrightarrow \begin{cases} 0 + \xi \leq \frac{r_1-r_2}{p+1} < 2 - \xi & \textit{if } \lfloor \frac{s}{p+1} \rfloor \equiv \frac{2e-d'-s}{p-1} \mod 2 \\ 1 - \xi \leq \frac{r_1-r_2}{p+1} < 1 + \xi & \textit{if } \lfloor \frac{s}{p+1} \rfloor \not\equiv \frac{2e-d'-s}{p-1} \mod 2. \end{cases}$$

Proof. This is just the case where the dimension of $\mathcal{GR}^{\mathbf{v},\mathrm{loc}}_{V_{\mathbb{F}},0}$ is zero. More explicitly: If $x_0 = \lfloor \frac{s}{p+1} \rfloor \equiv m(\mathbf{v}) \mod 2$, then $[x_0, m(\mathbf{v})]$ is the unique lattice with minimal distance d_1 from P_{irred}. We have

$$d_1([x_0, m(\mathbf{v})]_0, P_{\mathrm{irred}}) = \xi.$$

Thus this lattice is $\mathbf{v}$-admissible if and only if $\frac{r_1 - r_2}{p+1} \geq \xi$. There is no other $\mathbf{v}$-admissible lattice iff the lattices $\mathfrak{M}$ with $d_1(\mathfrak{M}, P_{\mathrm{irred}}) = 2 - \xi$ are not $\mathbf{v}$-admissible. This yields the claim.
The case $x_0 \not\equiv m(\mathbf{v}) \mod 2$ is similar. Instead of $[x_0, m(\mathbf{v})]_0$ we have to consider the lattice $[x_0 + 1, m(\mathbf{v})]_0$. □

4. The reducible case

In this section we want to analyze the case, where $(M_{\mathbb{F}}, \Phi)$ admits a proper Φ-stable subobject, at least after extending the scalars to some finite extension of $\mathbb{F}$. Before we start to determine the set of $\mathbf{v}$-admissible lattices in the building, we want to formulate the precise statement on the connected components of $\mathcal{GR}^{\mathbf{v},\mathrm{loc}}_{V_{\mathbb{F}},0}$. We first define some open and closed subschemes of $\mathcal{GR}^{\mathbf{v},\mathrm{loc}}_{V_{\mathbb{F}},0}$.

Definition 4.1. For $a \in \bar{\mathbb{F}}^\times$ and $j \in \mathbb{Z}_{\geq 0}$ define $(\mathfrak{M}^j(a), \Phi^j_a)$ by

$$\mathfrak{M}^j(a) = \bar{\mathbb{F}}[\![u]\!]\ , \qquad \Phi^j_a(1) = au^j.$$

Definition 4.2. A $\mathbf{v}$-admissible lattice $\mathfrak{M} \subset M_{\bar{\mathbb{F}}}$ is called $\mathbf{v}$-*ordinary* if there exists a short exact sequence

$$0 \to (\mathfrak{M}^{e-r_1}(a), \Phi^{e-r_1}_a) \to (\mathfrak{M}, \Phi) \to (\mathfrak{M}^{e-r_2}(b), \Phi^{e-r_2}_b) \to 0 \tag{4.0.1}$$

for some $a, b \in \bar{\mathbb{F}}^\times$.

Remark 4.3. The determinant condition in (1.2) implies that

$$u^{e-r_1}\mathfrak{M} \subset \langle \Phi(\mathfrak{M}) \rangle \subset u^{e-r_2}\mathfrak{M} \tag{4.0.2}$$

for all $\mathbf{v}$-admissible lattices $\mathfrak{M}$. Hence the $\mathbf{v}$-ordinary lattices are the lattices which admit a Φ-stable subobject with the minimal possible elementary divisors. If a $\mathbf{v}$-admissible lattice $(\mathfrak{M}, \Phi)$ admits a subobject isomorphic to $(\mathfrak{M}^{e-r_1}(a), \Phi^{e-r_1}_a)$ for some $a \in \bar{\mathbb{F}}^\times$, then the quotient has no u-torsion by (4.0.2) and is isomorphic to $(\mathfrak{M}^{e-r_2}(b), \Phi^{e-r_2}_b)$ for some $b \in \bar{\mathbb{F}}^\times$, because the sum of the elementary divisors is fixed by (1.2). Hence $(\mathfrak{M}, \Phi)$ is $\mathbf{v}$-ordinary in this case.

Denote by $\mathcal{S}(\mathbf{v})$ the set of isomorphism classes of one dimensional $\bar{\mathbb{F}}((u))$-modules M' with ϕ-linear map $\Phi' \neq 0$ such that M' admits a (unique) lattice $\mathfrak{M}_{[M']} \subset M'$ with $\langle \Phi(\mathfrak{M}_{[M']}) \rangle = u^{e-r_1}\mathfrak{M}_{[M']}$. The elements of $\mathcal{S}(\mathbf{v})$ are in bijection with the elements of $\bar{\mathbb{F}}^\times$: For each $a \in \bar{\mathbb{F}}^\times$ there is a unique isomorphism class represented by

$$(M_a, \Phi_a) = (\mathfrak{M}^{e-r_1}(a)[\tfrac{1}{u}], \Phi^{e-r_1}_a). \tag{4.0.3}$$

Set $X = \mathcal{GR}^{\mathbf{v},\mathrm{loc}}_{V_{\mathbb{F}},0} \otimes_{\mathbb{F}} \bar{\mathbb{F}}$. On X there is a universal sheaf of $\bar{\mathbb{F}}[\![u]\!]\widehat{\otimes}_{\bar{\mathbb{F}}}\mathcal{O}_X = \mathcal{O}_X[\![u]\!]$-lattices $\mathcal{M} \subset M_{\bar{\mathbb{F}}}\widehat{\otimes}_{\bar{\mathbb{F}}}\mathcal{O}_X$ satisfying

$$u^e\mathcal{M} \subset (\mathrm{id} \otimes \Phi)\phi^*\mathcal{M} \subset \mathcal{M}.$$

For each $[M'] \in \mathcal{S}(\mathbf{v})$ define a sheaf of $\mathcal{O}_X$-modules

$$\mathcal{F}_{[M']} = \mathcal{H}om_{\mathcal{O}_X[\![u]\!],\Phi}(\mathfrak{M}_{[M']}\widehat{\otimes}_{\bar{\mathbb{F}}}\mathcal{O}_X, \mathcal{M}) \tag{4.0.4}$$

where the subscript Φ indicates that the homomorphism have to commute with the semi-linear maps that are part of the data.

Proposition 4.4. *(i) For each $[M'] \in \mathcal{S}(\mathbf{v})$ the sheaf $\mathcal{F}_{[M']}$ is a coherent $\mathcal{O}_X$-module.*
(ii) A closed point $x \in \mathcal{GR}^{\mathbf{v},\mathrm{loc}}_{V_{\mathbb{F}},0}$ corresponds to a non-$\mathbf{v}$-ordinary lattice if and only if $\mathcal{F}_{[M']} \otimes \kappa(x) = 0$ for all $[M'] \in \mathcal{S}(\mathbf{v})$.

Proof. (i) For the isomorphism class $[M']$ we choose a representative of the form M_a defined in (4.0.3). Let $U = \operatorname{Spec} A \subset X$ an affine open. We claim
(a) $\operatorname{Hom}_{A[\![u]\!],\Phi}(\mathfrak{M}_{[M']}\widehat{\otimes}_{\bar{\mathbb{F}}}A, \mathcal{M}(U))$ *is a finitely generated A-module.*
(b) If $V = \operatorname{Spec} B \subset U$ is an affine open we have

$$\operatorname{Hom}_{B[\![u]\!],\Phi}(\mathfrak{M}_{[M']}\widehat{\otimes}_{\bar{\mathbb{F}}}B, \mathcal{M}(V)) \cong \operatorname{Hom}_{A[\![u]\!],\Phi}(\mathfrak{M}_{[M']}\widehat{\otimes}_{\bar{\mathbb{F}}}A, \mathcal{M}(U)) \otimes_A B. \tag{4.0.5}$$

This implies the first part of the Proposition.
Proof of (a): Because $\mathfrak{M}_{[M']}\widehat{\otimes}_{\bar{\mathbb{F}}}A$ is a free $A[\![u]\!]$-module of rank one, a morphism is given by the image of 1 and hence

$$\operatorname{Hom}_{A[\![u]\!],\Phi}(\mathfrak{M}_{[M']}\widehat{\otimes}_{\bar{\mathbb{F}}}A, \mathcal{M}(U)) \cong N_A \subset \mathcal{M}(U),$$

where N_A is the A-submodule of all $v \in \mathcal{M}(U)$ satisfying $\Phi(v) = au^{e-r_1}v$. We claim that the reduction modulo u^{e+1} induces an injective homomorphism

$$N_A \hookrightarrow \mathcal{M}(U)/u^{e+1}\mathcal{M}(U)$$

and hence N_A is finitely generated as an A-module, because the scheme X is noetherian. Now, if $0 \neq v = u^n w \in N_A$ with $n \geq 0$ and $w \in \mathcal{M}(U)\backslash u\mathcal{M}(U)$, then

$$u^{pn}\Phi(w) = \Phi(u^n w) = au^{e-r_1+n}w$$

and hence $0 \leq e - r_1 - (p-1)n \leq e$ which implies $n \leq e$.
Proof of (b): We have the following commutative diagram

$$\begin{tikzcd}
\operatorname{Hom}_{A[\![u]\!],\Phi}(\mathfrak{M}_{[M']}\widehat{\otimes}_{\bar{\mathbb{F}}}A, \mathcal{M}(U)) \arrow[r, "\cong"] \arrow[d] & N_A \arrow[r, hook] \arrow[d] & \mathcal{M}(U) \arrow[d] \\
\operatorname{Hom}_{B[\![u]\!],\Phi}(\mathfrak{M}_{[M']}\widehat{\otimes}_{\bar{\mathbb{F}}}B, \mathcal{M}(V)) \arrow[r, "\cong"] & N_B \arrow[r, hook] & \mathcal{M}(V) \cong \mathcal{M}(U)\widehat{\otimes}_A B
\end{tikzcd}$$

As N_A is a finitely generated A-module, we do not need to complete the tensor product to obtain N_B from N_A (there are only finitely many denominators). Hence (4.0.5) is an isomorphism.
(ii) Let $[M'] \in \mathcal{S}(\mathbf{v})$ be an isomorphism class and suppose that $x \in \mathcal{GR}^{\mathbf{v},\mathrm{loc}}_{V_{\mathbb{F}},0}$ is a closed point corresponding to a lattice $\mathfrak{M}$ such that $\mathcal{F}_{[M']} \otimes \kappa(x) \neq 0$, i.e. there exists a non trivial morphism

$$f : \mathfrak{M}_{[M']} \to \mathfrak{M}.$$

As both sides are free $\bar{\mathbb{F}}[\![u]\!]$-modules and the morphism is non trivial, it is injective. We have to convince ourselves that $\operatorname{coker} f$ has no u-torsion: in this case $\mathfrak{M}$ is the extension of free $\bar{\mathbb{F}}[\![u]\!]$-modules of rank 1 (an extension of $\operatorname{coker} f$ by $\operatorname{im} f$), and hence $\mathbf{v}$-ordinary.
We write $f(1) = u^n v$ for some $n \in \mathbb{Z}$ and $v \in \mathfrak{M}\backslash u\mathfrak{M}$ and claim $n = 0$. Because of $\Phi(f(1)) = f(\Phi(1))$ we find $\Phi(v) = au^{e-r_1-(p-1)n} \in \Phi(\mathfrak{M}) \subset u^{e-r_1}\mathfrak{M}$ for some $a \in \bar{\mathbb{F}}^\times$ and hence $n = 0$.

Conversely, if $\mathfrak{M}$ is $\mathbf{v}$-ordinary, then the inclusion of the Φ-stable subobject defines a nontrivial morphism $\mathfrak{M}_{[M']} \to \mathfrak{M}$ for some $[M'] \in \mathcal{S}(\mathbf{v})$. □

Definition 4.5. For each isomorphism class $[M'] \in \mathcal{S}(\mathbf{v})$ define

$$X^{\mathbf{v}}_{[M']} = \{x \in \mathcal{GR}^{\mathbf{v},\mathrm{loc}}_{V_{\mathbb{F}},0} \otimes_{\mathbb{F}} \bar{\mathbb{F}} \mid \mathcal{F}_{[M']} \otimes \kappa(x) \neq 0\}.$$

Further define

$$X^{\mathbf{v}}_0 = \mathcal{GR}^{\mathbf{v},\mathrm{loc}}_{V_{\mathbb{F}},0} \otimes_{\mathbb{F}} \bar{\mathbb{F}} \setminus \bigcup_{[M'] \in \mathcal{S}(\mathbf{v})} X^{\mathbf{v}}_{[M']}.$$

By the Proposition below these subsets are open and closed and hence they come along with a canonical scheme structure.

Proposition 4.6. *(i) The subset $X^{\mathbf{v}}_{[M']}$ is open and closed for each $[M'] \in \mathcal{S}(\mathbf{v})$.*
(ii) The subset $X^{\mathbf{v}}_0$ is open and closed.

Proof. (i) It is clear that $X^{\mathbf{v}}_{[M']}$ is closed, as $\mathcal{F}_{[M']}$ is coherent. We show that it is closed under cospecialization.
Let $\eta \rightsquigarrow x$ be a specialization with $x \in X^{\mathbf{v}}_{[M']}$ and assume that x is a closed point. We mark this specialization by $\operatorname{Spec} R \to X$, where R is a discrete valuation ring with uniformizer t and residue field $\bar{\mathbb{F}}$. Denote by $\mathfrak{M}_R$ the $R[\![u]\!]$-lattice in $M_{\bar{\mathbb{F}}} \widehat{\otimes}_{\bar{\mathbb{F}}} R$ defined by this morphism. Because of $\mathcal{F}_{[M']} \otimes \kappa(x) \neq 0$, there is a non trivial morphism $\mathfrak{M}_{[M']} \to \mathfrak{M}_x$ and hence there is a basis vector $b_1 \in M_{\bar{\mathbb{F}}}$ such that $\Phi(b_1) = au^{e-r_1}b_1$ for some $a \in \bar{\mathbb{F}}^\times$. As $\mathfrak{M}_R$ is a free $R[\![u]\!]$-module, there is a basis of $\mathfrak{M}_R$ such that

$$\mathfrak{M}_R \sim \begin{pmatrix} \alpha & \gamma \\ 0 & \delta \end{pmatrix},$$

for $\alpha, \gamma, \delta \in R[\![u]\!]$, with $\alpha \equiv au^{e-r_1} \mod t$. But the determinant condition in (1.2) implies $v_u(\alpha) \geq e - r_1$. Hence $v_u(\alpha) = e - r_1$ and $\eta \in X^{\mathbf{v}}_{[M'']}$ for some $[M''] \in \mathcal{S}(\mathbf{v})$. If $[M'] = [M'']$ we are done.
Assume $[M'] \neq [M'']$. As $X^{\mathbf{v}}_{[M'']}$ is closed, we have $x \in X^{\mathbf{v}}_{[M']} \cap X^{\mathbf{v}}_{[M'']}$. In this case $\mathfrak{M}_x$ admits two linear independent subspaces:

$$\mathfrak{M}_x \sim \begin{pmatrix} au^{e-r_1} & 0 \\ 0 & bu^{e-r_1} \end{pmatrix}$$

and hence $e - r_1 = e - r_2$. Now we easily deduce $\mathcal{GR}^{\mathbf{v},\mathrm{loc}}_{V_{\mathbb{F}},0} = \{\mathfrak{M}_x\}$ and the claim follows.
(ii) This follows from the first part of the Proposition together with the fact that the one-dimensional Φ-invariant subspaces of $M_{\bar{\mathbb{F}}}$ which admit an integral model $\mathfrak{M}$ with $\langle \Phi(\mathfrak{M}) \rangle = u^{e-r_1}\mathfrak{M}$ run over a finite set of isomorphism classes of one-dimensional objects:
Assume that there are two different one-dimensional Φ-stable subspaces $\langle b_1 \rangle$ and $\langle b_2 \rangle$ of $M_{\bar{\mathbb{F}}}$ such that $\Phi(b_i) = a_i u^{e-r_1} b_i$, for $i = 1, 2$. Then b_1 and b_2 are linear independent.
If $a_1 \neq a_2$, then $\langle b_1 + qb_2 \rangle$ is not Φ-stable for all $q \in \bar{\mathbb{F}}((u))^\times$ and hence there are only two isomorphism classes.
If $a_1 = a_2$, then there is a unique such isomorphism class given by $[M_a]$. □

We will see below that the open and closed subschemes $X^{\mathbf{v}}_{[M']}$ and $X^{\mathbf{v}}_0$ of $\mathcal{GR}^{\mathbf{v},\mathrm{loc}}_{V_{\mathbb{F}},0} \otimes_{\mathbb{F}} \bar{\mathbb{F}}$ are connected and hence turn out to be the connected components of $\mathcal{GR}^{\mathbf{v},\mathrm{loc}}_{V_{\mathbb{F}},0} \otimes_{\mathbb{F}} \bar{\mathbb{F}}$.

Now we want to determine the subset of $\mathbf{v}$-admissible lattices in the building. As we are assuming that $(M_{\mathbb{F}}, \Phi)$ is reducible, at least after extending scalars, there exists a finite extension $\mathbb{F}'$ of $\mathbb{F}$ and a basis e_1, e_2 of $M_{\mathbb{F}'} = M_{\mathbb{F}}\widehat{\otimes}_{\mathbb{F}}\mathbb{F}'$ such that

$$M_{\mathbb{F}'} \sim \begin{pmatrix} au^s & \gamma \\ 0 & bu^t \end{pmatrix}$$

for some $a, b \in \mathbb{F}'^{\times}$, $\gamma \in \mathbb{F}'((u))$ and $s, t \in \mathbb{Z}$ with $0 \leq s, t < p-1$. We choose this basis to be the standard basis.

Lemma 4.7. *(i) The map Φ extends to a map $\bar{\mathcal{B}} \to \bar{\mathcal{B}}$ also denoted by Φ.*
(ii) For $q \in \bar{\mathbb{F}}((u))$ and $[x,y]_q \in \mathcal{A}_q$ the map Φ is given by

$$\Phi([x,y]_q) = [px + s - t, py + s + t]_{q'}$$

with $q' = b^{-1}u^{-t}(au^s\phi(q) + \gamma)$.

Proof. *(i)* We can use the expressions in *(ii)* to extend Φ.
(ii) We have $\Phi(u^m e_1) = au^{pm+s}e_1$ and

$$\begin{aligned}\Phi(u^n(qe_1 + e_2)) &= u^{pn}(au^s\phi(q)e_1 + \gamma e_1 + bu^t e_2) \\ &= bu^{pn+t}(b^{-1}u^{-t}(\gamma + au^s\phi(q))e_1 + e_2).\end{aligned}$$

The Lemma follows from this. □

Corollary 4.8. *The scheme $\mathcal{GR}^{\mathbf{v},\mathrm{loc}}_{V_{\mathbb{F}},0}$ is empty if $2e - d' \not\equiv s + t \mod (p-1)$.*

Proof. This follows from Lemma 2.4 and Lemma 4.7: We have

$$d_2([x,y]_q, \Phi([x,y]_q)) = (p-1)y + s + t$$

and this distance must be equal to $2e - d'$ if $[x,y]_q$ is $\mathbf{v}$-admissible. □

We assume that the scheme is non empty and define

$$P_{\mathrm{red}} = [\tfrac{t-s}{p-1}, -\tfrac{t+s}{p-1}] \in \mathcal{A}_0 \subset \bar{\mathcal{B}} \tag{4.0.6}$$

$$m(\mathbf{v}) = \tfrac{2e-d'-(s+t)}{p-1} \in \mathbb{Z}. \tag{4.0.7}$$

These definitions imply $\mathcal{GR}^{\mathbf{v},\mathrm{loc}}_{V_{\mathbb{F}},0}(\bar{\mathbb{F}}) \subset \bar{\mathcal{B}}(m(\mathbf{v}))$.
There are three different cases which we have to study in order to determine the set of $\mathbf{v}$-admissible lattices. It makes a difference whether $(M_{\bar{\mathbb{F}}}, \Phi)$ is a split or a non-split extension of two one-dimensional objects. In the split case there are two possibilities: Either the direct summands are isomorphic or non-isomorphic.

4.1. **The case $(M_{\bar{\mathbb{F}}}, \Phi) \cong (M_1, \Phi_1) \oplus (M_1, \Phi_1)$.** In this section we want to analyze the case where $(M_{\mathbb{F}}, \Phi)$ becomes isomorphic to a direct sum of two isomorphic one-dimensional objects after possibly extending the scalars to some finite extension of $\mathbb{F}$, i.e. we want to assume that there exists $\mathbb{F}'/\mathbb{F}$ and an $\mathbb{F}'((u))$-basis e_1, e_2 of $M_{\mathbb{F}'}$ such that

$$M_{\mathbb{F}'} \sim \begin{pmatrix} au^s & 0 \\ 0 & au^s \end{pmatrix} \tag{4.1.1}$$

with $a \in \mathbb{F}'^{\times}$ and $0 \leq s < p-1$. We immediately find $\Phi(P_{\mathrm{red}}) = P_{\mathrm{red}}$.
For each $z \in \mathbb{P}^1(\bar{\mathbb{F}})$ we define a (half)-line $\mathcal{L}_z \subset \bar{\mathcal{B}}(m(\mathbf{v}))$ by

$$\begin{aligned}\mathcal{L}_z &= \{[x, m(\mathbf{v})]_z \mid x \geq 0\} \subset \bar{\mathcal{B}}(m(\mathbf{v})) &&\text{if } z \in \bar{\mathbb{F}} = \mathbb{A}^1(\bar{\mathbb{F}}) \\ \mathcal{L}_\infty &= \{[x, m(\mathbf{v})]_0 \mid x \leq 0\} \subset \bar{\mathcal{B}}(m(\mathbf{v})).\end{aligned}$$

These lines are defined in such a way that

$$\mathcal{T} := \bigcup_{z\in\mathbb{P}^1(\bar{\mathbb{F}})} \mathcal{L}_z = \bigcup_{z\in\bar{\mathbb{F}}} \mathcal{A}_z \cap \bar{\mathcal{B}}(m(\mathbf{v})). \tag{4.1.2}$$

The apartments on the right hand side are given by the basis $e_1, ze_1 + e_2$ and in this basis the semi-linear endomorphism Φ is of the form (4.1.1).

Lemma 4.9. *Let $Q \in \bar{\mathcal{B}}(m(\mathbf{v}))$ be an arbitrary point. Let $Q' \in \mathcal{T}$ be the unique point satisfying $d_1(Q, Q') = d_1(Q, \mathcal{T})$. Then*

$$\begin{aligned} d_1(Q, \Phi(Q)) &= (p+1)d_1(Q, P_{\mathrm{red}}) - 2d_1(Q', P_{\mathrm{red}}) \\ d_2(Q, \Phi(Q)) &= (p-1)d_2(Q, P_{\mathrm{red}}). \end{aligned}$$

Proof. The statement on d_2 follows immediately from Lemma 4.7. For the statement on d_1 we assume $Q' \in \mathcal{L}_0$. The cases $Q' \in \mathcal{L}_z$ for $z \in \bar{\mathbb{F}}$ are analogous and the case $Q' \in \mathcal{L}_\infty$ is obtained by interchanging e_1 and e_2.
First assume $Q = Q'$, i.e. $Q = [x, m(\mathbf{v})]_0 \in \mathcal{L}_0$. Then Lemma 4.7 implies $\Phi(Q) = [px, pm(\mathbf{v}) + 2s]_0$ and hence $d_1(Q, \Phi(Q)) = (p-1)x = (p-1)d_1(Q, P_{\mathrm{red}})$.
Now assume $Q \neq Q'$. We write

$$Q = [x, m(\mathbf{v})]_q\ , \qquad Q' = [x', m(\mathbf{v})]_0$$

with $x > x' = v_u(q) \in \mathbb{Z}_{>0}$. Then $\Phi(Q) = [px, pm(\mathbf{v}) + 2s]_{\phi(q)}$ by Lemma 4.7. Using $v_u(\phi(q)) = px'$, we find

$$\begin{aligned} d_1(Q, \Phi(Q)) &= (x - x') + (px' - x') + (px - px') \\ &= (p+1)x - 2x' = (p+1)d_1(Q, P_{\mathrm{red}}) - 2d_1(Q', P_{\mathrm{red}}). \end{aligned}$$

□

Remark 4.10. This Lemma shows that the case of a direct sum of two isomorphic objects corresponds to the case B 2 in [PR2] 6.d:
The unique point fixed by Φ is the point P_{red} and the projection of this point to the building for $PGL_2(\bar{\mathbb{F}}((u)))$ is a vertex. The link of this vertex is the projection of $\mathcal{T}$ and all the half-lines $\mathcal{L}_z$ of $\mathcal{T}$ (for $z \in \mathbb{P}^1(\bar{\mathbb{F}})$) are fixed by Φ.

Proposition 4.11. *With the notations of Definition* 4.1 *and* (4.0.3), (4.0.7) *assume that*

$$(M_{\bar{\mathbb{F}}}, \Phi) \cong (\mathfrak{M}^s(a)[\tfrac{1}{u}], \Phi_a^s) \oplus (\mathfrak{M}^s(a)[\tfrac{1}{u}], \Phi_a^s)$$

for some $a \in \bar{\mathbb{F}}^\times$ and $0 \leq s < p-1$.
(i) The schemes $X^{\mathbf{v}}_{[M']}$ are empty for all $[M'] \in \mathcal{S}(\mathbf{v})\backslash\{[M_a]\}$.
(ii) The scheme $X^{\mathbf{v}}_{[M_a]}$ is given by

$$X^{\mathbf{v}}_{[M_a]} \cong \begin{cases} \emptyset & \text{if } m(\mathbf{v}) + \frac{r_1-r_2}{p-1} \notin 2\mathbb{Z} \\ \{*\} & \text{if } 0 = \frac{r_1-r_2}{p-1} \in \mathbb{Z} \text{ and } \frac{r_1-r_2}{p-1} \equiv m(\mathbf{v}) \mod 2 \\ \mathbb{P}^1_{\bar{\mathbb{F}}} & \text{if } 0 \neq \frac{r_1-r_2}{p-1} \in \mathbb{Z} \text{ and } \frac{r_1-r_2}{p-1} \equiv m(\mathbf{v}) \mod 2. \end{cases}$$

(iii) If non empty, the scheme $X^{\mathbf{v}}_0$ is connected.

Proof. We first claim that every $\mathbf{v}$-admissible lattice $\mathfrak{M}$ can be linked to a $\mathbf{v}$-admissible lattice $\mathfrak{M}' \in \mathcal{T}$ by a chain of $\mathbb{P}^1$.
Assume $\mathfrak{M} = [x, m(\mathbf{v})]_q \notin \mathcal{T}$ and let $Q' \in \mathcal{T}$ be the unique point satisfying $d_1(\mathfrak{M}, Q') = d_1(\mathfrak{M}, \mathcal{T})$. Without loss of generality, we may again assume that

$Q' = [x', m(\mathbf{v})]_0 \in \mathcal{L}_0$. By construction we have $\mathfrak{M}, Q' \in \mathcal{A}_q$ and we choose the following basis b_1, b_2 of $\mathfrak{M}$:

$$b_1 = u^{(x+m(\mathbf{v}))/2} e_1\ , \quad b_2 = u^{(m(\mathbf{v})-x)/2}(qe_1 + e_2).$$

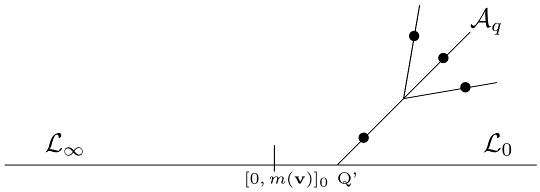

FIGURE 7. The fat points mark the image of the morphism $\bar{\chi}$ in the building in the case $p = 3$ and $\mathbb{F} = \mathbb{F}_3$.

Applying Lemma 3.7 with this basis yields a morphism $\bar{\chi} : \mathbb{P}^1_{\bar{\mathbb{F}}} \to \operatorname{Grass} M_{\bar{\mathbb{F}}}$ with $\bar{\chi}(z) = [x, m(\mathbf{v})]_{q+zu^{x-1}}$ for $z \in \bar{\mathbb{F}} = \mathbb{A}^1(\bar{\mathbb{F}})$ and $\bar{\chi}(\infty) = [x-2, m(\mathbf{v})]_q$. We have

$$d_1(\bar{\chi}(\infty), P_{\rm red}) < d_1(\bar{\chi}(z), P_{\rm red}) = d_1(\mathfrak{M}, P_{\rm red})$$

for all $z \in \bar{\mathbb{F}}$, while $d_1(\bar{\chi}(z), \mathcal{T}) \leq d_1(\mathfrak{M}, \mathcal{T})$ for all $z \in \mathbb{P}^1(\bar{\mathbb{F}})$ and, by construction, $d_1(\bar{\chi}(\infty), \mathcal{T}) < d_1(\mathfrak{M}, \mathcal{T})$.
By Lemma 4.9 and Lemma 2.4, the morphism $\bar{\chi}$ factors through $\mathcal{GR}^{\mathbf{v},\rm loc}_{V_{\mathbb{F}},0}$ and the claim follows by induction on the distance $d_1(\mathfrak{M}, \mathcal{T})$.
Now we assume that $\mathfrak{M} \in \mathcal{T}$ is a $\mathbf{v}$-admissible lattice and we are looking for a $\mathbf{v}$-admissible lattice $\mathfrak{M}'$ that can be linked with $\mathfrak{M}$ by a $\mathbb{P}^1$ and that has strictly smaller distance d_1 from $P_{\rm red} = [0, \frac{-2s}{p-1}]_0$ than $\mathfrak{M}$, i.e. $d_1(\mathfrak{M}', P_{\rm red}) < d_1(\mathfrak{M}, P_{\rm red})$. We may assume $\mathfrak{M} = [x, m(\mathbf{v})]_0 \in \mathcal{L}_0$. Assuming $x > 1$, our candidate for $\mathfrak{M}'$ is $[x-2, m(\mathbf{v})]_0$. Fixing a basis

$$b_1 = u^{(x+m(\mathbf{v}))/2} e_1\ ,\ b_2 = u^{(m(\mathbf{v})-x)/2} e_2$$

of $\mathfrak{M}$ so that $\mathfrak{M}' = \langle u^{-1}b_1, ub_2\rangle$, yields a morphism $\bar{\chi} : \mathbb{P}^1_{\bar{\mathbb{F}}} \to \operatorname{Grass} M_{\bar{\mathbb{F}}}$ with $\bar{\chi}(0) = \mathfrak{M}$ and $\bar{\chi}(\infty) = \mathfrak{M}'$. This morphism factors through $\mathcal{GR}^{\mathbf{v},\rm loc}_{V_{\mathbb{F}},0}$ iff the lattices $\bar{\chi}(z) = [x, m(\mathbf{v})]_{zu^{x-1}}$ are $\mathbf{v}$-admissible for all $z \in \bar{\mathbb{F}}\backslash\{0\}$.This is the case iff

$$\begin{aligned} d_1(\bar{\chi}(z), \Phi(\bar{\chi}(z))) &= (p+1)d_1(\bar{\chi}(z), P_{\rm red}) - 2d_1([x-1, m(\mathbf{v})]_0, P_{\rm red}) \\ &= (p+1)x - 2(x-1) = (p-1)x + 2 \leq r_1 - r_2. \end{aligned}$$

Consider the following subset of $\mathbf{v}$-admissible lattices

$$\mathcal{N} = \{\mathfrak{M} \in \mathcal{GR}^{\mathbf{v},\rm loc}_{V_{\mathbb{F}},0}(\bar{\mathbb{F}}) \mid \mathfrak{M} \notin \mathcal{T} \text{ or } (\mathfrak{M} \in \mathcal{T} \text{ and } d_1(\mathfrak{M}, P_{\rm red}) \leq \tfrac{r_1-r_2-2}{p-1})\}.$$

So far, we have shown that all $\mathbf{v}$-admissible lattices $\mathfrak{M} \in \mathcal{N}$ can either be linked to the lattice $[0, m(\mathbf{v})]_0$ or to one of the lattices

$$\{[1, m(\mathbf{v})]_z \mid z \in \bar{\mathbb{F}}\} \cup \{[-1, m(\mathbf{v})]_0\} = \{\mathfrak{M} \in \bar{\mathcal{B}}(m(\mathbf{v})) \mid d_1(\mathfrak{M}, P_{\rm red}) = 1\}. \tag{4.1.3}$$

by a chain of $\mathbb{P}^1$. Here, the two different cases depend on $m(\mathbf{v}) \mod 2$. Hence the subset of $\mathcal{GR}^{\mathbf{v},\rm loc}_{V_{\mathbb{F}},0} \otimes_{\mathbb{F}} \bar{\mathbb{F}}$ given by the lattices in $\mathcal{N}$ is connected: using Remark 3.8 again, the set in (4.1.3) forms a $\mathbb{P}^1$. The Proposition now follows from the following two facts:
(a) If $\mathfrak{M} \in \mathcal{N}$, then $\mathfrak{M}$ is not $\mathbf{v}$-ordinary, i.e. $\mathcal{N} \subset X^{\mathbf{v}}_0(\bar{\mathbb{F}})$.
(b) If $\mathfrak{M} \notin \mathcal{N}$ is $\mathbf{v}$-admissible, then $\mathfrak{M} \in X^{\mathbf{v}}_{[M_a]}(\bar{\mathbb{F}})$ and

$$\mathfrak{M} \in \{[\tfrac{r_1-r_2}{p-1}, m(\mathbf{v})]_z \mid z \in \bar{\mathbb{F}}\} \cup \{[-\tfrac{r_1-r_2}{p-1}, m(\mathbf{v})]_0\}. \tag{4.1.4}$$

By Remark 3.8, this set forms a $\mathbb{P}^1$ if $r_1 \neq r_2$. Otherwise it is a single point.
Proof of (a): If $\mathfrak{M} \in \mathcal{T}$, then $d_1(\mathfrak{M}, \langle\Phi(\mathfrak{M})\rangle) \leq r_1 - r_2 - 2 < r_1 - r_2$ and hence the elementary divisors of $\langle\Phi(\mathfrak{M})\rangle$ with respect to $\mathfrak{M}$ are not given by $(e - r_2, e - r_1)$. If $\mathfrak{M} \notin \mathcal{T}$, say $\mathfrak{M} = [x, m(\mathbf{v})]_q$ with $x > v_u(q) > 0$ for example, then $\mathfrak{M} = \langle b_1, b_2\rangle$ with

$$b_1 = u^{(x+m(\mathbf{v}))/2}e_1 \ , \qquad b_2 = u^{(m(\mathbf{v})-x)/2}(qe_1 + e_2)$$

and one finds

$$\mathfrak{M} \sim (a_{ij})_{ij} = \begin{pmatrix} au^{\frac{p-1}{2}(x+m(\mathbf{v}))+s} & a\phi(q)u^{\frac{p-1}{2}m(\mathbf{v})-\frac{p+1}{2}x+s} \\ 0 & au^{\frac{p-1}{2}(m(\mathbf{v})-x)+s} \end{pmatrix}$$

with $v_u(a_{12}) < v_u(a_{11})$, because $v_u(q) < x$, and hence the minimal elementary divisor of $\langle\Phi(\mathfrak{M})\rangle$ with respect to $\mathfrak{M}$ is not given by a Φ-stable subspace.
Proof of (b): Let $\mathfrak{M} \notin \mathcal{N}$ be $\mathbf{v}$-admissible. Then $\mathfrak{M} \in \mathcal{T}$ and

$$\tfrac{r_1-r_2-2}{p-1} < d_1(\mathfrak{M}, P_{\mathrm{red}}) \leq \tfrac{r_1-r_2}{p-1}.$$

We show that $d_1(\mathfrak{M}, P_{\mathrm{red}}) = \frac{r_1-r_2}{p-1}$ which implies (4.1.4).
Suppose that $\mathfrak{M} = [x, m(\mathbf{v})]_z$ with $z \in \bar{\mathbb{F}}$ and

$$x = \pm\tfrac{r_1-r_2-1}{p-1} \in \mathbb{Z} \ , \qquad m(\mathbf{v}) = \tfrac{2e-d'-2s}{p-1} = \tfrac{2e-r_1-r_2-2s}{p-1}.$$

In this case we find

$$x + m(\mathbf{v}) = \tfrac{2e-2s-(r_1+r_2)\pm(r_1-r_2)\mp 1}{p-1} \notin 2\mathbb{Z},$$

contradiction. We are left to show that $\mathfrak{M} \in X^{\mathbf{v}}_{[M_a]}(\bar{\mathbb{F}})$, i.e. that there exists a vector $e_{\mathfrak{M}} \in \mathfrak{M}$ and a Φ-stable subspace $\bar{\mathbb{F}}[\![u]\!]e_{\mathfrak{M}} \subset \mathfrak{M}$ with $\Phi(e_{\mathfrak{M}}) = au^{e-r_1}e_{\mathfrak{M}}$. An easy computation shows that we may choose

$$\begin{aligned} e_{\mathfrak{M}} &= u^{\frac{e-r_1-s}{p-1}}(ze_1 + e_2) && \text{if } \mathfrak{M} = [\tfrac{r_1-r_2}{p-1}, m(\mathbf{v})]_z,\ z \in \bar{\mathbb{F}} \\ e_{\mathfrak{M}} &= u^{\frac{e-r_1-s}{p-1}}e_1 && \text{if } \mathfrak{M} = [-\tfrac{r_1-r_2}{p-1}, m(\mathbf{v})]_0. \end{aligned}$$

□

We conclude the discussion by determining the cases where $\mathcal{GR}^{\mathbf{v},\mathrm{loc}}_{V_{\mathbb{F}},0}$ is reduced to a single point.

Corollary 4.12. *(i) If $m(\mathbf{v}) \equiv 0 \mod 2$, then $\mathcal{GR}^{\mathbf{v},\mathrm{loc}}_{V_{\mathbb{F}},0} = \{*\}$ iff $\frac{r_1-r_2}{p-1} < 2$.*
(ii) If $m(\mathbf{v}) \equiv 1 \mod 2$, then $\mathcal{GR}^{\mathbf{v},\mathrm{loc}}_{V_{\mathbb{F}},0}$ can not be a single point.

$$\begin{aligned} &\mathcal{GR}^{\mathbf{v},\mathrm{loc}}_{V_{\mathbb{F}},0} = \emptyset && \Leftrightarrow 0 \leq \tfrac{r_1-r_2}{p-1} < 1 \\ &\mathcal{GR}^{\mathbf{v},\mathrm{loc}}_{V_{\mathbb{F}},0} \otimes_{\mathbb{F}} \bar{\mathbb{F}} \cong \mathbb{P}^1_{\bar{\mathbb{F}}} && \Leftrightarrow 1 \leq \tfrac{r_1-r_2}{p-1} < 3. \end{aligned}$$

Proof. *(i)* As $m(\mathbf{v}) \equiv 0 \mod 2$, the lattice $[0, m(\mathbf{v})]_0$ is always $\mathbf{v}$-admissible. It is the unique point of $\mathcal{GR}^{\mathbf{v},\mathrm{loc}}_{V_{\mathbb{F}},0}$ if the lattices $\mathfrak{M}$ with $d_1(\mathfrak{M}, P_{\mathrm{red}}) = 2$ are not $\mathbf{v}$-admissible. By Lemma 4.9 this is the case iff $\frac{r_1-r_2}{p-1} < 2$.
(ii) The scheme is empty if the lattices $\mathfrak{M}$ with $d_1(\mathfrak{M}, P_{\mathrm{red}}) = 1$ are not $\mathbf{v}$-admissible. By Lemma 4.9, this is the case iff $\frac{r_1-r_2}{p-1} < 1$.
If $\frac{r_1-r_2}{p-1} \geq 1$, then the lattices

$$\{[1, 0]_z \mid z \in \bar{\mathbb{F}}\} \cup \{[-1, 0]_0\}$$

are $\mathbf{v}$-admissible and form a $\mathbb{P}^1_{\bar{\mathbb{F}}}$. Again by Lemma 4.9 there are no other $\mathbf{v}$-admissible lattices iff $\frac{r_1-r_2}{p-1} < 3$. $\square$

4.2. **The case** $(M_{\bar{\mathbb{F}}}, \Phi) \cong (M_1, \Phi_1) \oplus (M_2, \Phi_2)$**.** In this section we treat the case where $(M_{\mathbb{F}}, \Phi)$ becomes isomorphic to the direct sum of two non-isomorphic one-dimensional objects after extending the scalars to some finite extension. The situation is the following: There exists a finite extension $\mathbb{F}'$ of $\mathbb{F}$ and a basis e_1, e_2 of $M_{\mathbb{F}'}$ such that

$$M_{\mathbb{F}'} \sim \begin{pmatrix} au^s & 0 \\ 0 & bu^t \end{pmatrix}$$

with $a, b \in \mathbb{F}'^{\times}$ and $0 \leq s, t < p-1$. As we are assuming that the direct summands are not isomorphic, we further have $s \neq t$ or $a \neq b$. Again we find $\Phi(P_{\mathrm{red}}) = P_{\mathrm{red}}$.

Lemma 4.13. *Let $Q \in \bar{\mathcal{B}}(m(\mathbf{v}))$ be an arbitrary point. Let $Q' \in \mathcal{A}_0 \cap \bar{\mathcal{B}}(m(\mathbf{v}))$ be the unique point satisfying $d_1(Q, Q') = d_1(Q, \mathcal{A}_0)$. Then*

$$d_1(Q, \Phi(Q)) = (p+1)d_1(Q, P_{\mathrm{red}}) - 2d_1(Q', P_{\mathrm{red}})$$
$$d_2(Q, \Phi(Q)) = (p-1)d_2(Q, P_{\mathrm{red}}).$$

Proof. This is similar to Lemma 4.9. Again the statement on d_2 is an immediate consequence of Lemma 4.7. Let Q be any point. We may assume that the unique point $Q' \in \mathcal{A}_0 \cap \bar{\mathcal{B}}(m(\mathbf{v}))$ satisfying $d_1(Q, Q') = d_1(Q, \mathcal{A}_0)$ is given by $[x, m(\mathbf{v})]_0$ with $x \geq \frac{t-s}{p-1}$. The case $x \leq \frac{t-s}{p-1}$ is obtained by interchanging e_1 and e_2.
If $Q = Q'$, then $Q \in \mathcal{A}_0$ and the statement is a consequence of Lemma 4.7.
Assume $Q \neq Q'$ and put

$$Q = [x, m(\mathbf{v})]_q\ , \qquad Q' = [x', m(\mathbf{v})]_0$$

with $x > x' = v_u(q) \geq \frac{t-s}{p-1}$. Now $\Phi(Q) = [px + s - t, pm(\mathbf{v}) + s + t]_{q'}$ with $q' = ab^{-1}u^{-(t-s)}\phi(q)$.
If $s \neq t$, then $x' = v_u(q) > \frac{t-s}{p-1}$ or equivalently $x' = v_u(q) < v_u(q') = pv_u(q) - (t-s)$, and we find

$$\begin{aligned} d_1(Q, \Phi(Q)) &= (x - x') + (px' - (t-s) - x') + (px + s - t - (px' + s - t)) \\ &= (p+1)(x - \tfrac{t-s}{p-1}) - 2(x' - \tfrac{t-s}{p-1}) \\ &= (p+1)d_1(Q, P_{\mathrm{red}}) - 2d_1(Q', P_{\mathrm{red}}). \end{aligned}$$

If $s = t$, then $a \neq b$. We find $v_u(q) \neq v_u(q')$ if $v_u(q) \neq 0$ and in this case the computation is the same as above.
If $q = a_0 + a_1 u + \dots$ with $a_0 \neq 0$, then $q' = ab^{-1}a_0 + \dots$ and hence the absolute coefficient of q is different from the absolute coefficient of q'. The geodesic between Q and the projection of $\Phi(Q)$ to $\bar{\mathcal{B}}(m(\mathbf{v}))$ contains the point $Q' = [0, m(\mathbf{v})]_0 = [\frac{t-s}{p-1}, m(\mathbf{v})]_0$. Hence

$$\begin{aligned} d_1(Q, \Phi(Q)) &= x + (px + s - t) = (p+1)x \\ &= (p+1)d_1(Q, P_{\mathrm{red}}) - 2d_1(Q', P_{\mathrm{red}}). \end{aligned}$$

$\square$

Remark 4.14. Again, this Lemma shows the connection to [PR2] 6.d. The point fixed by Φ is again the point P_{red}.
If $s = t$, then we are in the case B 2 of loc. cit.: The projection of the fixed point to the building for $PGL_2(\bar{\mathbb{F}}((u)))$ is a vertex. Exactly two of the half-lines of the

link of this vertex are fixed by Φ.
If $s \neq t$ we are in the case A 2 of loc. cit.: The projection of the fixed point P_{red} is not a vertex but it lies on an edge and the projections of the two half-lines $\{[x, m(\mathbf{v})]_0 \mid x \leq \frac{t-s}{p-1}\}$ and $\{[x, m(\mathbf{v})]_0 \mid x \geq \frac{t-s}{p-1}\}$ to the building for $PGL_2(\bar{\mathbb{F}}((u)))$ are fixed by Φ.

Proposition 4.15. *With the notations of Definition* 4.1 *and* (4.0.3), (4.0.7) *assume that*

$$(M_{\bar{\mathbb{F}}}, \Phi) \cong (\mathfrak{M}^s(a)[\tfrac{1}{u}], \Phi^s_a) \oplus (\mathfrak{M}^t(b)[\tfrac{1}{u}], \Phi^t_b)$$

with $a, b \in \bar{\mathbb{F}}^\times$ *and* $0 \leq s, t < p-1$. *Further assume* $a \neq b$ *or* $s \neq t$.
(i) The schemes $X^{\mathbf{v}}_{[M']}$ *are empty for all* $[M'] \in \mathcal{S}(\mathbf{v}) \backslash \{[M_a], [M_b]\}$.
(ii) If $s = t$, *then*

$$X^{\mathbf{v}}_{[M_a]} \cong X^{\mathbf{v}}_{[M_b]} = \begin{cases} \emptyset & \text{if } m(\mathbf{v}) + \frac{r_1 - r_2}{p-1} \notin 2\mathbb{Z} \\ \{*\} & \text{if } m(\mathbf{v}) + \frac{r_1 - r_2}{p-1} \in 2\mathbb{Z}, \end{cases}$$

further $X^{\mathbf{v}}_{[M_a]} = X^{\mathbf{v}}_{[M_b]}$ *iff* $r_1 = r_2$.
(iii) If $s \neq t$, *then*

$$X^{\mathbf{v}}_{[M_a]} = \begin{cases} \emptyset & \text{if } \frac{t-s}{p-1} - \frac{r_1 - r_2}{p-1} + m(\mathbf{v}) \notin 2\mathbb{Z} \\ \{*\} & \text{if } \frac{t-s}{p-1} - \frac{r_1 - r_2}{p-1} + m(\mathbf{v}) \in 2\mathbb{Z} \end{cases}$$

$$X^{\mathbf{v}}_{[M_b]} = \begin{cases} \emptyset & \text{if } \frac{t-s}{p-1} + \frac{r_1 - r_2}{p-1} + m(\mathbf{v}) \notin 2\mathbb{Z} \\ \{*\} & \text{if } \frac{t-s}{p-1} + \frac{r_1 - r_2}{p-1} + m(\mathbf{v}) \in 2\mathbb{Z}. \end{cases}$$

(iv) If non empty the scheme $X^{\mathbf{v}}_0$ *is connected.*

Proof. Again, this is similar to the proof of Proposition 4.11.
First, we link any $\mathbf{v}$-admissible lattice to a $\mathbf{v}$-admissible lattice in $\mathcal{A}_0$ by a chain of $\mathbb{P}^1$.
Let $\mathfrak{M}$ be any $\mathbf{v}$-admissible lattice and let $Q' \in \mathcal{A}_0 \cap \bar{\mathcal{B}}(m(\mathbf{v}))$ be the unique point with $d_1(Q, Q') = d_1(Q, \mathcal{A}_0)$. Again, we may assume without loss of generality $Q' = [x', m(\mathbf{v})]_0$ with $x' \geq \frac{t-s}{p-1}$. Completely analogous to the proof of Proposition 4.11, we find a morphism

$$\bar{\chi} : \mathbb{P}^1_{\bar{\mathbb{F}}} \to \mathcal{GR}^{\mathbf{v},\mathrm{loc}}_{V_{\mathbb{F}},0}$$

such that $\bar{\chi}(0) = \mathfrak{M}$ and $d_1(\bar{\chi}(\infty), \mathcal{A}_0) < d_1(\mathfrak{M}, \mathcal{A}_0)$. By induction on the distance $d_1(\mathfrak{M}, \mathcal{A}_0)$, we find that we can link any $\mathbf{v}$-admissible lattice to a $\mathbf{v}$-admissible lattice in $\mathcal{A}_0$.
Now assume $\mathfrak{M} = [x, m(\mathbf{v})]_0 \in \mathcal{A}_0$.
If $x > \frac{t-s}{p-1}$, we find a map $\mathbb{P}^1 \to \mathcal{GR}^{\mathbf{v},\mathrm{loc}}_{V_{\mathbb{F}},0}$, as in the proof of Proposition 4.11, whose image contains $[x, m(\mathbf{v})]_0$ and $[x-2, m(\mathbf{v})]_0$, if $(p-1)(x - \frac{t-s}{p-1}) \leq r_1 - r_2 - 2$.
If $x < \frac{t-s}{p-1}$, we find a map $\mathbb{P}^1 \to \mathcal{GR}^{\mathbf{v},\mathrm{loc}}_{V_{\mathbb{F}},0}$ whose image contains $[x, m(\mathbf{v})]_0$ and $[x+2, m(\mathbf{v})]_0$, if $(p-1)(\frac{t-s}{p-1} - x) \leq r_1 - r_2 - 2$.
Similarly as in Proposition 4.11, one can proof the following two facts:
(a) The set

$$\mathcal{N} = \{\mathfrak{M} \in \mathcal{GR}^{\mathbf{v},\mathrm{loc}}_{V_{\mathbb{F}},0}(\bar{\mathbb{F}}) \mid \mathfrak{M} \notin \mathcal{A}_0 \text{ or } (\mathfrak{M} \in \mathcal{A}_0 \text{ and } d_1(\mathfrak{M}, P_{\mathrm{red}}) \leq \tfrac{r_1 - r_2 - 2}{p-1})\}$$

consists of non-$\mathbf{v}$-ordinary lattices.
(b) If $\mathfrak{M} \in \mathcal{A}_0$ is $\mathbf{v}$-admissible, then

$$d_1(\mathfrak{M}, P_{\mathrm{red}}) > \tfrac{r_1 - r_2 - 2}{p-1} \Rightarrow d_1(\mathfrak{M}, P_{\mathrm{red}}) = \tfrac{r_1 - r_2}{p-1}.$$

Now $\mathcal{N}$ defines a connected subset of $X_0^{\mathbf{v}}$:
If $s \neq t$, then there is a unique lattice with minimal distance from $[\frac{t-s}{p-1}, m(\mathbf{v})]_0$ and every $\mathbf{v}$-admissible lattice in $\mathcal{N}$ can be linked to this lattice by a chain of $\mathbb{P}^1$.
If $s = t$, then either $[\frac{t-s}{p-1}, m(\mathbf{v})]_0$ is a lattice itself and any $\mathbf{v}$-admissible lattice can be linked to this lattice by a chain of $\mathbb{P}^1$, or there are two $\mathbf{v}$-admissible lattices $[\pm 1, m(\mathbf{v})]_0$ in $\mathcal{N}$ with distance 1 from $[0, m(\mathbf{v})]_0$ and by the above there is a morphism

$$\mathbb{P}^1 \to \mathcal{GR}^{\mathbf{v},\mathrm{loc}}_{V_{\mathbb{F}},0}$$

containing both lattices in its image. Thus $\mathcal{N}$ defines a connected subset.
Consider the following points:

$$\begin{aligned} Q_+ &= [\tfrac{t-s}{p-1} + \tfrac{r_1-r_2}{p-1}, m(\mathbf{v})]_0 \\ Q_- &= [\tfrac{t-s}{p-1} - \tfrac{r_1-r_2}{p-1}, m(\mathbf{v})]_0. \end{aligned}$$

We are left to show that, if one of these points defines a lattice $\mathfrak{M}_+ = Q_+$ (resp. $\mathfrak{M}_- = Q_-$), then this point lies in $X^{\mathbf{v}}_{[M_b]}$ (resp $X^{\mathbf{v}}_{[M_a]}$).
If $\frac{t-s}{p-1} - \frac{r_1-r_2}{p-1} + m(\mathbf{v}) \in 2\mathbb{Z}$, then $Q_- = \mathfrak{M}_-$ is a lattice and

$$e_{\mathfrak{M}_-} = u^{(e-r_1-s)/(p-1)} e_1$$

defines a Φ-stable subspace satisfying $\Phi(e_{\mathfrak{M}_-}) = a u^{e-r_1} e_{\mathfrak{M}_-}$, i.e. $\mathfrak{M}_- \in X^{\mathbf{v}}_{[M_a]}$.
If $\frac{t-s}{p-1} + \frac{r_1-r_2}{p-1} + m(\mathbf{v}) \in 2\mathbb{Z}$, then $Q_+ = \mathfrak{M}_+$ is a lattice and

$$e_{\mathfrak{M}_+} = u^{(e-r_1-t)/(p-1)} e_2$$

defines a Φ-stable subspace satisfying $\Phi(e_{\mathfrak{M}_+}) = b u^{e-r_1} e_{\mathfrak{M}_+}$, i.e. $\mathfrak{M}_+ \in X^{\mathbf{v}}_{[M_b]}$.
We have two different cases:
If $s = t$ and $\frac{r_1-r_2}{p-1} + m(\mathbf{v}) \in 2\mathbb{Z}$, then the lattices $\mathfrak{M}_+$ and $\mathfrak{M}_-$ define points $\mathfrak{M}_- \in X^{\mathbf{v}}_{[M_a]}$ and $\mathfrak{M}_+ \in X^{\mathbf{v}}_{[M_b]}$ which coincide iff $r_1 = r_2$.
If $s \neq t$, then

$$\begin{aligned} &Q_- \text{ defines an isolated point in } X^{\mathbf{v}}_{[M_a]} \Leftrightarrow \tfrac{t-s-(r_1-r_2)}{p-1} + m(\mathbf{v}) \in 2\mathbb{Z} \\ &Q_+ \text{ defines an isolated point in } X^{\mathbf{v}}_{[M_b]} \Leftrightarrow \tfrac{t-s+(r_1-r_2)}{p-1} + m(\mathbf{v}) \in 2\mathbb{Z}. \end{aligned}$$

This cannot happen at the same time, as $\frac{t-s}{p-1} \notin \mathbb{Z}$. This finishes the proof of the Proposition. □

Corollary 4.16. *(i) Assume $s = t$.*
(a) If $m(\mathbf{v}) \equiv 0 \mod 2$, then $\mathcal{GR}^{\mathbf{v},\mathrm{loc}}_{V_{\mathbb{F}},0} = \{*\}$ iff $\frac{r_1-r_2}{p-1} < 2$.
(b) If $m(\mathbf{v}) \equiv 1 \mod 2$, then $\mathcal{GR}^{\mathbf{v},\mathrm{loc}}_{V_{\mathbb{F}},0}$ cannot be a single point.

$$\begin{aligned} \mathcal{GR}^{\mathbf{v},\mathrm{loc}}_{V_{\mathbb{F}},0} &= \emptyset && \Leftrightarrow 0 \le \tfrac{r_1-r_2}{p-1} < 1 \\ \mathcal{GR}^{\mathbf{v},\mathrm{loc}}_{V_{\mathbb{F}},0} &= \{*\} \cup \{*\} && \Leftrightarrow 1 \le \tfrac{r_1-r_2}{p-1} < 3. \end{aligned}$$

(ii) Assume $s \neq t$.
Define $x_0 = \lfloor \frac{t-s}{p-1} \rfloor$ and write $\xi = \frac{t-s}{p-1} - x_0$ for the fractional part of $\frac{t-s}{p-1}$.

$$\mathcal{GR}^{\mathbf{v},\mathrm{loc}}_{V_{\mathbb{F}},0} = \{*\} \Leftrightarrow \begin{cases} 0 + \xi \le \frac{r_1-r_2}{p-1} < 2 - \xi & \text{if } m(\mathbf{v}) \equiv x_0 \mod 2 \\ 1 - \xi \le \frac{r_1-r_2}{p-1} < 1 + \xi & \text{if } m(\mathbf{v}) \not\equiv x_0 \mod 2. \end{cases}$$

Proof. *(i)* This is nearly identical to Corollary 4.12.
(ii) Assume $m(\mathbf{v}) \equiv x_0 \mod 2$. Then $[x_0, m(\mathbf{v})]_0$ is the unique lattice with minimal distance d_1 from $P_{\rm red}$. By Lemma 4.13 it is $\mathbf{v}$-admissible iff $\frac{r_1-r_2}{p-1} \geq \xi$.
Again by Lemma 4.13 it is the only $\mathbf{v}$-admissible lattice iff $[x_0+2, m(\mathbf{v})]_0$ is not $\mathbf{v}$-admissible. This is the case iff $\frac{r_1-r_2}{p-1} < 2-\xi$.
The case $m(\mathbf{v}) \not\equiv x_0 \mod 2$ is similar. □

4.3. **The case of a non split extension.** Finally, we analyze the case where $(M_{\bar{\mathbb{F}}}, \Phi)$ is a non split extension of two one dimensional objects. There is a basis e_1, e_2 such that

$$M_{\bar{\mathbb{F}}} \sim \begin{pmatrix} au^s & \gamma \\ 0 & bu^t \end{pmatrix}$$

with $0 \leq s, t < p-1$ and $a, b \in \bar{\mathbb{F}}^\times$, $\gamma \in \bar{\mathbb{F}}((u))$. In any basis of the form e_1, qe_1+e_2 defining the apartment $\mathcal{A}_q$, the endomorphism Φ is upper triangular with diagonal entries au^s and bu^t, and we fix the basis such that the valuation of the upper right entry $k := v_u(\gamma)$ is maximal.

Lemma 4.17. *(i) The integer $k = v_u(\gamma)$ satisfies*

$$k \leq \tfrac{pt-s}{p-1}.$$

(ii) If $\mathfrak{M} = [x, y]_q$ with $\min\{x, v_u(q)\} \geq \frac{k-s}{p}$, then

$$\begin{aligned} d_1(\mathfrak{M}, \langle\Phi(\mathfrak{M})\rangle) &= (p+1)x + s + t - 2k \\ d_2(\mathfrak{M}, \langle\Phi(\mathfrak{M})\rangle) &= (p-1)d_2(\mathfrak{M}, P_{\rm red}). \end{aligned}$$

(iii) If $\mathfrak{M} = [x, y]_q$ with $x < \frac{k-s}{p}$ or $v_u(q) < \frac{k-s}{p}$, let $Q' \in \mathcal{A}_0 \cap \bar{\mathcal{B}}(y)$ be the unique point such that $d_1(\mathfrak{M}, Q') = d_1(\mathfrak{M}, \mathcal{A}_0)$. Then

$$\begin{aligned} d_1(\mathfrak{M}, \langle\Phi(\mathfrak{M})\rangle) &= (p+1)d_1(\mathfrak{M}, P_{\rm red}) - 2d_1(Q', P_{\rm red}) \\ d_2(\mathfrak{M}, \langle\Phi(\mathfrak{M})\rangle) &= (p-1)d_2(\mathfrak{M}, P_{\rm red}). \end{aligned}$$

Proof. *(i)* This follows from the maximality of $k = v_u(\gamma)$: We have

$$\Phi(qe_1+e_2) = (\gamma + au^s\phi(q) - bu^tq)e_1 + bu^t(qe_1+e_2). \tag{4.3.1}$$

And

$$v_u(au^s\phi(q) - bu^tq) = \begin{cases} v_u(q)+t & \text{if } v_u(q) > \frac{t-s}{p-1} \\ pv_u(q)+s & \text{if } v_u(q) < \frac{t-s}{p-1}. \end{cases}$$

If we had $k = v_u(\gamma) = v_u(q)+t$ for any q with $v_u(q) > \frac{t-s}{p-1}$, we could delete the leading coefficient of γ in (4.3.1) which contradicts the maximality of $v_u(\gamma)$. Hence we have $k < v_u(q)+t$ for all q with $v_u(q) > \frac{t-s}{p-1}$ which yields the first claim.
(ii) The first part of the lemma implies $\frac{k-s}{p} \geq k-t$ and hence our assumptions on $v_u(q)$ imply $k \leq \min\{v_u(q)+t, pv_u(q)+s\}$. We find $v_u(\gamma + au^s\phi(q) - bu^tq) = v_u(\gamma)$ and we may assume $q = 0$, i.e. $\mathfrak{M} \in \mathcal{A}_0$, as the situation is the same as in the standard apartment. Now we have $\langle\Phi(\mathfrak{M})\rangle = [px+s-t, py+s+t]_{b^{-1}u^{-t}\gamma}$, and $x \geq \frac{k-s}{p}$ implies

$$px+s-t \geq k-t\ , \qquad x \geq k-t.$$

Thus $d_1(\mathfrak{M}, \langle\Phi(\mathfrak{M})\rangle) = (px + s - t - (k - t)) + (x - (k - t)) = (p + 1)x + s + t - 2k$. The statement on d_2 is easy.
(iii) If $\mathfrak{M} \notin \mathcal{A}_0$, then $v_u(q) < \frac{k-s}{p} \leq \frac{t-s}{p-1}$ and hence

$$v_u(\gamma + au^s\phi(q) - bu^t q) = v_u(au^s\phi(q) - bu^t q)$$

and the situation is the same as in the split case, i.e. the case $\gamma = 0$. If $\mathfrak{M} \in \mathcal{A}_0$, then $\langle\Phi(\mathfrak{M})\rangle \in \mathcal{A}_0$ and the statement is easy. □

Remark 4.18. In the case of a non split extension we are in the case B 2 or A 3 of [PR2] 6.d. More precisely, if $\frac{k-s}{p} \notin \mathbb{Z}$, then the unique fixed point of ([PR2], Prop. 6.1) is not in the building $\bar{\mathcal{B}}$. It is only visible after extending $\bar{\mathbb{F}}((u))$ to some separable wildly ramified extension (the apartment containing the fixed point will branch off from $\mathcal{A}_0$ at the line $x = \frac{k-s}{p}$, because we can successively delete the leading coefficient of γ in (4.3.1) if there is some q with $v_u(q) = \frac{k-s}{p}$). The image of the half line $\{[x, m(\mathbf{v})]_0 \mid x \leq \frac{k-s}{p}\}$ in the building for $PGL_2(\bar{\mathbb{F}}((u)))$ is stable under Φ and the geodesic between $[\lfloor\frac{k-s}{p}\rfloor + 1, m(\mathbf{v})]_0$ and its image under Φ contains the (projection of the) point $[x_0, m(\mathbf{v})]_0$ in the building for $PGL_2(\bar{\mathbb{F}}((u)))$. This is the case A 3 of [PR2] 6.d.
If $\frac{k-s}{p} \in \mathbb{Z}$, then we are in the case B 2 of [PR2] 6.d.: In this case the maximality of $k = v_u(q)$ implies $k - t = \frac{k-s}{p} = \frac{t-s}{p-1} = 0$ (otherwise we could delete the leading coefficient of γ) and we find that P_{red} is the fixed point in the building. In this case there is a unique half-line in the apartment for $PGL_2(\bar{\mathbb{F}}((u)))$ that is fixed by Φ, namely the image of the half-line $\{[x, m(\mathbf{v})]_0 \mid x \leq 0\}$ under the projection.

Proposition 4.19. *With the notations of Definition* 4.1 *and* (4.0.3), (4.0.7)*, assume that* $(M_{\bar{\mathbb{F}}}, \Phi)$ *is a non split extension*

$$0 \to (\mathfrak{M}^s(a)[\tfrac{1}{u}], \Phi_a^s) \to (M_{\bar{\mathbb{F}}}, \Phi) \to (\mathfrak{M}^t(b)[\tfrac{1}{u}], \Phi_b^t) \to 0$$

for some $a, b \in \bar{\mathbb{F}}^\times$ *and* $0 \leq s, t < p - 1$.
(i) The schemes $X^{\mathbf{v}}_{[M']}$ *are empty for all* $[M'] \in \mathcal{S}(\mathbf{v})\backslash\{[M_a]\}$.
(ii) For $X^{\mathbf{v}}_{[M_a]}$ *the following holds:*

$$X^{\mathbf{v}}_{[M_a]} = \begin{cases} \emptyset & \textit{if } \frac{t-s}{p-1} - \frac{r_1-r_2}{p-1} + m(\mathbf{v}) \notin 2\mathbb{Z} \\ \{*\} & \textit{if } \frac{t-s}{p-1} - \frac{r_1-r_2}{p-1} + m(\mathbf{v}) \in 2\mathbb{Z}. \end{cases}$$

(iii) If non empty, the scheme $X^{\mathbf{v}}_0$ *is connected.*

Proof. Lemma 4.17 *(i)* implies $\frac{k-s}{p} \leq \frac{t-s}{p-1}$ and an easy computation using the same inequality shows that

$$\tfrac{t-s}{p-1} - \tfrac{r_1-r_2}{p-1} \leq \tfrac{k-s}{p} \Leftrightarrow \tfrac{k-s}{p} \leq \tfrac{1}{p+1}(r_1 - r_2 - s - t + 2k),$$

and hence

$$\tfrac{t-s}{p-1} - \tfrac{r_1-r_2}{p-1} \leq \tfrac{k-s}{p} \leq \tfrac{1}{p+1}(r_1 - r_2 - s - t + 2k),$$

if $\mathcal{GR}^{\mathbf{v},\mathrm{loc}}_{V_{\mathbb{F}},0} \neq \emptyset$. Further denote by $\widetilde{\mathcal{N}}$ the set of $\mathbf{v}$-admissible lattices $\mathfrak{M} = [x, m(\mathbf{v})]_q$ with $\frac{k-s}{p} \leq \min\{x, v_u(q)\}$.

As we have seen above, the situation for the $\mathbf{v}$-admissible lattices $\mathfrak{M} \notin \widetilde{\mathcal{N}}$ is the same as in the split case. Hence we can link all $\mathbf{v}$-admissible lattices to $\mathbf{v}$-admissible lattices in $\mathcal{A}_0$ by a chain of $\mathbb{P}^1$. If $\mathfrak{M} = [x, m(\mathbf{v})]_0$ is a $\mathbf{v}$-admissible lattice in $\mathcal{A}_0$

with $x+2<\frac{k-s}{p}$, then there is a $\mathbb{P}^1$ in $\mathcal{GR}^{\mathbf{v},\mathrm{loc}}_{V_{\mathbb{F}},0}$ containing $\mathfrak{M}=[x,m(\mathbf{v})]_0$ and $[x+2,m(\mathbf{v})]_0$, except if $\mathfrak{M}=\mathfrak{M}_-=[\frac{t-s-(r_1-r_2)}{p-1},m(\mathbf{v})]_0$ which defines an isolated point in $X^{\mathbf{v}}_{[M_a]}$ if $\frac{t-s}{p-1}-\frac{r_1-r_2}{p-1}+m(\mathbf{v})\in 2\mathbb{Z}$ (compare Proposition 4.15).
Let $\mathfrak{M}'=[x_0,m(\mathbf{v})]_0$ be the lattice where x_0 is the maximal integer smaller than $\frac{k-s}{p}$ that is congruent to $m(\mathbf{v}) \mod 2$. We claim:
(a) If $\mathfrak{M}'$ is $\mathbf{v}$-admissible and $x_0\neq\frac{t-s}{p-1}-\frac{r_1-r_2}{p-1}$, then any lattice in $\widetilde{\mathcal{N}}$ can be linked to $\mathfrak{M}'$ by a chain of $\mathbb{P}^1$.
(b) The lattices $\mathfrak{M}\in\widetilde{\mathcal{N}}$ are non-$\mathbf{v}$-ordinary.
This finishes the proof of the proposition.
Proof of (a): Let $\mathfrak{M}=[x,m(\mathbf{v})]_q\in\widetilde{\mathcal{N}}$ be a lattice. Without loss of generality, we may assume $\mathfrak{M}\in\mathcal{A}_0$, as the situation is the same in all apartments $\mathcal{A}_q$ with $v_u(q)\geq\frac{k-s}{p}$. By Lemma 4.17, we have

$$\tfrac{k-s}{p}\leq x\leq\tfrac{1}{p+1}(r_1-r_2-s-t+2k).$$

We consider the basis

$$b_1=u^{(x+m(\mathbf{v}))/2}e_1\ ,\ b_2=u^{(m(\mathbf{v})-x)/2}e_2$$

of $\mathfrak{M}$ and by Lemma 3.7, there is a morphism

$$\bar{\chi}:\mathbb{P}^1_{\bar{\mathbb{F}}}\to\operatorname{Grass}M_{\bar{\mathbb{F}}}$$

with $\bar{\chi}(z)=[x,m(\mathbf{v})]_{zu^{x-1}}$ for $z\in\bar{\mathbb{F}}$ and $\bar{\chi}(\infty)=[x-2,m(\mathbf{v})]_0$.
If $x-1\geq\frac{k-s}{p}$, then the morphism factors over $\mathcal{GR}^{\mathbf{v},\mathrm{loc}}_{V_{\mathbb{F}},0}$. Consider the following two cases:
If $\frac{k-s}{p}\leq x_0+1$, then this argument shows that we can link all $\mathfrak{M}\in\widetilde{\mathcal{N}}$ to the lattice $[x_0,m(\mathbf{v})]_0$ by a chain of $\mathbb{P}^1$.
If $\frac{k-s}{p}>x_0+1$, then this argument shows that we can link all $\mathfrak{M}\in\widetilde{\mathcal{N}}$ to the lattice $\mathfrak{M}''=[x_0+2,m(\mathbf{v})]_0$ by a chain of $\mathbb{P}^1$. We can link the lattice $\mathfrak{M}''$ to the lattice $\mathfrak{M}'=[x_0,m(\mathbf{v})]_0$ if the lattices $\mathfrak{M}_z=[x_0,m(\mathbf{v})]_{zu^{x-1}}$ are $\mathbf{v}$-admissible for all $z\in\bar{\mathbb{F}}$. For $z\neq 0$ we have

$$d_1(\mathfrak{M}_z,\langle\Phi(\mathfrak{M}_z)\rangle)=(p+1)d_1(\mathfrak{M}_z,P_{\mathrm{red}})-2d_1(Q',P_{\mathrm{red}}),$$

where $Q'=[x_0+1,m(\mathbf{v})]_0$ is the unique point in $\mathcal{A}_0$ with minimal distance from $\mathfrak{M}_z$. Hence $d_1(\mathfrak{M}_z,\langle\Phi(\mathfrak{M}_z)\rangle)=d_1(\mathfrak{M}',\langle\Phi(\mathfrak{M}')\rangle)+2$ and the morphism factors through $\mathcal{GR}^{\mathbf{v},\mathrm{loc}}_{V_{\mathbb{F}},0}$ as $x_0\neq\frac{t-s}{p-1}-\frac{r_1-r_2}{p-1}$. (Otherwise $\mathfrak{M}'$ is the unique isolated point in $X^{\mathbf{v}}_{[M_a]}$).
Proof of (b): Let $\mathfrak{M}\in\widetilde{\mathcal{N}}$ be a lattice. Similarly to the proof of Proposition 4.11, we find

$$\mathfrak{M}\sim\begin{pmatrix}a_{11}&a_{12}\\0&a_{22}\end{pmatrix}$$

with $v_u(a_{12})<v_u(a_{11})$ and hence the minimal elementary divisor of $\langle\Phi(\mathfrak{M})\rangle$ with respect to $\mathfrak{M}$ is not defined by a Φ-stable subspace. $\square$

Summarizing the results on the connected components we find the following Theorem.

Theorem 4.20. *Assume that $(M_{\mathbb{F}},\Phi)$ becomes reducible after extending the scalars to some finite extension $\mathbb{F}'$ of $\mathbb{F}$.*
(i) The subschemes $X^{\mathbf{v}}_0$ and $X^{\mathbf{v}}_{[M']}$ are open and closed in $\mathcal{GR}^{\mathbf{v},\mathrm{loc}}_{V_{\mathbb{F}},0}\otimes_{\mathbb{F}}\bar{\mathbb{F}}$ for all

isomorphism classes $[M'] \in \mathcal{S}(\mathbf{v})$.
(ii) If non empty, the scheme $X_0^{\mathbf{v}}$ *is connected.*
(iii) For each $[M'] \in \mathcal{S}(\mathbf{v})$ *the scheme* $X_{[M']}^{\mathbf{v}}$ *is either empty, a single point or isomorphic to* $\mathbb{P}^1_{\bar{\mathbb{F}}}$.
(iv) There are at most two isomorphism classes $[M'] \in \mathcal{S}(\mathbf{v})$ *such that* $X_{[M']}^{\mathbf{v}} \neq \emptyset$.

Proof. This is a summary of the Propositions 4.11, 4.15 and 4.19. □

This theorem implies a modified version of the conjecture of Kisin stated in ([Ki], 2.4. 16).

Definition 4.21. For an integer s denote by $\mathcal{GR}_{V_{\mathbb{F}},0}^{\mathbf{v},\mathrm{loc},s}$ the open and closed subscheme of $\mathcal{GR}_{V_{\mathbb{F}},0}^{\mathbf{v},\mathrm{loc}}$ consisting of all $\mathbf{v}$-admissible lattices $\mathfrak{M}$, where the rank of the maximal Φ-stable subobject $\mathfrak{M}_1$ satisfying $\langle\Phi(\mathfrak{M}_1)\rangle = u^{e-r_1}\mathfrak{M}_1$ is equal to s.

Corollary 4.22. *Assume* $p \neq 2$ *and let* $\rho : G_K \to V_{\mathbb{F}}$ *be any two-dimensional continuous representation of* G_K *that admits a finite flat model after possibly extending the scalars to some finite extension of* $\mathbb{F}$.
Assume that $\mathrm{End}_{\mathbb{F}'[G_K]}(V_{\mathbb{F}'})$ *is a simple algebra for all finite extensions* $\mathbb{F}'$ *of* $\mathbb{F}$. *Then* $\mathcal{GR}_{V_{\mathbb{F}},0}^{\mathbf{v},\mathrm{loc},s}$ *is geometrically connected for all* s. *Furthermore*
(i) If $s = 1$ *and* $\mathrm{End}_{\mathbb{F}'[G_K]}(V_{\mathbb{F}'}) = \mathbb{F}'$ *for all finite extensions* $\mathbb{F}'$ *of* $\mathbb{F}$, *then* $\mathcal{GR}_{V_{\mathbb{F}},0}^{\mathbf{v},\mathrm{loc},s}$ *is either empty or a single point.*
If $s = 1$ *and* $\mathrm{End}_{\mathbb{F}'[G_K]}(V_{\mathbb{F}'}) = M_2(\mathbb{F}')$ *for some finite extension* $\mathbb{F}'$ *of* $\mathbb{F}$, *then* $\mathcal{GR}_{V_{\mathbb{F}},0}^{\mathbf{v},\mathrm{loc},s}$ *is either empty or becomes isomorphic to* $\mathbb{P}^1_{\mathbb{F}'}$ *after extending the scalars to* $\mathbb{F}'$.
(ii) If $s = 2$, *then* $\mathcal{GR}_{V_{\mathbb{F}},0}^{\mathbf{v},\mathrm{loc},s}$ *is either empty or a single point.*

Proof. Our definitions imply

$$\mathcal{GR}_{V_{\mathbb{F}},0}^{\mathbf{v},\mathrm{loc},0} \otimes_{\mathbb{F}} \bar{\mathbb{F}} = X_0^{\mathbf{v}}.$$

Further

$$\bigcup_{[M'] \in \mathcal{S}(\mathbf{v})} X_{[M']}^{\mathbf{v}} = \begin{cases} \mathcal{GR}_{V_{\mathbb{F}},0}^{\mathbf{v},\mathrm{loc},1} & \text{if } r_1 > r_2 \\ \mathcal{GR}_{V_{\mathbb{F}},0}^{\mathbf{v},\mathrm{loc},2} & \text{if } r_1 = r_2. \end{cases}$$

By ([Br], Thm. 3.4.3) we have $\mathrm{End}_{\mathbb{F}'[G_K]}(V_{\mathbb{F}'}) = \mathrm{End}_{\mathbb{F}'((u)),\Phi}(M_{\mathbb{F}'})$. The same Theorem implies that the image of the category of finite flat G_K-representations on finite length $\mathbb{Z}_p$-algebras under the restriction to G_{K_∞} is closed under subobjects and quotients. Hence $V_{\mathbb{F}'}$ is irreducible (resp. reducible, resp. split reducible) if and only if $(M_{\mathbb{F}'}, \Phi)$ is. An easy computation yields:
$\mathrm{End}_{\mathbb{F}'[G_K]}(V_{\mathbb{F}'}) = \mathbb{F}'$ if $V_{\mathbb{F}'}$ is irreducible or non-split reducible.
$\mathrm{End}_{\mathbb{F}'[G_K]}(V_{\mathbb{F}'}) = \mathbb{F}' \times \mathbb{F}'$ if $V_{\mathbb{F}'}$ is the direct sum of two non-isomorphic one-dimensional representations.
$\mathrm{End}_{\mathbb{F}'[G_K]}(V_{\mathbb{F}'}) = M_2(\mathbb{F}')$ if $V_{\mathbb{F}'}$ is the direct sum of two isomorphic one-dimensional representations.
The Corollary now follows from Theorem 4.20 and Propositions 4.11, 4.15 and 4.19. □

5. The structure of $X_0^{\mathbf{v}}$

In this section we want to analyze the structure of the connected component $X_0^{\mathbf{v}}$ of non-$\mathbf{v}$-ordinary lattices. In the absolutely simple case we have

$$X_0^{\mathbf{v}} = \mathcal{GR}^{\mathbf{v},\mathrm{loc}}_{V_{\mathbb{F}},0} \otimes_{\mathbb{F}} \bar{\mathbb{F}}$$

and this is isomorphic to a Schubert variety. In the reducible case it turns out that this component has a quite complicated structure. It is in general not irreducible and its irreducible components have varying dimensions.

5.1. **The case** $(M_{\bar{\mathbb{F}}}, \Phi) \cong (M_1, \Phi_1) \oplus (M_1, \Phi_1)$**.** We assume that $(M_{\bar{\mathbb{F}}}, \Phi)$ is a direct summand of two isomorphic one-dimensional objects and we will use the notations of section 4.1. First we define some subsets of the affine Grassmannian.
Denote by n the maximal integer congruent to $m(\mathbf{v}) \mod 2$, such that

$$n \le \tfrac{r_1 - r_2 + 2}{p+1}. \tag{5.1.1}$$

Denote by l the minimal integer such that

$$n + 2 \le \tfrac{r_1 - r_2 + 2l}{p+1}. \tag{5.1.2}$$

For $z \in \mathbb{P}^1(\bar{\mathbb{F}})$ and $j \ge 0$ we define the following points:

$$\begin{aligned} Q_j^z &= [l + (p+1)j, m(\mathbf{v})]_z \quad \text{if } z \in \bar{\mathbb{F}} \\ Q_j^\infty &= [-l - (p+1)j, m(\mathbf{v})]_0. \end{aligned}$$

We define the following subschemes $Z, Z_j \subset X_0^{\mathbf{v}}$ for $j \ge 0$ by specifying its closed points:

$$\begin{aligned} Z(\bar{\mathbb{F}}) &= \{\mathfrak{M} \in \bar{\mathcal{B}}(m(\mathbf{v})) \mid d_1(\mathfrak{M}, P_{\mathrm{red}}) \le n\} \\ Z_j(\bar{\mathbb{F}}) &= \bigcup_{z \in \mathbb{P}^1(\bar{\mathbb{F}})} \{\mathfrak{M} \in \bar{\mathcal{B}}(m(\mathbf{v})) \mid d_1(\mathfrak{M}, Q_j^z) \le n + 2 - l - (p-1)j\}. \end{aligned} \tag{5.1.3}$$

We want to consider these subsets as subschemes with the reduced scheme structure.

Lemma 5.1. *With the notations of* (5.1.1)-(5.1.3)*:*

$$X_0^{\mathbf{v}}(\bar{\mathbb{F}}) = (\bigcup_{j \ge 0} Z_j(\bar{\mathbb{F}})) \cup Z(\bar{\mathbb{F}}).$$

Proof. Let $\mathfrak{M} = [x, m(\mathbf{v})]_q$ be a non-$\mathbf{v}$-ordinary lattice and denote by $Q' = [x', m(\mathbf{v})]_z$ the unique lattice with $d_1(\mathfrak{M}, Q') = d_1(\mathfrak{M}, \mathcal{T})$. Without loss of generality, we may assume that $Q' \in \mathcal{L}_0$, i.e. $z = 0$ and $v_u(q) = x' > 0$.
If $1 \le x' = v_u(q) < l$, then by Lemma 4.9 and the definition of n and l we find that $\mathfrak{M}$ is $\mathbf{v}$-admissible (and non-$\mathbf{v}$-ordinary) iff $d_1(\mathfrak{M}, P_{\mathrm{red}}) \le n$.
If $v_u(q) \ge l$, then there is a unique j such that $l + (p+1)j \le x' < l + (p+1)(j+1)$. By Lemma 4.9, we find that $\mathfrak{M}$ is $\mathbf{v}$-admissible iff $d_1(\mathfrak{M}, P_{\mathrm{red}}) \le \frac{r_1 - r_2 + 2x'}{p+1}$. Now

$$d_1(\mathfrak{M}, P_{\mathrm{red}}) = d_1(\mathfrak{M}, Q_j^0) + (l + (p+1)j)$$

and hence $\mathfrak{M}$ is $\mathbf{v}$-admissible iff

$$d_1(\mathfrak{M}, Q_j^0) \le \tfrac{r_1 - r_2 + 2l}{p+1} + \tfrac{2(x' - l - (p+1)j)}{p+1} - l - (p-1)j.$$

By the definition of n and l and the fact $x' - l - (p+1)j < (p+1)$ we find that $\mathfrak{M}$ is $\mathbf{v}$-admissible iff

$$d_1(\mathfrak{M}, Q_j^0) \le n + 2 - l - (p-1)j.$$

This yields the claim. $\square$

Proposition 5.2. *With the notation of* (5.1.1)-(5.1.3)*:*

$$X_0^{\mathbf{v}} = (\bigcup_{j\geq 0} Z_j) \cup Z. \tag{5.1.4}$$

(i) The scheme Z is isomorphic to an n-dimensional Schubert variety.
(ii) For $j \geq 0$ there is a projective, surjective and birational morphism

$$f_j : \mathbb{P}^1_{\bar{\mathbb{F}}} \times Y_j \to Z_j$$

where Y_j is an $n+2-l-(p-1)j$ dimensional Schubert variety. Especially Z_j is closed and irreducible.
(iii) If $l \neq 2$, then (5.1.4) *is the decomposition of $X_0^{\mathbf{v}}$ into its irreducible components.*
(iv) If $l = 2$, then the decomposition of $X_0^{\mathbf{v}}$ into its irreducible components is given by

$$X_0^{\mathbf{v}} = \bigcup_{j\geq 0} Z_j.$$

(v) The dimension of $X_0^{\mathbf{v}}$ is given by

$$\dim\, X_0^{\mathbf{v}} = \begin{cases} n+1 & \textit{if } l=2 \\ n & \textit{if } l \neq 2. \end{cases}$$

Proof. *(i)* The closed points of the scheme Z are the lattices with distance smaller than n from the point $[0, m(\mathbf{v})]_0$. By the same argument as in the proof of Theorem 3.9 *(b)*, this is an n-dimensional Schubert variety.
(ii) The scheme Z_j is the union of the Schubert varieties consisting of the lattices $\mathfrak{M}$ with distance $d_1(\mathfrak{M}, Q_j^z) \leq n+2-l-(p-1)j =: n_j$ for $z \in \mathbb{P}^1(\bar{\mathbb{F}})$.
Let us first assume that $m(\mathbf{v}) \equiv x_j := l + (p+1)j \mod 2$, i.e. Q_j^z is a lattice for all $z \in \mathbb{P}^1(\bar{\mathbb{F}})$. For any linearly independent vectors b_1 and b_2 denote by

$$\psi(b_1, b_2) : Y_j \hookrightarrow \mathrm{Grass}\, M_{\bar{\mathbb{F}}}$$

the inclusion of the Schubert variety of lattices $\mathfrak{M}$ with

$$\begin{aligned} d_1(\mathfrak{M}, \langle b_1, b_2\rangle) &\leq n_j \\ d_2(\mathfrak{M}, \langle b_1, b_2\rangle) &= 0. \end{aligned}$$

First we construct a morphism

$$\widetilde{f}_j : \mathbb{P}^1_{\bar{\mathbb{F}}} \times Y_j \to \mathrm{Grass}\, M_{\bar{\mathbb{F}}}.$$

The inclusion $\psi(e_1, e_2)$ defines a sheaf $\mathcal{M}_{Y_j}$ of $\mathcal{O}_{Y_j}[\![u]\!]$-lattices in $M_{\bar{\mathbb{F}}} \widehat{\otimes}_{\bar{\mathbb{F}}} \mathcal{O}_{Y_j}$. If $U = \mathrm{Spec}\, A \subset Y_j$ is an affine open we write $\mathfrak{M}_A = \Gamma(U, \mathcal{M}_{Y_j})$ for the $A[\![u]\!]$-lattice in $M_{\bar{\mathbb{F}}} \widehat{\otimes}_{\bar{\mathbb{F}}} A$ defined by $\mathcal{M}_{Y_j}$. To define the morphism $\widetilde{f}_j$ we define a sheaf $\widetilde{\mathcal{M}}$ of $\mathcal{O}_{\mathbb{P}^1_{\bar{\mathbb{F}}} \times Y_j}[\![u]\!]$-lattices in $M_{\bar{\mathbb{F}}} \widehat{\otimes}_{\bar{\mathbb{F}}} \mathcal{O}_{\mathbb{P}^1_{\bar{\mathbb{F}}} \times Y_j}$. Let $\mathbb{P}^1_{\bar{\mathbb{F}}} = V_0 \cup V_\infty$ with $V_0 = \mathrm{Spec}\, \bar{\mathbb{F}}[T]$ and $V_\infty = \mathrm{Spec}\, \bar{\mathbb{F}}[T^{-1}]$. We define $\widetilde{\mathcal{M}}$ by specifying its sections over the open subsets $V \times U$ of $\mathbb{P}^1_{\bar{\mathbb{F}}} \times Y_j$ where $V \subset \mathbb{P}^1_{\bar{\mathbb{F}}}$ and $U = \mathrm{Spec}\, A \subset Y_j$ are affine open subschemes. If $V' = \mathrm{Spec}\, \bar{\mathbb{F}}[T]_g \subset V_0$ for some $g \in \bar{\mathbb{F}}[T]$, then $\Gamma(V' \times U, \widetilde{\mathcal{M}})$ is the pushout of $\mathfrak{M}_A \widehat{\otimes}_A A[T]_g$ via the endomorphism of $M_{\bar{\mathbb{F}}} \widehat{\otimes}_{\bar{\mathbb{F}}} A[T]_g$ defined by the matrix

$$C_A^0 = \begin{pmatrix} u^{(m(\mathbf{v})+x_j)/2} & Tu^{(m(\mathbf{v})-x_j)/2} \\ Tu^{(m(\mathbf{v})+x_j)/2} & u^{(m(\mathbf{v})-x_j)/2} \end{pmatrix}.$$

If $V'' = \operatorname{Spec} \bar{\mathbb{F}}[T^{-1}]_h \subset V_\infty$ for some $h \in \bar{\mathbb{F}}[T^{-1}]$, then $\Gamma(V'' \times U, \widetilde{\mathcal{M}})$ is the pushout of $\mathfrak{M}_A \widehat{\otimes}_A A[T^{-1}]_h$ via the endomorphism of $M_{\bar{\mathbb{F}}} \widehat{\otimes}_{\bar{\mathbb{F}}} A[T^{-1}]_h$ defined by the matrix

$$C_A^\infty = \begin{pmatrix} T^{-1} u^{(m(\mathbf{v})+x_j)/2} & u^{(m(\mathbf{v})-x_j)/2} \\ u^{(m(\mathbf{v})+x_j)/2} & T^{-1} u^{(m(\mathbf{v})-x_j)/2} \end{pmatrix}.$$

These definitions are compatible: if $V' \subset V_0 \cap V_\infty = \operatorname{Spec} \bar{\mathbb{F}}[T, T^{-1}]$, then the matrices C_A^0 and C_A^∞ differ by a unit (namely T resp. T^{-1}). Further this definitions are compatible with localization in the following sense.
If $U' = \operatorname{Spec} B \subset U = \operatorname{Spec} A$ is an affine open, then

$$\Gamma(V' \times U', \widetilde{\mathcal{M}}) = \Gamma(V' \times U, \widetilde{\mathcal{M}}) \widehat{\otimes}_A B,$$

as $\mathfrak{M}_B = \mathfrak{M}_A \widehat{\otimes}_A B$. And similarly for V'' and for localization on $\mathbb{P}^1_{\bar{\mathbb{F}}}$. As the sets $\{V' \times U, V'' \times U \mid V' \subset V_0\ ,\ V'' \subset V_\infty\ ,\ U \subset Y_j \text{ affine open}\}$ form a basis of the topology this indeed defines a sheaf of $\mathcal{O}_{\mathbb{P}^1_{\bar{\mathbb{F}}} \times Y_j}[\![u]\!]$-lattices on $\mathbb{P}^1_{\bar{\mathbb{F}}} \times Y_j$.

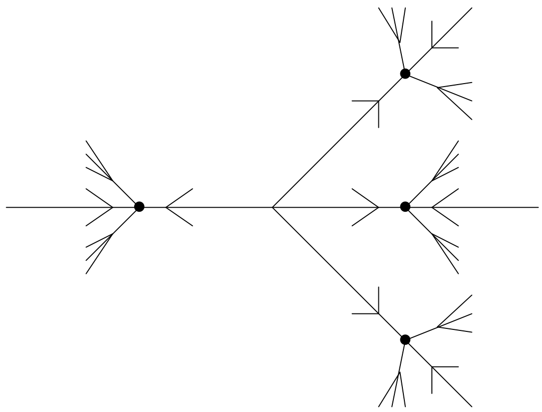

FIGURE 8. The closed points of Z_j in the building in the case $p = 3$, $\mathbb{F} = \mathbb{F}_3$. The fat points mark the points Q_j^z for $z \in \mathbb{P}^1(\mathbb{F})$.

By construction the values of $\widetilde{f}_j$ on closed points are given by

$$\widetilde{f}_j\left((z_1 : z_2), x\right) = \psi\left(u^{(m(\mathbf{v})+x_j)/2}(z_1 e_2 + z_2 e_1), u^{(m(\mathbf{v})-x_j)/2}(z_1 e_1 + z_2 e_2)\right)(x).$$

If we set $T = z \in \bar{\mathbb{F}}$ (resp. $T^{-1} = 0$), then we pushout the Schubert variety Y_j along the automorphism

$$\begin{aligned} e_1 &\mapsto u^{(m(\mathbf{v})+x_j)/2} e_1 \\ e_2 &\mapsto u^{(m(\mathbf{v})-x_j)/2}(z e_1 + e_2). \end{aligned}$$

This is the Schubert variety consisting of the lattices $\mathfrak{M}$ with $d_1(\mathfrak{M}, Q_j^z) \leq n_j$, where

$$Q_j^z = \langle u^{(m(\mathbf{v})+x_j)/2} e_1, u^{(m(\mathbf{v})-x_j)/2}(z e_1 + e_2)\rangle.$$

The conclusion for the point at infinity in $\mathbb{P}^1(\bar{\mathbb{F}})$ is similar. This also shows that the image of $\widetilde{f}_j$ is Z_j. As $\mathbb{P}^1_{\bar{\mathbb{F}}} \times Y_j$ is reduced, the morphism $\widetilde{f}_j$ factors through Z_j and we obtain a surjective morphism $f_j : \mathbb{P}^1_{\bar{\mathbb{F}}} \times Y_j \to Z_j$. As the source of this

morphism is projective, it follows that Z_j is a closed irreducible subset of the affine Grassmannian and that the morphism f_j is projective.
We have to show that it is birational. Denote by $\widetilde{U} \subset Y_j$ the subset of all lattices

$$\{\mathfrak{M} = \langle u^{n_j/2}e_1, u^{-n_j/2}(qe_1 + e_2) \mid q = \sum_{i=0}^{n_j-1} a_i u^i \rangle\},$$

(our assumptions guarantee that n_j is even in this case). This subscheme is isomorphic to the affine space $A_{\bar{\mathbb{F}}}^{n_j}$ and is a maximal dimensional affine subspace of Y_j. Now $\widetilde{V} = f_j(V_0 \times \widetilde{U}) \subset Z_j$ is the subset of all lattices

$$\left\{ \mathfrak{M} = \langle u^{(n+2j+2)/2}e_1, u^{-(n+2j+2)/2}(qe_1 + e_2)\rangle \mid q = a_0 + \sum_{i=l+(p+1)j}^{n+2j+1} a_i u^i \right\}.$$

This is again an affine space and f_j maps $V_0 \times \widetilde{U}$ isomorphically onto $\widetilde{V}$. Thus it is birational.
The case $m(\mathbf{v}) \not\equiv x_j \mod 2$ is similar. We have to consider the lattices

$$\begin{aligned} &[l + (p+1)j, m(\mathbf{v}) - 1]_z \quad \text{for } z \in \bar{\mathbb{F}} \\ &[-l - (p+1)j, m(\mathbf{v}) - 1]_0 \end{aligned}$$

instead of Q_j^z. Now $Y_j \hookrightarrow \operatorname{Grass} M_{\bar{\mathbb{F}}}$ is the inclusion of the Schubert variety of lattices $\mathfrak{M}$ with $d_2(\mathfrak{M}, \langle b_1, b_2\rangle) = 1$ and the same condition on d_1 as above. The conclusion is now similar.
(iii) For $i \geq 0$ we always have

$$Z_i \not\subset (\bigcup_{j=0}^{i-1} Z_j) \cup Z,$$

because for example $[n + 2i + 2, m(\mathbf{v})]_0 \in Z_i(\bar{\mathbb{F}})$ but not in the latter union, as we can see from the definitions. If $l \neq 2$, then

$$Z \not\subset \bigcup_{j \geq 0} Z_j$$

because for example $[n, m(\mathbf{v})]_u \in Z(\bar{\mathbb{F}})$ but not in the latter union. The claim follows from that and the computation of the dimensions: At first consider $Z \subset X_0^{\mathbf{v}}$. This is irreducible and its complement has dimension less or equal to dim Z. Now consider $Z \cup Z_0 \not\supset Z$. The complement of this subscheme has dimension (strictly) less than dim Z_0. Proceeding by induction on j yields the claim.
(iv) In the case $l = 2$ we have $Z \subset Z_0$: If $\mathfrak{M} = [x, m(\mathbf{v})]_q \in Z(\bar{\mathbb{F}})$ and if we assume again $v_u(q) > 0$, then $d_1(\mathfrak{M}, \mathcal{T}) \leq n - 1$ and hence

$$d_1(\mathfrak{M}, Q_0^0) = d_1(\mathfrak{M}, Q') + d_1(Q', Q_0^0) \leq n - 1 + l - 1 = n + 2 - l.$$

Thus each point $\mathfrak{M} \in Z(\bar{\mathbb{F}})$ is also contained in Z_0. The statement now follows by the same argument as in *(iii)*.
(v) This is a consequence of *(i)*-*(iv)*. □

Remark 5.3. On each of the half lines $\mathcal{L}_z$ for $z \in \mathbb{P}^1(\bar{\mathbb{F}})$ we find Schubert varieties with decreasing dimensions. This behavior is called *"thinning tubes"* in [PR2] 6.d. compare loc. cit. B 2.

5.2. **The case $(M_{\bar{\mathbb{F}}},\Phi)\cong(M_1,\Phi_1)\oplus(M_2,\Phi_2)$.** In this section we assume that

$$M_{\bar{\mathbb{F}}}\sim\begin{pmatrix}au^s & 0\\ 0 & bu^t\end{pmatrix}$$

with $a,b\in\bar{\mathbb{F}}^\times$ and $0\le s,t<p-1$. Further we assume $s\neq t$ or $a\neq b$.
Assume $s=t$ and let n be the largest integer that is congruent to $m(\mathbf{v}) \mod 2$ and that satisfies

$$n\le\tfrac{r_1-r_2}{p+1}. \tag{5.2.1}$$

Denote by l the smallest integer satisfying

$$n+2\le\tfrac{r_1-r_2+2l}{p+1}. \tag{5.2.2}$$

Define the points

$$Q_j^\pm=[\pm(l+(p+1)j),m(\mathbf{v})]_0,$$

and the subschemes $Z,Z_j^\pm\subset X_0^{\mathbf{v}}$ by:

$$\begin{aligned}Z(\bar{\mathbb{F}})\ &=\{\mathfrak{M}\in\bar{\mathcal{B}}(m(\mathbf{v}))\mid d_1(\mathfrak{M},P_{\mathrm{red}})\le n\}\\ Z_j^\pm(\bar{\mathbb{F}})&=\{\mathfrak{M}\in\bar{\mathcal{B}}(m(\mathbf{v}))\mid d_1(\mathfrak{M},Q_j^\pm)\le n+2-l-(p-1)j\}.\end{aligned}$$

Proposition 5.4. *Assume $s=t$ and define n and l as in* (5.2.1) *and* (5.2.2).
(i) The scheme Z is isomorphic to an n-dimensional Schubert variety.
(ii) The schemes $Z_j^\pm$ are isomorphic to $n+2-l-(p-1)j$ dimensional Schubert varieties.
(iii) If $l\neq1$, then

$$X_0^{\mathbf{v}}=Z\cup(\bigcup_{j\ge0}Z_j^+)\cup(\bigcup_{j\ge0}Z_j^-)$$

is the decomposition of $X_0^{\mathbf{v}}$ into its irreducible components.
(iv) If $l=1$, then

$$X_0^{\mathbf{v}}=(\bigcup_{j\ge0}Z_j^+)\cup(\bigcup_{j\ge0}Z_j^-)$$

is the decomposition of $X_0^{\mathbf{v}}$ into its irreducible components.
(v) The dimension of $X_0^{\mathbf{v}}$ is given by

$$\dim X_0^{\mathbf{v}}=\begin{cases}n+1 & \text{if } l=1\\ n & \text{if } l\neq1.\end{cases}$$

Proof. *(i)* and *(ii)* follow immediately from the definitions.
As in Lemma 5.1 we easily find

$$X_0^{\mathbf{v}}=Z\cup(\bigcup_{j\ge0}Z_j^+)\cup(\bigcup_{j\ge0}Z_j^-) \tag{5.2.3}$$

and as in Proposition 5.2 we find

$$Z_i^\pm\not\subset Z\cup(\bigcup_{j\ge0}^{i-1}Z_j^+)\cup(\bigcup_{j\ge0}^{i-1}Z_j^-).$$

Further

$$\begin{aligned}Z&\not\subset Z_0^\pm &&\text{if } l\neq1\\ Z&\subset Z_0^\pm &&\text{if } l=1.\end{aligned}$$

The computations are the same as in the proof of Proposition 5.2 with the only difference that we have to replace $\mathcal{T}$ by $\mathcal{L}_0 \cup \mathcal{L}_\infty = \mathcal{A}_0 \cap \bar{\mathcal{B}}(m(\mathbf{v}))$. Part *(iii)* and *(iv)* now follow exactly as in the proof of Proposition 5.2.
Finally *(v)* follows from *(i)-(iv)*. □

In the case $s \neq t$ we have to distinguish more different cases. We only sketch the structure of the irreducible components.
Denote by $x_0 = \lfloor \frac{t-s}{p-1} \rfloor$ the integral part of $\frac{t-s}{p-1}$. Let n_+ be the largest integer congruent to $m(\mathbf{v}) \mod 2$ such that

$$n_+ \leq \tfrac{t-s}{p-1} + \tfrac{1}{p+1}(r_1 - r_2 + 2(x_0 + 1 - \tfrac{t-s}{p-1})). \tag{5.2.4}$$

Let n_- be the smallest integer congruent to $m(\mathbf{v}) \mod 2$ such that

$$n_- \geq \tfrac{t-s}{p-1} - \tfrac{1}{p+1}(r_1 - r_2 + 2(\tfrac{t-s}{p-1} - x_0)). \tag{5.2.5}$$

By Lemma 4.13, these numbers have the following meaning: The maximal distance d_1 of a $\mathbf{v}$-admissible lattice in $\mathcal{A}_q \backslash \mathcal{A}_0$ with $v_u(q) = x_0 + 1$ from P_{red} is $n_+ - \frac{t-s}{p-1}$; the maximal distance d_1 of a $\mathbf{v}$-admissible lattice in $\mathcal{A}_q \backslash \mathcal{A}_0$ with $v_u(q) = x_0$ from P_{red} is $\frac{t-s}{p-1} - n_-$. We define

$$\begin{aligned} x_1 &= \tfrac{1}{2}(n_+ + n_-) \\ n &= \tfrac{1}{2}(n_+ - n_-). \end{aligned} \tag{5.2.6}$$

Let Z be the subscheme whose closed points are given by

$$Z(\bar{\mathbb{F}}) = \{\mathfrak{M} \in \bar{\mathcal{B}}(m(\mathbf{v})) \mid d_1(\mathfrak{M}, [x_1, m(\mathbf{v})]_0) \leq n\}.$$

This is a n-dimensional Schubert variety. Let l_+ be the smallest integer such that

$$n_+ + 2 \leq \tfrac{t-s}{p-1} + \tfrac{1}{p+1}(r_1 - r_2 + 2(l_+ - \tfrac{t-s}{p-1})),$$

i.e. the smallest integer such that there are $\mathbf{v}$-admissible lattices with x-coordinate $n_+ + 2$ in the apartments branching of from $\mathcal{A}_0$ at the line $x = l_+$.
Similarly, let l_- the largest integer such that

$$n_- - 2 \geq \tfrac{t-s}{p-1} - \tfrac{1}{p+1}(r_1 - r_2 + 2(\tfrac{t-s}{p-1} - l_-)).$$

For $j \geq 0$ we define the following points

$$\begin{aligned} Q_j^+ &= [l_+ + (p+1)j, m(\mathbf{v})]_0 \\ Q_j^- &= [l_- - (p+1)j, m(\mathbf{v})]_0. \end{aligned}$$

Again, we define the following subschemes of $X_0^{\mathbf{v}}$:

$$\begin{aligned} Z_j^+(\bar{\mathbb{F}}) &= \{\mathfrak{M} \in \bar{\mathcal{B}}(m(\mathbf{v})) \mid d_1(\mathfrak{M}, Q_j^+) \leq n_+ + 2 - l_+ - (p-1)j\} \\ Z_j^-(\bar{\mathbb{F}}) &= \{\mathfrak{M} \in \bar{\mathcal{B}}(m(\mathbf{v})) \mid d_1(\mathfrak{M}, Q_j^-) \leq l_- + 2 - n_- - (p-1)j\}. \end{aligned} \tag{5.2.7}$$

These subschemes are isomorphic to Schubert varieties.

Lemma 5.5. *With the above notation we have*

$$X_0^{\mathbf{v}}(\bar{\mathbb{F}}) = Z(\bar{\mathbb{F}}) \cup (\bigcup_{j \geq 0} Z_j^+(\bar{\mathbb{F}})) \cup (\bigcup_{j \geq 0} Z_j^-(\bar{\mathbb{F}})). \tag{5.2.8}$$

Proof. The proof of this fact is similar to the proof of Lemma 5.1.
If $\mathfrak{M} = [x, m(\mathbf{v})]_q$ is a lattice, we denote by Q' the unique point in $\mathcal{A}_0 \cap \bar{\mathcal{B}}(m(\mathbf{v}))$ with $d_1(\mathfrak{M}, Q') = d_1(\mathfrak{M}, \mathcal{A}_0)$. We assume that $Q' = [x', m(\mathbf{v})]_0$ with $x' > \frac{t-s}{p-1}$.
If $x' < l_+$, then $\mathfrak{M}$ is $\mathbf{v}$-admissible (and non-$\mathbf{v}$-ordinary) if and only if $\mathfrak{M} \in Z(\bar{\mathbb{F}})$. This is a direct consequence of the definitions.
If $x' \geq l_+$, there is a unique integer $j \geq 0$ such that

$$l_+ + (p+1)j \leq x' < l_+ + (p+1)(j+1).$$

In this case the definitions imply that $\mathfrak{M}$ is $\mathbf{v}$-admissible (and non-$\mathbf{v}$-ordinary) if and only if $\mathfrak{M} \in Z_j^+(\bar{\mathbb{F}})$ (compare the proof of Lemma 5.1).
The conclusions for $x' < \frac{t-s}{p-1}$ are similar. In the set of coordinates considered above, the computations become more complicated, but we can also deduce this result by interchanging e_1 and e_2. ☐

Proposition 5.6. *With the notations of* (5.2.4)-(5.2.7) *:*

$$X_0^{\mathbf{v}} = Z \cup (\bigcup_{j\geq 0} Z_j^+) \cup (\bigcup_{j \geq 0} Z_j^-). \tag{5.2.9}$$

(i) The scheme Z is isomorphic to an n-dimensional Schubert variety.
(ii) For $j \geq 0$ the schemes $Z_j^{\pm}$ are isomorphic to Schubert varieties of dimension

$$\begin{aligned} \dim Z_j^+ &= n_+ + 2 - l_+ - (p-1)j \\ \dim Z_j^- &= l_- + 2 - n_- - (p-1)j. \end{aligned}$$

(iii) If $Z \not\subset Z_0^+ \cup Z_0^-$, then (5.2.9) *is the decomposition of $X_0^{\mathbf{v}}$ into its irreducible components.*
(iv) If $Z \subset Z_0^+ \cup Z_0^-$, then

$$X_0^{\mathbf{v}} = (\bigcup_{j\geq 0} Z_j^+) \cup (\bigcup_{j\geq 0} Z_j^-)$$

is the decomposition of $X_0^{\mathbf{v}}$ into its irreducible components.

In this case we would have to consider many cases in order to determine whether $Z \subset Z_0^+ \cup Z_0^-$ or not from the given integers r_1, r_2, s, t.

Proof. By use of Lemma 5.5, this is nearly the same as in the proof of Proposition 5.4. ☐

Remark 5.7. Again, we find that the Schubert varieties of decreasing dimension defined above correspond to the *"thinning tubes"* in [PR2] 6.d along the Φ-stable half lines $\{[x, m(\mathbf{v})]_0 \mid x \leq \frac{t-s}{p-1}\}$ and $\{[x, m(\mathbf{v})]_0 \mid x \geq \frac{t-s}{p-1}\}$ in the building for $PGL_2(\bar{\mathbb{F}}((u)))$.

5.3. **The case of a non split extension.** As in section 4.3, we assume that there is a basis e_1, e_2 of $M_{\bar{\mathbb{F}}}$ such that

$$M_{\bar{\mathbb{F}}} \sim \begin{pmatrix} au^s & \gamma \\ 0 & bu^t \end{pmatrix}$$

for some $a, b \in \bar{\mathbb{F}}^\times$, $\gamma \in \bar{\mathbb{F}}((u))$ and $0 \leq s, t < p-1$. We assume that $(M_{\bar{\mathbb{F}}}, \Phi)$ is a non-split extension and use the notations of section 4.3.

Denote by x_0 the largest integer $x_0 < \frac{k-s}{p}$. Let n_+ be the largest integer congruent $m(\mathbf{v}) \mod 2$ such that

$$n_+ \leq \tfrac{1}{p+1}(r_1 - r_2 - s - t + 2k). \tag{5.3.1}$$

Let n_- be the smallest integer congruent $m(\mathbf{v}) \mod 2$ such that

$$n_- \geq \tfrac{t-s}{p-1} - \tfrac{1}{p+1}(r_1 - r_2 + 2(\tfrac{t-s}{p-1} - x_0)). \tag{5.3.2}$$

These numbers have the following meaning: The integer n_+ is the maximal x-coordinate of a $\mathbf{v}$-admissible lattice in $\mathcal{A}_q$ with $v_u(q) \geq \frac{k-s}{p}$; further $\frac{t-s}{p-1} - n_-$ is the maximal distance d_1 of a $\mathbf{v}$-admissible lattice in $\mathcal{A}_q \backslash \mathcal{A}_0$ with $v_u(q) = x_0$ from P_{red}. As above, we define the following integers

$$\begin{aligned} x_1 &= \tfrac{1}{2}(n_+ + n_-) \\ n &= \tfrac{1}{2}(n_+ - n_-). \end{aligned} \tag{5.3.3}$$

We have $x_1 \in \{x_0, x_0 + 1\}$ which can be deduced from the equation

$$\tfrac{1}{p+1}(r_1 - r_2 - s - t + 2k) - \tfrac{k-s}{p} = \tfrac{k-s}{p} - (\tfrac{t-s}{p-1} - \tfrac{1}{p+1}(r_1 - r_2 + 2(\tfrac{t-s}{p-1} - \tfrac{k-s}{p}))).$$

(Here, we compute the distance from the point $\frac{k-s}{p}$ and write $\frac{k-s}{p}$ instead of x_0 as in (5.3.2)). We define the following subset

$$Z(\bar{\mathbb{F}}) = \{\mathfrak{M} \in \bar{\mathcal{B}}(m(\mathbf{v})) \mid d_1(\mathfrak{M}, [x_1, m(\mathbf{v})]_0) \leq n\}. \tag{5.3.4}$$

Let l_- be the largest integer such that

$$n_- - 2 \geq \tfrac{t-s}{p-1} - \tfrac{1}{p+1}(r_1 - r_2 + 2(\tfrac{t-s}{p-1} - l_-)). \tag{5.3.5}$$

For $j \geq 0$ we define the points $Q_j^- = [l_- - (p+1)j, m(\mathbf{v})]_0$ and the subsets

$$Z_j^-(\bar{\mathbb{F}}) = \{\mathfrak{M} \in \bar{\mathcal{B}}(m(\mathbf{v})) \mid d_1(\mathfrak{M}, Q_j^-) \leq l_- + 2 - n_- - (p-1)j\}. \tag{5.3.6}$$

Lemma 5.8. *With the above notations*

$$X_0^{\mathbf{v}}(\bar{\mathbb{F}}) = Z(\bar{\mathbb{F}}) \cup (\bigcup_{j \geq 0} Z_j^-(\bar{\mathbb{F}})).$$

Proof. Let $\mathfrak{M} = [x, m(\mathbf{v})]_q$ be a lattice.
If $v_u(q) \geq \frac{k-s}{p}$ (or equivalently if $v_u(q) > x_0$), then (by Lemma 4.17) $\mathfrak{M}$ is $\mathbf{v}$-admissible (and non-$\mathbf{v}$-ordinary) iff

$$x \leq \tfrac{1}{p+1}(r_1 - r_2 - s - t + 2k),$$

or equivalently iff $x \leq n_+$.
If $v_u(q) = x_0$ and $x > v_u(q)$, then (by Lemma 4.17) $\mathfrak{M}$ is $\mathbf{v}$-admissible and non-$\mathbf{v}$-ordinary iff

$$d_1(\mathfrak{M}, [\tfrac{t-s}{p-1}, m(\mathbf{v})]_0) \leq \tfrac{1}{p+1}(r_1 - r_2 + 2(\tfrac{t-s}{p-1} - x_0)),$$

or equivalently iff $d_1(\mathfrak{M}, [\frac{t-s}{p-1}, m(\mathbf{v})]_0) \leq \frac{t-s}{p-1} - n_-$. By the definitions of x_1 and n and the fact $x_1 \in \{x_0, x_0 + 1\}$, we find that in both cases $\mathfrak{M}$ is $\mathbf{v}$-admissible (and non $\mathbf{v}$-ordinary) iff $\mathfrak{M} \in Z(\bar{\mathbb{F}})$.
For the $\mathbf{v}$-admissible lattices $\mathfrak{M} = [x, m(\mathbf{v})]_q$ with $v_u(q) < x_0$ we proceed as in the proof of Lemma 5.5. $\square$

Proposition 5.9. *With the notations of* (5.3.1)-(5.3.6)*:*

$$X_0^{\mathbf{v}} = Z \cup (\bigcup_{j\geq 0} Z_j^-). \tag{5.3.7}$$

(i) The scheme Z is isomorphic to an n-dimensional Schubert variety.
(ii) For $j \geq 0$ the schemes Z_j^- are isomorphic to Schubert varieties of dimension

$$\dim Z_j^- = l_- + 2 - n_- - (p-1)j.$$

(iii) If $Z \not\subset Z_0^-$, then (5.3.7) *is the decomposition of $X_0^{\mathbf{v}}$ into its irreducible components.*
(iv) If $Z \subset Z_0^-$, then the decomposition of $X_0^{\mathbf{v}}$ into its irreducible components is given by

$$X_0^{\mathbf{v}} = \bigcup_{j\geq 0} Z_j^-.$$

Proof. (i),(ii) This follows from the definitions.
(iii),(iv) As in the discussion of the other cases, our Schubert varieties are constructed in a way such that

$$Z_i^- \not\subset Z \cup (\bigcup_{j=0}^{i-1} Z_j^-)$$

for all $i \geq 0$. The claim follows from this and the computation of the dimension (compare the proof of Proposition 5.2). □

Remark 5.10. We find a sequence of Schubert varieties of decreasing dimension along the unique Φ-stable half line $\{[x, m(\mathbf{v})]_0 \mid x < \frac{k-s}{p}\}$ corresponding to the *"thinning tubes"* in [PR2] 6.d.
Further, we find a Schubert variety Z corresponding to a ball with given radius around a given point as in loc. cit. A 3 resp. B 2.

The discussion of this section implies the following result.

Theorem 5.11. *If $(M_{\mathbb{F}}, \Phi)$ is not isomorphic to the direct sum of two isomorphic one-dimensional ϕ-modules, then the irreducible components of $X_0^{\mathbf{v}}$ are Schubert varieties. Especially they are normal.*

6. Relation to Raynaud's theorem

In this section, we assume $p \neq 2$. In [Ra], Raynaud introduces a partial order on the set of finite flat models for $V_{\mathbb{F}}$ (i.e. the set of $\mathbb{F}$-valued points of $\mathcal{GR}_{V_{\mathbb{F}},0}$) by defining $\mathcal{G}_1 \preceq \mathcal{G}_2$ if there exists a morphism $\mathcal{G}_2 \to \mathcal{G}_1$ inducing the identity on the generic fiber of $\operatorname{Spec} \mathcal{O}_K$. By (loc. cit. 2.2.3 and 3.3.2), this order admits a minimal and maximal object (if the set is non-empty) which agree if $e < p-1$.
In our case, Raynaud's partial order is given by the inclusion of lattices in $M_{\mathbb{F}}$: Inclusion of two admissible lattices is a morphism that commutes with the semi-linear map Φ and induces the identity of $M_{\mathbb{F}}$ after inverting u. Here, a lattice $\mathfrak{M}$ is called *admissible* if it defines a finite flat group scheme, i.e. if $u^e\mathfrak{M} \subset \langle \Phi(\mathfrak{M}) \rangle \subset \mathfrak{M}$.

Proposition 6.1. *Let $\rho : G_K \to V_{\mathbb{F}}$ be a continuous representation of G_K. Assume that there exists a finite extension $\mathbb{F}'$ of $\mathbb{F}$ such that there is a finite flat group scheme* *model over* $\operatorname{Spec}\mathcal{O}_K$ *for the induced* G_K *representation on* $V_{\mathbb{F}'} = V_{\mathbb{F}} \otimes_{\mathbb{F}} \mathbb{F}'$. *Then there exists a finite flat model for* $V_{\mathbb{F}}$*, i.e.*

$$\mathcal{GR}_{V_{\mathbb{F}},0} \neq \emptyset \Rightarrow \mathcal{GR}_{V_{\mathbb{F}},0}(\mathbb{F}) \neq \emptyset.$$

Proof. Our assumptions imply $\mathcal{GR}_{V_{\mathbb{F}},0}(\mathbb{F}') \neq \emptyset$ for some finite extension $\mathbb{F}'$ of $\mathbb{F}$. Hence, by Raynaud's theorem, the set $\mathcal{GR}_{V_{\mathbb{F}},0}(\mathbb{F}')$ has a unique minimal element. The natural action of the Galois group $\mathrm{Gal}(\mathbb{F}'/\mathbb{F})$ on $\mathcal{GR}^{\mathbf{v},\mathrm{loc}}_{V_{\mathbb{F}},0}(\mathbb{F}')$ preserves the partial order (it preserves inclusion of lattices) and hence the minimal element is stable under this action. Consequently, the minimal object is already defined over $\mathbb{F}$. ☐

We now want to reprove Raynaud's theorem in our context: we will show that there is a minimal and a maximal lattice for the order induced by inclusion on the set $\mathcal{GR}_{V_{\mathbb{F}},0}(\bar{\mathbb{F}})$.

Proposition 6.2. *There exists a minimal and a maximal admissible lattice* $\mathfrak{M}_{\min}$ *resp.* $\mathfrak{M}_{\max}$ *for the order defined by the inclusion.*

Proof. We only prove the statement about the maximal lattice. The other one is analogue. We choose a basis e_1, e_2. The proposition follows from the following two observations:
(a) There exists a unique admissible lattice with minimal y*-coordinate.*
(b) If $\mathfrak{M}$ *is a admissible lattice with non-minimal* y*-coordinate, then it is contained in an admissible lattice with strictly smaller* y*-coordinate.*
Proof of (a): First it is clear that the y-coordinates of admissible lattices are bounded below: If

$$\langle e_1, e_2\rangle \sim A' = \begin{pmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{pmatrix}$$

and if $\mathfrak{M} = [x, y]_q$ is admissible, then $2e - d' = (p-1)y + v_u(\det A')$ with

$$0 \leq d' = \dim \langle \Phi(\mathfrak{M})\rangle / u^e\, \mathfrak{M} \leq 2e.$$

Suppose now that $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are admissible lattices with the same y-coordinate. There is a basis b_1, b_2 such that

$$\begin{aligned} \mathfrak{M}_1 &= \langle b_1, b_2 \rangle \\ \mathfrak{M}_2 &= \langle u^n b_1, u^{-n} b_2 \rangle. \end{aligned}$$

for some $n \geq 0$. We have

$$\begin{aligned} \mathfrak{M}_1 \sim A &= \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \\ \mathfrak{M}_2 \sim B &= \begin{pmatrix} u^{n(p-1)}\alpha & u^{-n(p+1)}\beta \\ u^{n(p+1)}\gamma & u^{-n(p-1)}\delta \end{pmatrix} \end{aligned}$$

for some $\alpha, \beta, \gamma, \delta \in \bar{\mathbb{F}}((u))$. Define

$$\mathfrak{M}_3 = \langle b_1, u^{-n} b_2 \rangle = \begin{pmatrix} 1 & 0 \\ 0 & u^{-n} \end{pmatrix} \mathfrak{M}_1.$$

Then $\mathfrak{M}_3$ has strictly smaller y-coordinate and is admissible. Indeed:

$$\mathfrak{M}_3 \sim C = \begin{pmatrix} \alpha & u^{-np}\beta \\ u^n\gamma & u^{-n(p-1)}\delta \end{pmatrix}$$

and we have to show: $\min_{i,j} c_{ij} \geq 0$ and $v_u(\det C) - \min_{i,j} c_{ij} \leq e$.
But because $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are admissible we know:

$$\begin{aligned} &v_u(\alpha) \geq 0 && v_u(u^{-np}\beta) \geq n \\ &v_u(u^n\gamma) \geq n && v_u(u^{-n(p-1)}\delta) \geq 0. \end{aligned}$$

Similarly $v_u(\det C) = v_u(\det A) - (p-1)n = v_u(\det B) - (p-1)n$ and hence:

$$\begin{aligned} &v_u(\det C) - v_u(\alpha) \leq e - (p-1)n && v_u(\det C) - v_u(u^{-np}\beta) \leq e - pn \\ &v_u(\det C) - v_u(u^n\gamma) \leq e - pn && v_u(\det C) - v_u(u^{-n(p-1)}\delta) \leq e - (p-1)n. \end{aligned}$$

Proof of (b): Let $\mathfrak{M}$ be an admissible lattice with non-minimal y-coordinate. Then there exists an admissible lattice $\mathfrak{M}'$ with strictly smaller y-coordinate. These lattices are contained in a common apartment and hence there is a basis b_1, b_2 such that

$$\begin{aligned} \mathfrak{M} &= \langle b_1, b_2 \rangle \\ \mathfrak{M}' &= \langle u^m b_1, u^n b_2 \rangle \end{aligned}$$

for some integers m, n with $m + n < 0$, because the y-coordinate of $\mathfrak{M}'$ is strictly smaller than the y-coordinate of $\mathfrak{M}$.
Without loss of generality, we assume $m - n \geq 0$. If $m \leq 0$, then $n \leq 0$ and we are done, since then $\mathfrak{M} \subset \mathfrak{M}'$.
If $m > 0$ we claim that the lattice $\mathfrak{M}_1 = \langle b_1, u^{m+n} b_2 \rangle$ is admissible. Indeed

$$\begin{aligned} \mathfrak{M} &\sim \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \\ \mathfrak{M}' &\sim \begin{pmatrix} u^{(p-1)m}\alpha & u^{pn+m}\beta \\ u^{pm-n}\gamma & u^{(p-1)n}\delta \end{pmatrix} \\ \mathfrak{M}_1 &\sim \begin{pmatrix} \alpha & u^{p(m+n)}\beta \\ u^{-m-n}\gamma & u^{(p-1)(m+n)}\delta \end{pmatrix} \end{aligned}$$

and the claim follows by a similar argument as in the proof of *(a)*. □

Proposition 6.3. *If $e < p-1$, then the minimal and the maximal lattice coincide.*

Proof. Denote the minimal lattice by $\mathfrak{M}_{\min}$, the maximal by $\mathfrak{M}_{\max}$. There is a apartment containing both lattices and we may assume $\mathfrak{M}_{\max} = \langle e_1, e_2 \rangle = [0,0]_0$ and $\mathfrak{M}_{\min} = [x,y]_0$ for some $y \geq 0$.
Let $A \in GL_2(\bar{\mathbb{F}}((u)))$ be a matrix such that

$$\mathfrak{M}_{\max} \sim A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}.$$

Define $d_{\min} = \dim\ \mathfrak{M}_{\min}/\langle \Phi(\mathfrak{M}_{\min}) \rangle$ and similarly $d_{\max}$. Then

$$\begin{aligned} 2e - d_{\max} &= 2e - \dim\ \mathfrak{M}_{\max}/\langle \Phi(\mathfrak{M}_{\max}) \rangle = v_u(\det A) \\ 2e - d_{\min} &= 2e - \dim\ \mathfrak{M}_{\min}/\langle \Phi(\mathfrak{M}_{\min}) \rangle = v_u(\det A) + (p-1)y. \end{aligned}$$

Thus we have $(p-1)y = d_{\min} - d_{\max} \leq 2e < 2(p-1)$ and hence $y = 0$ or $y = 1$. If $y = 0$ we are done, as $\mathfrak{M}_{\max}$ is the unique lattice with minimal y-coordinate.

Assume $y = 1$. In this case $\mathfrak{M}_{\min} \subset \mathfrak{M}_{\max}$ implies $\mathfrak{M}_{\min} = [\pm 1, 1]_0$. Without loss of generality, we assume $\mathfrak{M}_{\min} = [-1, 1]_0 = \langle ue_1, e_2 \rangle$. Then

$$\mathfrak{M}_{\min} \sim B = \begin{pmatrix} u^{p-1}\alpha & u^{-1}\beta \\ u^p\gamma & \delta \end{pmatrix}.$$

As both lattices are admissible, we have

$$\max\{v_u(\det B) - v_u(u^{-1}\beta), v_u(\det B) - v_u(\delta)\} \leq e$$

and hence:

$$\begin{array}{ll} v_u(\alpha) \geq 0 & v_u(\beta) - 1 \geq v_u(\det B) - e = v_u(\det A) + (p-1) - e > v_u(\det A) \\ v_u(\gamma) \geq 0 & v_u(\delta) \geq v_u(\det B) - e = v_u(\det A) + (p-1) - e > v_u(\det A) \end{array}$$

It follows that

$$v_u(\det A) = v_u(\alpha\delta - \beta\gamma) \geq \min\{v_u(\alpha) + v_u(\delta), v_u(\beta) + v_u(\gamma)\} > v_u(\det A).$$

Contradiction. □

Finally we want to determine the elementary divisors of $\langle \Phi(\mathfrak{M}) \rangle$ with respect to $\mathfrak{M}$ for the minimal and the maximal lattice in the cases where $(M_{\bar{\mathbb{F}}}, \Phi)$ is simple resp. split reducible. If $(M_{\bar{\mathbb{F}}}, \Phi)$ is non-split reducible, the computations turn out to be very difficult and are omitted.

6.1. **The absolutely simple case.** As in section 3, we fix a basis e_1, e_2 such that

$$M_{\bar{\mathbb{F}}} \sim \begin{pmatrix} 0 & au^s \\ 1 & 0 \end{pmatrix}$$

with $a \in \bar{\mathbb{F}}^\times$ and $0 \leq s < p^2 - 1$. Let $\mathfrak{M}_{\min}$ be the minimal and $\mathfrak{M}_{\max}$ be the maximal lattice. Denote by s_1, s_2 the unique integers $0 \leq s_1 < p+1$ resp. $0 \leq s_2 < p-1$ with

$$\begin{aligned} s_1 &\equiv s \mod (p+1) \\ s_2 &\equiv s \mod (p-1). \end{aligned}$$

Because $p-1$ and $p+1$ are both even, we find $s_1 \equiv s_2 \mod 2$.
Similarly let s_2' be the unique integer $0 \leq s_2' < p-1$ with $2e - s \equiv s_2' \mod (p-1)$.

Proposition 6.4. *Denote by $m = \frac{s-s_1}{p+1}$ the integral part of $\frac{s}{p+1}$ and by $l = \frac{s-s_2}{p-1}$ resp. $l' = \frac{2e-s-s_2'}{p-1}$ the integral part of $\frac{s}{p-1}$ resp. $\frac{2e-s}{p-1}$.*
(i) The elementary divisors $(a_{\max}, b_{\max})$ of $\langle \Phi(\mathfrak{M}_{\max}) \rangle$ with respect to $\mathfrak{M}_{\max}$ are given by

$$\begin{cases} (\frac{s_1+s_2}{2}, \frac{s_2-s_1}{2}) & \textit{if } l+m \in 2\mathbb{Z} \textit{ and } s_2 \geq s_1 \\ (\frac{s_2-s_1}{2} + p, \frac{s_1+s_2}{2} - 1) & \textit{if } l+m \in 2\mathbb{Z} \textit{ and } s_2 < s_1 \\ (\frac{s_2-s_1+(p+1)}{2}, \frac{s_1+s_2-(p-1)}{2}) & \textit{if } l+m \notin 2\mathbb{Z} \textit{ and } s_1+s_2 \geq p+1 \\ (\frac{s_1+s_2+(p-1)}{2}, \frac{s_2-s_1+(p-1)}{2}) & \textit{if } l+m \notin 2\mathbb{Z} \textit{ and } s_1+s_2 < p+1. \end{cases}$$

(ii) The elementary divisors $(a_{\min}, b_{\min})$ *of* $\langle \Phi(\mathfrak{M}_{\min}) \rangle$ *with respect to* $\mathfrak{M}_{\min}$ *are given by*

$$\begin{cases} (e + \frac{s_1 - s_2'}{2}, e - \frac{s_1+s_2}{2}) & \textit{if } \ l' + m \in 2\mathbb{Z} \textit{ and } s_1 \leq s_2' \\ (e + 1 - \frac{s_1+s_2}{2}, e - p + \frac{s_1-s_2}{2}) & \textit{if } \ l' + m \in 2\mathbb{Z} \textit{ and } s_1 > s_2' \\ (e + \frac{(p+1)-s_1-s_2'}{2}, e + \frac{s_1-s_2'-(p+1)}{2}) & \textit{if } \ l' + m \notin 2\mathbb{Z} \textit{ and } s_1 + s_2' \geq p+1 \\ (e + \frac{s_1-s_2'-(p-1)}{2}, e + \frac{-s_1-s_2'-(p-1)}{2}) & \textit{if } \ l' + m \notin 2\mathbb{Z} \textit{ and } s_1 + s_2 < p+1. \end{cases}$$

Proof. We only prove the statement on $\mathfrak{M}_{\max}$. From Corollary 3.10 we know that the lattice $\mathfrak{M}_{\max}$ is contained in the apartment defined by e_1, e_2. For a lattice $\mathfrak{M}$, denote the elementary divisors of $\langle \Phi(\mathfrak{M}) \rangle$ with respect to $\mathfrak{M}$ by (a, b). Then $\mathfrak{M}$ is admissible if $0 \leq b, a \leq e$. As we are assuming that there exist admissible lattices we only have to check the condition $a, b \geq 0$ for $\mathfrak{M}_{\max}$ (and the condition $a, b \leq e$ for $\mathfrak{M}_{\min}$).
If $l + m \in 2\mathbb{Z}$, the candidate for the maximal lattice is $[m, -l]_0$. If this is not admissible, then we take $[m+1, -l+1]_0$. Computing the elementary divisors a, b by use of Lemma 3.2 and Definition 2.2 we find the above expressions. In the case $l + m \notin 2\mathbb{Z}$ we deal with the lattices $[m+1, -l]_0$ and $[m, -l+1]_0$. $\square$

6.2. **The split reducible case.** As in section 4, we fix a basis e_1, e_2 such that

$$M_{\bar{\mathbb{F}}} \sim \begin{pmatrix} au^s & 0 \\ 0 & bu^t \end{pmatrix}$$

with $a, b \in \bar{\mathbb{F}}^\times$ and $0 \leq s, t < p-1$.

Proposition 6.5. *(i) The elementary divisors* $(a_{\max}, b_{\max})$ *of* $\langle \Phi(\mathfrak{M}_{\max}) \rangle$ *with respect to* $\mathfrak{M}_{\max}$ *are given by*

$$(a_{\max}, b_{\max}) = \begin{cases} (t, s) & \textit{if } t \geq s \\ (s, t) & \textit{if } s \geq t. \end{cases}$$

(ii) The elementary divisors $(a_{\min}, b_{\min})$ *of* $\langle \Phi(\mathfrak{M}_{\min}) \rangle$ *with respect to* $\mathfrak{M}_{\min}$ *are given by*

$$(a_{\min}, b_{\min}) = \begin{cases} ((p-1)\lfloor \frac{e-t}{p-1} \rfloor + t, (p-1)\lfloor \frac{e-s}{p-1} \rfloor + s) & \textit{if } \frac{t-s}{p-1} \geq \lfloor \frac{e-s}{p-1} \rfloor - \lfloor \frac{e-t}{p-1} \rfloor \\ ((p-1)\lfloor \frac{e-s}{p-1} \rfloor + s, (p-1)\lfloor \frac{e-t}{p-1} \rfloor + t) & \textit{if } \frac{t-s}{p-1} \leq \lfloor \frac{e-s}{p-1} \rfloor - \lfloor \frac{e-t}{p-1} \rfloor. \end{cases}$$

Proof. From Corollary 4.12 and Corollary 4.16 we know that the minimal and the maximal lattice are contained in the apartment defined by e_1, e_2. Now

$$\Phi(u^m e_1) = au^{(p-1)m+s}(u^m e_1)$$
$$\Phi(u^n e_2) = bu^{(p-1)n+t}(u^n e_2).$$

The first part of the Proposition follows from the fact that s (resp. t) are the smallest positive integers that are congruent to s (resp. t) modulo $p-1$.
The second part follows from the fact that $(p-1)\lfloor \frac{e-s}{p-1} \rfloor + s$ is the largest integer smaller than e that is congruent to s modulo $p-1$ (and similar for t). $\square$

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