DENSITY OF AUTOMORPHIC POINTS IN DEFORMATION RINGS OF POLARIZED GLOBAL GALOIS REPRESENTATIONS

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Abstract. Conjecturally, the Galois representations that are attached to essentially selfdual regular algebraic cuspidal automorphic representations are Zariski-dense in a polarized Galois deformation ring. We prove new results in this direction in the context of automorphic forms on definite unitary groups over totally real fields. This generalizes the infinite fern argument of Gouvea-Mazur and Chenevier, and relies on the construction of non-classical p-adic automorphic forms, and the computation of the tangent space of the space of trianguline Galois representations. This boils down to a surprising statement about the linear envelope of intersections of Borel subalgebras.

Contents

1. Introduction

Let F be a number field, and fix a positive integer $n \geqslant 1$ and a prime number p. The goal of this paper is to study some properties of deformation spaces of continuous representations

$$\overline{\rho} : \mathrm{Gal}(\overline{F}/F) \longrightarrow \mathrm{GL}_n(\mathbf{F})$$

where $\mathbf{F}$ is a finite extension of $\mathbf{F}_p$. Assume that $\overline{\rho}$ is absolutely irreducible and unramified outside a finite set of places S containing places dividing p. Mazur proved in [Maz89] that there exists a universal deformation of $\overline{\rho}$ unramified outside of S, that is, for $F_S \subset \overline{F}$ the maximal algebraic extension of F unramified outside of S, a complete local noetherian ring $R_{\overline{\rho},S}$ and a continuous representation

$$\rho_S^{\mathrm{univ}} : \mathrm{Gal}(F_S/F) \longrightarrow \mathrm{GL}_n(R_{\overline{\rho},S})$$

pro-representing the functor of deformations of $\overline{\rho}$ unramified outside S. The generic fiber $\mathcal{X}_{\overline{\rho},S}$ of the formal scheme $\mathrm{Spf}\, R_{\overline{\rho},S}$ is a rigid analytic space over $W(\mathbf{F})[\frac{1}{p}]$ whose closed points can be canonically identified with liftings of $\overline{\rho}$ to finite extensions of the p-adic field $W(\mathbf{F})[\frac{1}{p}]$.

When F is a totally real field or a CM field, it is known that we can attached to each regular algebraic cuspidal automorphic representation π of $\mathrm{GL}_n(\mathbb{A}_F)$ an n-dimensional p-adic continuous representation

$$\rho_\pi : \mathrm{Gal}(\overline{F}/F) \longrightarrow \mathrm{GL}_n(\overline{\mathbf{Q}_p}).$$

This representation is characterized by some local compatibility with π at almost all finite places of F. As a consequence it is unramified outside a finite number of places. A very natural problem with regard to rigid analytic spaces $\mathcal{X}_{\overline{\rho},S}$ concerns the distribution of automorphic points in $\mathcal{X}_{\overline{\rho},S}$, that is points corresponding to regular algebraic cuspidal automorphic representations π of $\mathrm{GL}_n(\mathbb{A}_F)$ such that $\overline{\rho}_\pi$ reduces to $\overline{\rho}$ modulo p.

Beyond the case $n = 1$ which is a consequence of class field theory, the case $n = 2$, $F = \mathbf{Q}$ and $\overline{\rho}$ attached to a modular form has been completely solved by works of Gouvea-Mazur ([GM98]) and Böckle ([Bö01]). It follows from their results that the space $\mathcal{X}_{\overline{\rho},S}$ is equidimensional of dimension 3 and that the automorphic points are Zariski-dense inside $\mathcal{X}_{\overline{\rho},S}$.

For general values of n, the case of polarized representations has been studied by Chenevier in the paper [Che11]. Let $\varepsilon : \mathrm{Gal}(\overline{F}/F) \to \mathbf{Z}_p^\times$ be the cyclotomic character. Assume moreover that F is a totally imaginary quadratic extension of a totally real number field F^+ and let c be the non trivial element of $\mathrm{Gal}(F/F^+)$. We recall that an n-dimensional p-adic representation $\rho : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_n(\overline{\mathbf{Q}_p})$ is *polarized* if there exists an isomorphism

$$\rho^\vee \circ c \simeq \rho \otimes \varepsilon^{n-1}.$$

When the regular algebraic cuspidal automorphic representation π of $\mathrm{GL}_n(\mathbb{A}_F)$ is *conjugate self dual*, that is $\pi^{\vee,c} \simeq \pi$, the representation ρ_π is polarized. Moreover the representation ρ_π is crystalline at p if and only if the representations π_v are unramified for $v \mid p$. In this situation, we have the following conjecture of Chenevier ([Che11, Conj. 1.15]) :

Conjecture 1.1. *Assume that $\overline{\rho}$ is absolutely irreducible and that there exists a regular conjugate self dual algebraic cuspidal automorphic representation π of $\mathrm{GL}_n(\mathbb{A}_E)$ such that $\overline{\rho_\pi} \simeq \overline{\rho}$, then the set of points of the form $\rho_{\pi'}$ for π' a regular conjugate self dual algebraic cuspidal automorphic representation unramified at p is Zariski dense in $\mathcal{X}_{\overline{\rho},S}$.*

When $n = 3$, $F_v = \mathbf{Q}_p$ for $v \mid p$ and the deformation functor of $\overline{\rho}$ is *unobstructed*, this conjecture has been proven by Chenevier.

The main result of this paper is the following.

Theorem 1.2. *Assume $p > 2$ and the following assumptions*

- *the extension F/F^+ is unramified;*
- $2 \mid [F^+ : \mathbf{Q}]$ *if $n \equiv 2$ (mod. 4);*
- *S contains only places which are split in F ;*
- *the representation $\overline{\rho}$ is absolutely irreducible and the group $\overline{\rho}(\mathrm{Gal}(\overline{F}/F(\zeta_p))$ is* adequate *in the sense of* [Tho12].

Assume moreover that there exists some regular conjugate self dual cuspidal automorphic representation π which is unramified outside of $S \setminus S_p$ and such that $\overline{\rho_\pi} \simeq \overline{\rho}$. Then the Zariski closure of automorphic points in $\mathcal{X}_{\overline{\rho},S}$ is a union of irreducible components.

In the paper [All], Patrick Allen proved that, assuming standard automorphy lifting conjectures, it is true that all irreducible components of the space $\mathcal{X}_{\overline{\rho},S}$ contain some regular conjugate self dual cuspidal automorphic point. As such points are smooth by [All16], Theorem 1.2 reaches substantial new cases of Conjecture 1.1 under the standard automorphy lifting conjectures.

Following [Che11], our strategy to prove Theorem 1.2 is to use base change results to deduce this result from an analogous result concerning automorphic forms on some definite unitary group G.

For this definite unitary group we can rely on a well developed theory of families of p-adic automorphic forms, so called eigenvarieties: there exists a rigid analytic space called the eigenvariety $Y(U^p,\overline{\rho})$ parametrizing overconvergent p-adic eigenforms on G. This space is a generalization of the eigencurve of Coleman and Mazur, and was first introduced by Chenevier in the setting of definite unitary groups. The existence of a family of Galois representations on $Y(U^p,\overline{\rho})$ gives rise to a map $Y(U^p,\overline{\rho})\to\mathcal{X}_{\overline{\rho},S}$. The image of this map is the so called "infinite fern".

The main idea is to consider the Zariski-closure $\mathcal{X}_{\overline{\rho},S}^{\mathrm{aut}}\subset\mathcal{X}_{\overline{\rho},S}$ of all automorphic points and show that each of its irreducible components contains a smooth point ρ such that there is an equality of tangent spaces

$$T_\rho\mathcal{X}_{\overline{\rho},S}^{\mathrm{aut}}=T_\rho\mathcal{X}_{\overline{\rho},S}.$$

We are hence reduced to proving that the left hand side is large enough.

It is standard to prove that automorphic points form a Zariski dense subset of the eigenvariety and hence the canonical map

$$\bigoplus_x T_xY(U^p,\overline{\rho})\longrightarrow T_\rho\mathcal{X}_{\overline{\rho},S} \tag{1.1}$$

factors through the tangent space $T_\rho\mathcal{X}_{\overline{\rho},S}^{\mathrm{aut}}$, where the direct sum is indexed by all the preimages $x\in Y(U^p,\overline{\rho})$ of ρ. Hence it will essentially suffice to prove that (1.1) is surjective.

One of the main results of [BHS] is the precise determination of this index set. In [Che11] it is shown that the map in question is surjective, if the restriction of ρ to the local Galois groups at places dividing p satisfies some genericity assumption: roughly, the representations should be crystalline and the Hodge filtration in general position with respect to all possible Frobenius stable flags. The main problem is that in higher dimensions there is (for the time being) no way to guarantee that $\mathcal{X}_{\overline{\rho},S}$ contains any point satisfying this assumption. The point of our paper is the proof of the surjectivity of (1.1) without this genericity assumption.

As in [Che11] we do so by proving a similar surjectivity result for local avatars of the spaces $\mathcal{X}_{\overline{\rho},S}$ and $Y(U^p,\overline{\rho})$: the global deformation ring is replaced by a local deformation ring, and the eigenvariety is replaced by the so called space of trianguline Galois representations. The key construction of [BHS] is a local model for the space of trianguline representations. This local model allows us to reduce the surjectivity of (1.1) to a problem in linear algebra.

The problem is to determine the linear envelope of the intersection of a Borel algebra $\mathfrak{b}$ in the Lie-algebra $\mathfrak{gl}_n$ with the Weyl group translates of a fixed Borel $\mathfrak{b}_0$. This statement seems to be a very nice and interesting statement in its own right:

Theorem 1.3. *Let n be a positive integer, $\mathfrak{S}_n$ the symmetric group of order n, and $\mathfrak{gl}_n$ the algebra of $n\times n$ matrices with entries in a fixed field k. Let $\mathrm{GL}_n(k)$ be the group of the non-singular elements in $\mathfrak{gl}_n$ and $\mathfrak{b}_0\subset\mathfrak{gl}_n$ the Borel subalgebra* *of upper triangular matrices. For any element $g \in \mathrm{GL}_n(k)$ let $\mathfrak{b}_g = g^{-1}\mathfrak{b}_0 g$ denote the Borel subalgebra conjugate to $\mathfrak{b}_0$ by g^{-1}.*

Any Borel subalgebra $\mathfrak{b}$ coincides with the linear envelope of its intersections with the conjugate of $\mathfrak{b}_0$ under $\mathfrak{S}_n$

$$\mathfrak{b} = \sum_{w \in \mathfrak{S}_n} \mathfrak{b} \cap \mathfrak{b}_w.$$

The plan of the paper is the following. In a first section, we prove Theorem 1.3 and prove as an application a surjectivity result for a map between tangent spaces of our local models. The second section is purely local and its purpose is to prove our main local result concerning the sum of tangent spaces of quasi-trianguline deformation spaces. Finally the last section is of global nature and contains the proof of our main global theorem.

Notation : We fix a prime number p. Let K be a finite extension of $\mathbf{Q}_p$, $\overline{K}$ an algebraic closure of K and $\widehat{\overline{K}}$ its completion for the unique valuation extending the p-adic valuation. We use the notation $\mathcal{G}_K$ for the Galois group $\mathrm{Gal}(\overline{K}/K)$.

We fix L a finite extension of $\mathbf{Q}_p$. Let Σ be the set of $\mathbf{Q}_p$-algebra homomorphisms from K to L. We choose L big enough so that $|\Sigma| = [K : \mathbf{Q}_p]$, or equivalently $L \otimes_{\mathbf{Q}_p} K \simeq L^{[K:\mathbf{Q}_p]}$, and we denote by $\mathcal{O}_L$ the ring of integers of L, by $\mathfrak{m}_L$ the unique maximal ideal of $\mathcal{O}_L$ and by $k_L \coloneqq \mathcal{O}_L/\mathfrak{m}_L$ its residue field. Let $|\cdot|_p$ be the unique norm on L inducing the p-adic norm on $\mathbf{Q}_p$. Let ε_L be the character $N_{L/\mathbf{Q}_p}|N_{L/\mathbf{Q}_p}|$ from $L^\times$ to $\mathbf{Z}_p^\times$. Let $\mathrm{rec}_L : L^\times \to W_L^{\mathrm{ab}}$ be the local reciprocity isomorphism sending a uniformizer of L onto a *geometric* Frobenius element. We have $\varepsilon_L = \chi_{\mathrm{cyc}} \circ \mathrm{rec}_L$.

If X is some algebraic variety defined over K, we will use the notation $X_{K/\mathbf{Q}_p}$ for the Weil restriction of X from K to $\mathbf{Q}_p$. Moreover if Y is some algebraic variety defined over $\mathbf{Q}_p$, we will use the notation Y_L for the base change of Y from $\mathbf{Q}_p$ to L, so that $X_{K/\mathbf{Q}_p,L} = (X_{K/\mathbf{Q}_p}) \times_{\mathrm{Spec}\,\mathbf{Q}_p} \mathrm{Spec}\, L$. As we have $L \otimes_{\mathbf{Q}_p} K \simeq L^\Sigma$, we have an isomorphism of algebraic varieties over L

$$X_{K/\mathbf{Q}_p,L} \simeq \prod_{\tau \in \Sigma} X_\tau \tag{1.2}$$

where X_τ is the base change of X from K to L via the embedding τ. If x is some L-point of $X_{K/\mathbf{Q}_p}$ we will denote $(x_\tau) \in \prod_{\tau\in\Sigma} X_\tau$ its image by the isomorphism (1.2).

If $\mathbf{k} \in (\mathbf{Z}^n)^{[K:\mathbf{Q}_p]}$, we define the algebraic character $\delta_{\mathbf{k}} : T(K) \to L^\times$ by the formula

$$(a_1, \dots, a_n) \mapsto \prod_{i=1}^{n} \prod_{\tau \in \Sigma} \tau(a_i)^{k_{i,\tau}}$$

If X is a scheme, or a rigid analytic space and $x \in X$ is a point, we write $T_x X$ for the tangent space of X at x. Similarly, if $\mathfrak{X}$ is a deformation functor defined on local Artin rings with fixed residue field (or a formal scheme pro-representing such a functor), we write $T\mathfrak{X}$ for the tangent space of $\mathfrak{X}$ at the unique closed point.

2. On intersections of Borel algebras

2.1. **Envelopes of intersections of Borel subalgebras.** Let n be a positive integer and k a field. We denote by $\mathfrak{gl}_n$ the Lie algebra of $n \times n$-matrices with coefficients in k and by $\mathfrak{b}_0 \subset \mathfrak{gl}_n$ the Borel subalgebra of upper triangular matrices. Given an element $g \in \mathrm{GL}_n(k)$ we write $\mathfrak{b}_g = g^{-1}\mathfrak{b}_0 g$ for the conjugate of $\mathfrak{b}_0$ by g. We denote by B the subgroup of upper triangular matrices in $\mathrm{GL}_n(k)$ and by W the Weyl group N/T, where N stands for the subgroup of matrices with exactly one non-zero entry in each row and each column and T for the subgroup of diagonal matrices, $T = B \cap N$; the Weyl group W identifies with the subgroup of $\mathrm{GL}_n(k)$ of $n \times n$ permutation matrices, and as such is isomorphic to the group of permutations over n elements, $\mathfrak{S}_n$. When speaking of elements of maximal length in W we refer to the generating set S of W whose elements are (the permutation matrices associated with) the transpositions $(j, j+1)$, $j \in [[1, n-1]]$, so that the quadruple $(\mathrm{GL}_n(k), B, N, S)$ forms a Tits system ([Bou68, IV, 2.2]). Hereafter we freely identify an element w of W with its image in $\mathrm{GL}_n(k)$, the associated permutation matrix, and with its image in $\mathfrak{S}_n$, the underlying permutation. All scalars to be considered will be taken in k.

By its very definition the linear envelope $\sum_{w\in W} \mathfrak{b} \cap \mathfrak{b}_w$ of the intersection of any Borel subalgebra $\mathfrak{b} \subset \mathfrak{gl}_n$ with the conjugates of $\mathfrak{b}_0$ under the Weyl group, is contained in $\mathfrak{b}$; we discuss here the reverse inclusion and show the nice identity,

$$\mathfrak{b} = \sum_{w\in W} \mathfrak{b} \cap \mathfrak{b}_w,$$

that states the envelope does coincide with $\mathfrak{b}$.

Since any Borel subalgebra of $\mathfrak{gl}_n$ is a conjugate of the standard Borel subalgebra $\mathfrak{b}_0$, it will be enough to establish the identity for $\mathfrak{b} = \mathfrak{b}_g$, for an arbitrary element $g \in \mathrm{GL}_n(k)$. By the Bruhat decomposition, every such element g splits as a product $g = u_1 s u_2$ of two (invertible) upper triangular matrices, u_1 and u_2 and a permutation matrix s associated to a permutation $s \in \mathfrak{S}_n$, so that the identity to discuss reads

$$\mathfrak{b}_{su} = \sum_{w\in W} \mathfrak{b}_{su} \cap \mathfrak{b}_w,$$

which is to hold for an arbitrary $n \times n$ permutation matrix s and an arbitrary upper triangular matrix $u \in B = \mathrm{GL}_n(k) \cap \mathfrak{b}_0$.

In a first part we settle this identity for w_0 *the* permutation of maximal length in $\mathfrak{S}_n$, i.e. the involution $w_0 = (1,n)(2,n-1)...(\lfloor \frac{n}{2} \rfloor, n - \lfloor \frac{n}{2} \rfloor + 1)$. Since conjugation by any element $g \in \mathrm{GL}_n(k)$, is a linear isomorphism of $\mathfrak{gl}_n$, the conjugate of a linear envelope coincides with the envelope of the conjugates, and since intersection and conjugation trivially commute, we find

$$\sum_{w \in W} \mathfrak{b}_{w_0 u} \cap \mathfrak{b}_w = \sum_{w \in W} (\mathfrak{b}_{w_0} \cap \mathfrak{b}_{wu^{-1}})_u.$$

Hence the envelope $\sum_{w \in W} \mathfrak{b}_{w_0 u} \cap \mathfrak{b}_w$ coincides with the Borel subalgebra $\mathfrak{b}_{w_0 u}$ if and only if $\mathfrak{b}_{w_0} = \sum_{w \in W} \mathfrak{b}_{w_0} \cap \mathfrak{b}_{wu^{-1}}$ (the reader will note the Borel subalgebra $\mathfrak{b}_{w_0}$ coincides with the Borel algebra of lower triangular matrices).

The proof proceeds through an explicit "dévissage", which the following lemma will make clear; It does *not* rely on any induction on the dimension, nor does it require any further assumption on the fixed base field k: anyone will do. The elementary $n \times n$ matrix whose (l,m)-entry is given by $\delta_{i,l}\delta_{j,m}$ for $l, m \in [\![1,n]\!]$ will be denoted by $e^{i,j}$ and by $x_{i,j}$, $i,j \in [\![1,n]\!]$, we will denote scalars in the base field k.

Lemma 2.1. *Let k be an arbitrary field. For any $u \in B$, and for any ordered pair (i,j) in $[\![1,n]\!]^2$, $i \geqslant j$, there is some permutation $s_{i,j} \in \mathfrak{S}_n$, and $i-j$ scalars $(x_{i,l})_{l=j+1}^{i}$ in k, such that the matrix $a^{i,j} := e^{i,j} + \sum_{l=j+1}^{i} x_{i,l} e^{i,l}$ lies in the Borel subalgebra $\mathfrak{b}_{s_{i,j}u^{-1}}$.*

The matrices $a^{i,j}$ with (i,j) in $[\![1,n]\!]^2$, $i \geqslant j$, then form a basis of the Borel subalgebra $\mathfrak{b}_{w_0}$ of lower triangular matrices in $\mathfrak{gl}_n$ all elements of which lie in the envelope $\sum_{w \in \mathfrak{S}_n} \mathfrak{b}_{w_0} \cap \mathfrak{b}_{wu^{-1}}$.

The permutation $s_{i,j}$ may be chosen to be the (i,j)-transposition.

Proof of Lemma 2.1. Let's denote by $u_{i,j}$ the entries of the upper triangular matrix u, so that $u_{i,j} \in k$, $u_{i,j} = 0$ if $i > j$, and $\prod_{i=1}^{n} u_{i,i} \neq 0$. From the properties of the inverse matrice u^{-1} of u we will only use it is some non-singular upper triangular matrix, which we'll denote by v.

Because of obvious support properties of the involved matrices, the general entry of the product matrix $p := va^{i,j}u$ vanishes – i.e. $p_{l,m}$=0 – for all $l, l \in [\![1,n]\!]$, as soon as $m \in [\![1, j-1]\!]$ and for all $m, m \in [\![1,n]\!]$, as soon as $l \in [\![i+1, n]\!]$: in words, the first $j-1$ columns and last $n-i$ rows of the product p vanish identically. One may further observe all entries of the matrix $q := a^{i,j}u$ vanish on the same ground, but for the entries $q_{i,l}$, $l \in [\![j,n]\!]$, which are given by the identities: $q_{i,l} = u_{j,l} + \sum_{m=j+1}^{\min(k,i)} u_{m,l} x_{i,m}$. From these identities we can then easily deduce the following expression for the entries $p_{a,b}$, $a, b \in [\![j,i]\!]$:

$$p_{a,b} = v_{a,i}(u_{j,b} + u_{j+1,b}x_{i,j+1} + ... + u_{b,b}x_{i,b}) = v_{a,i}(u_{j,b} + \sum_{l=j+1}^{b} u_{l,b}x_{i,l}). \tag{2.1}$$

It can be further remarked that the matrix $r \coloneqq p_{s_{(i,j)}}$, obtained by swapping both the i-th and j-th columns and rows, – so that $p_{s_{(i,j)}}$ coincides with the conjugate of p by the permutation matrix $s_{(i,j)}$ associated with the (i,j)-transposition, consistently with previously introduced notations –, shares with p the property that all entries below the main diagonal and outside the block $l \in [\![j,i]\!]$, $m \in [\![j,i]\!]$ – a $(i-j+1) \times (i-j+1)$-submatrix –, vanish, to the effect that r will lie in $\mathfrak{b}_0$, p in $\mathfrak{b}_{s_{(i,j)}}$ and so $a^{i,j}$ in $\mathfrak{b}_{s_{(i,j)}u^{-1}}$ as soon as all sub-diagonal entries of this block in r cancel.

Identity (2.1) above gives precisely the following expressions for the entries $r_{a,b}$ in the block: first, for all $b \in [\![j+1, i-1]\!]$ and all $a \in [\![j+1, i-1]\!]$,

$$r_{j,b} = p_{i,b} = v_{i,i}\ (u_{j,b} + \sum_{l=j+1}^{b} u_{l,b}x_{i,l},)$$

$$r_{a,b} = p_{a,b} = v_{a,i}\ (u_{j,b} + \sum_{l=j+1}^{b} u_{l,b}x_{i,l}), \text{ and}$$

$$r_{i,b} = p_{j,b} = v_{j,i}\ (u_{j,b} + \sum_{l=j+1}^{b} u_{l,b}x_{i,l});$$

As for the first column of the block, which is to extract from the j-th column of r, itself coinciding with the i-th column of p – up to swapping the i-th and j-th entries –, we get

$$r_{j,j} = p_{i,i} = v_{i,i}\ (u_{j,i} + \sum_{l=j+1}^{i} u_{l,i}x_{i,l}),$$

$$r_{a,j} = p_{a,i} = v_{a,i}\ (u_{j,i} + \sum_{l=j+1}^{i} u_{l,i}x_{i,l}),$$

$$r_{i,j} = p_{j,i} = v_{j,i}\ (u_{j,i} + \sum_{l=j+1}^{i} u_{l,i}x_{i,l}).$$

At this point it should be clear the b-th column, $b \in [\![j+1, i-1]\!]$, vanishes identically as soon as the following affine relation is satisfied by the scalars $(x_{i,l})_{l=j+1}^{i}$

$$u_{j,b} + \sum_{l=j+1}^{b} u_{l,b} x_{i,l} = 0. \tag{2.2}$$

From the second set of equations, it follows the j-th column will also cancel provided the following affine relation is satisfied by the scalars $(x_{i,l})_{l=j+1}^{i}$

$$u_{j,i} + \sum_{l=j+1}^{i} u_{l,i} x_{i,l} = 0. \tag{2.3}$$

It remains to organize the system of equations (2.2) – for $b \in [\![j+1, i-1]\!]$ –, and the equation (2.3) into some lower triangular linear system of size $i-j$ as follows

$$\begin{pmatrix} u_{j+1,j+1} & 0 & 0 & \dots & 0 \\ u_{j+1,j+2} & u_{j+2,j+2} & 0 & \dots & 0 \\ u_{j+1,j+3} & u_{j+2,j+3} & u_{j+3,j+3} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ u_{j+1,i} & u_{j+2,i} & u_{j+3,i} & \dots & u_{i,i} \end{pmatrix} \begin{pmatrix} x_{i,j+1} \\ x_{i,j+2} \\ x_{i,j+3} \\ \vdots \\ x_{i,i} \end{pmatrix} = - \begin{pmatrix} u_{j,j+1} \\ u_{j,j+2} \\ u_{j,j+3} \\ \vdots \\ u_{j,i} \end{pmatrix}. \tag{2.4}$$

Since the matrix u is non-singular by assumption, the product $\prod_{l=j+1}^{i} u_{l,l}$ does not vanish and the previous system admits a (unique) solution; for such a solution $\underline{x} = (x_{il})_{l=j+1}^{i}$ the matrix $a^{i,j} := a^{i,j}(\underline{x})$, $i, j \in [\![1, n]\!]$, $i \geqslant j$, lies in $\mathfrak{b}_{s_{(i,j)}u^{-1}} \cap \mathfrak{b}_{w_0}$, where as above we denote by $s_{(i,j)}$ the permutation matrix in $GL_n(k)$ associated with the (i,j)-transposition in $\mathfrak{S}_n$. This closes the proof of the first and main claim in Lemma 2.1.

By construction one then passes from the matrices $e^{i,j}$ to the matrices $a^{i,j}$, $i, j \in [\![1, n]\!]$, $i \geqslant j$, by some unipotent triangular matrix (with many zeros, since it is n-block diagonal), provided the order we choose on the set $\{(i,j), i, j \in [\![1, n]\!], i \geqslant j\}$ is compatible with the row order, i.e. such that for all $i \in [\![1, n]\!]$ $(i,j) < (i,k)$ if $j < k$ (the lexicographic order clearly has the property); the matrices $a^{i,j}$, $i, j \in [\![1, n]\!]$, $i \geqslant j$, then form a basis of the Borel subalgebra $\mathfrak{b}_{w_0}$, while each one lying in exactly one of the generators $\mathfrak{b}_{w_0} \cap \mathfrak{b}_{s_{(i,j)}u^{-1}}$ of the envelope $\sum_{w \in \mathfrak{W}} \mathfrak{b}_{w_0} \cap \mathfrak{b}_{wu^{-1}}$, the second and last claim in Lemma 2.2. □

The reader may want to notice the above argument reaches the stronger statement that for all u, $u \in B$, $\mathfrak{b}_{w_0 u} = \sum_{t \in \mathfrak{T}_n} \mathfrak{b}_{w_0 u} \cap \mathfrak{b}_t$, where the sum is taken over the subset $\mathfrak{T}_n \subset W_0$ consisting of the identity and the i,j-transpositions, $n \geqslant i > j \geqslant 1$, a small subset of $\mathfrak{S}_n$ with only $(n^2 - n + 2)/2$ elements.

We now discuss the details of the reduction of the general statement to Lemma 2.1, although it may look less surprizing to trained minds. Let's proceed to the reduction. The following lemma is a direct consequence of a refined version of the Bruhat decomposition established in [BT65, Thm. 5.15].

Lemma 2.2. *Let k be an arbitrary field. Any $n \times n$-matrix in $\mathfrak{gl}_n$ splits as the product of an upper triangular matrix by a lower triangular matrix by a permutation matrix: for all matrices $m \in \mathfrak{gl}_n$, there exist an upper triangular matrix u, a lower triangular matrix l, and a permutation matrix p, such that $m = ulp$.*

If useful, one may require the upper triangular u or the lower triangular l to be unipotent (but not both simultaneously of course).

Let's turn back to our main object, realizing a Borel subalgebra as the envelope of its intersections with the conjugates of any fixed Borel subalgebra under the Weyl group. From Lemmas 2.1 and 2.2 we can deduce the following statement.

Theorem 2.3. *Let n be a positive integer, $\mathfrak{S}_n$ the symmetric group of order n, and $\mathfrak{gl}_n$ the algebra of $n \times n$ matrices with entries in a fixed field k. Let $\mathrm{GL}_n(k)$ be the group of the non-singular elements in $\mathfrak{gl}_n$ and $\mathfrak{b}_0 \subset \mathfrak{gl}_n$ the Borel subalgebra of upper triangular matrices. For any element $g \in \mathrm{GL}_n(k)$ let $\mathfrak{b}_g = g^{-1}\mathfrak{b}_0 g$ denote the Borel subalgebra conjugate to $\mathfrak{b}_0$ by g^{-1}.*

Any Borel subalgebra $\mathfrak{b}$ coincides with the linear envelope of its intersections with the conjugate of $\mathfrak{b}_0$ under $\mathfrak{S}_n$

$$\mathfrak{b} = \sum_{w \in \mathfrak{S}_n} \mathfrak{b} \cap \mathfrak{b}_w.$$

Proof. All Borel subalgebras are known to be conjugate, and it is enough to prove the identity in the theorem for $\mathfrak{b} = \mathfrak{b}_g = g^{-1}\mathfrak{b}_0 g$ for all $g \in GL_n(k)$. By Lemma 2.2 the element g splits as a product of an upper triangular matrix, u, by a lower triangular matrix, l, by a permutation matrix, p,i.e. we can write $g = ulp$. The matrix l can be written as the conjugate $l = w_0 u_2 w_0^{-1}$ of an upper triangular matrix u_2 by the permutation matrix w_0 associated with the permutation of maximal length in $\mathfrak{S}_n$, that is

$$w_0 = (1, n)(2, n-1) \dots (\lfloor \frac{n}{2} \rfloor, n - \lfloor \frac{n}{2} \rfloor + 1).$$

Substituting for l in $g = ulp$ accordingly, and introducing the permutation matrix $q \coloneqq w_0^{-1}p$ we get $g = uw_0u_2w_0^{-1}p = uw_0u_2q$, and for the Borel subalgebra $\mathfrak{b}_g = \mathfrak{b}_{w_0u_2q}$.

Now, as observed above, conjugation trivially commutes with taking linear envelope and intersection so that $\sum_{w \in \mathfrak{S}_n} \mathfrak{b}_{w_0u_2} \cap \mathfrak{b}_w$ coincides with $(\sum_{w \in \mathfrak{S}_n} \mathfrak{b}_{w_0} \cap \mathfrak{b}_{wu_2^{-1}})_{u_2}$ and the identity $\mathfrak{b}_{w_0u_2} = \sum_{w \in \mathfrak{S}_n} \mathfrak{b}_{w_0u_2} \cap \mathfrak{b}_w$ is equivalent to $\mathfrak{b}_{w_0} = \sum_{w \in \mathfrak{S}_n} \mathfrak{b}_{w_0} \cap \mathfrak{b}_{wu_2^{-1}}$, which, in turn, is precisely the conclusion of Lemma 2.1.

Again, conjugation commutes with taking linear envelope and intersection, to the effect that the identity $\mathfrak{b}_{w_0u_2} = \sum_{w\in\mathfrak{S}_n} \mathfrak{b}_{w_0u_2} \cap \mathfrak{b}_w$ reads

$$\mathfrak{b}_g = \mathfrak{b}_{w_0u_2q} = (\sum_{w\in\mathfrak{S}_n} \mathfrak{b}_{w_0u_2}\cap\mathfrak{b}_w)_q = \sum_{w\in\mathfrak{S}_n} (\mathfrak{b}_{w_0u_2}\cap\mathfrak{b}_w)_q = \sum_{w\in\mathfrak{S}_n} \mathfrak{b}_{w_0u_2q}\cap\mathfrak{b}_{wq} = \sum_{w\in\mathfrak{S}_n} \mathfrak{b}_g\cap\mathfrak{b}_{wq}.$$

Since in any group (right-) translation by any element is a bijection, the latter sum can be rewritten $\sum_{w\in\mathfrak{S}_n} \mathfrak{b}_g \cap \mathfrak{b}_w$, which reaches the claim that

$$\mathfrak{b}_g = \sum_{w\in\mathfrak{S}_n} \mathfrak{b}_g \cap \mathfrak{b}_w.$$

and closes the proof of Aequatio Praeclara. □

One will note the remark closing the proof of Lemma 2.1 that conjugations under only a small subset of $\mathfrak{S}_n$ are needed to recover $\mathfrak{b}_{w_0u}$ implies, when introduced in the previous discussion, that in general it is enough to consider the envelope of the intersections of the Borel subalgebra $\mathfrak{b}_g$ with the Borel subalgebras $\mathfrak{b}_{tq}$, $t \in \mathfrak{T}_n$, where $q = w_0^{-1}p$, for p the permutation factor of the *ulp* decomposition of g: the sum in the above decomposition can be taken on a prescribed translate of $\mathfrak{T}_n$, a small subset (of cardinal $(n^2 - n + 2)/2$) of $\mathfrak{S}_n$.

2.2. **A surjectivity result.** Let k be a field and let $G := \mathrm{GL}_{n,k}$. Let B be a Borel subgroup in G and let $\mathfrak{g} := \operatorname{Lie} G$ and $\mathfrak{b} := \operatorname{Lie} B$. The quotient scheme G/B is identified to the projective scheme classifying the complete flag in k^n. As two complete flags in k^n are conjugate under $G(k)$, we have a natural isomorphism of sets $(G/B)(k) \simeq G(k)/B(k)$, so that we will identify k-points of G/B with right cosets $gB(k)$, for $g \in G(k)$. Let $\widetilde{\mathfrak{g}}$ be the Grothendieck simultaneous resolution of $\mathfrak{g}$: it coincides with the closed subscheme $\{(A, gB) \mid \mathrm{Ad}(g^{-1})A \in \mathfrak{b}\}$ of the product $\mathfrak{g} \times_k G/B$. Let π_1 and π_2 be the projections of $\widetilde{\mathfrak{g}} \times_{\mathfrak{g}} \widetilde{\mathfrak{g}}$ onto $\widetilde{\mathfrak{g}}$ with respect to the first and second factors.

Lemma 2.4. *Let $g \in G(k)$. The tangent space of $\widetilde{\mathfrak{g}}$ at the point $(0, gB(k))$ of $\mathfrak{g} \times G/B$ is the k-linear subspace $g\mathfrak{b}g^{-1} \oplus T_{gB(k)}G/B$ of $\mathfrak{g} \oplus T_{gB(k)}G/B$.*

Proof. Let B^- be the Borel subgroup of G, opposite to B and let U^- be the unipotent radical of B^-. There is an open embedding of U^- into G/B sending $u \in U^-$ to guB (see for example [Jan87, II.1.10]). This open embedding induces an isomorphism $T_{gB(k)}G/B \simeq T_{\mathrm{Id}}U^-$. This implies that the tangent space $T_{(0,gB(k))}\widetilde{\mathfrak{g}}$ can be identified with the set of pairs (A, C) in $\mathfrak{g} \times \operatorname{Lie} U^-$ such that $(\varepsilon A, g(\mathrm{Id} + \varepsilon C)) \in \widetilde{\mathfrak{g}}(L[\varepsilon])$, which means

$$(g(\mathrm{Id} + \varepsilon B))^{-1}\varepsilon A g(\mathrm{Id} + \varepsilon C) \in k[\varepsilon] \otimes_k \mathfrak{b}. \tag{2.5}$$

Using the fact that $\varepsilon^2 = 0$, (2.5) is equivalent to $g^{-1}Ag \in \mathfrak{b}$. □

The same kind of computation shows the following result.

Lemma 2.5. *The tangent space of $\widetilde{\mathfrak{g}} \times_{\mathfrak{g}} \widetilde{\mathfrak{g}}$ at the point $(gB(k), 0, hB(k)) \in G/B \times \mathfrak{g} \times G/B$ is the subspace*

$$T_{gB(k)}G/B \oplus (g\mathfrak{b}g^{-1} \cap h\mathfrak{b}h^{-1}) \oplus T_{hB(k)}G/B$$

of $T_{gB(k)}G/B \oplus \mathfrak{g} \oplus T_{hB(k)}G/B$.

Let T be a maximal split torus in G and let $\mathfrak{t}$ be its Lie algebra. The following result will prove essential later. Let us write $(G/B)^T$ for the set of k-points of G/B which are fixed by the group $T(k)$. Note that this set is in bijection with the Weyl group W of (G, T).

Theorem 2.6. *Let $hB(k) \in (G/B)(k)$. We have*

$$\sum_{gB(k) \in (G/B)^T} d\pi_2(T_{(gB(k),0,hB(k))}(\widetilde{\mathfrak{g}} \times_{\mathfrak{g}} \widetilde{\mathfrak{g}})) = T_{(0,hB(k))}\widetilde{\mathfrak{g}}$$

Proof. Let us remark that $gB(k) \in (G/B)^T$ if and only if $T \subset gB(k)g^{-1}$ which is equivalent to $\mathfrak{t} \subset g\mathfrak{b}g^{-1}$. Let $\mathcal{B}$ be the set of Borel sub-algebras of $\mathfrak{g}$ containing $\mathfrak{t}$. Using Lemmas 2.4 and 2.5, we see that the statement is equivalent to the following identity :

$$\sum_{\mathfrak{b}' \in \mathcal{B}} (\mathfrak{b}' \cap h\mathfrak{b}h^{-1}) = h\mathfrak{b}h^{-1}. \tag{2.6}$$

This, in turn, is a consequence of Theorem 2.3. □

3. Local deformation rings

Let $\mathcal{C}$ be the category of finite local L-algebras A with residue field isomorphic to L. If A is an object of $\mathcal{C}$ we denote by $\mathfrak{m}_A$ its unique maximal ideal.

3.1. **(φ, Γ_K)-modules.** Let K' be the maximal unramified extension of $\mathbf{Q}_p$ contained in $K(\mu_{p^\infty})$, it is a finite extension of $\mathbf{Q}_p$. Let $\mathcal{R}$ be the Robba ring of K defined as $\varinjlim_{r<1} \mathcal{R}^{]r,1[}$ where $\mathcal{R}^{]r,1[}$ is the ring of rigid analytic functions on the open annulus $\{r < |X| < 1\}$ over K'. This ring is a Bezout domain. If A is a finite dimensional $\mathbf{Q}_p$-algebra let $\mathcal{R}_A := A \otimes_{\mathbf{Q}_p} \mathcal{R}$. The ring $\mathcal{R}$ is endowed with a Frobenius endomorphism ϕ and a continuous action of the group $\Gamma_K := \mathrm{Gal}(K(\zeta_{p^\infty})/K)$ commuting with ϕ (see [KPX14, Def. 2.2.2]). The ring $\mathcal{R}$ contains an element t which is the image in $\mathcal{R}$ of the rigid analytic function $x \mapsto \log(1 + x)$ defined over the open unit disc over $\mathbf{Q}_p$. This element has the properties $\phi(t) = pt$ and $[\mathrm{rec}_K(a)] \cdot t = \chi_{\mathrm{cyc}}(\mathrm{rec}_K(a))t = at$, for $a \in K^\times$.

When A is an object of $\mathcal{C}$, we define a (φ, Γ_K)-module over $\mathcal{R}_A$ as a pair $(\mathcal{D}_A, \varphi)$ where $\mathcal{D}_A$ is a finite free $\mathcal{R}_A$-module, φ is a ϕ-semilinear endomorphism of $\mathcal{D}_A$ inducing an isomorphism $\mathcal{R}_A \otimes_{\mathcal{R}_A,\phi} \mathcal{D}_A \xrightarrow{\sim} \mathcal{D}_A$, and $\mathcal{D}_A$ is equipped with a continuous semilinear action of Γ_K commuting with φ (here $\mathcal{D}_A$ is a $\mathcal{R}$-module of finite type and has the canonical topology coming from the topology of $\mathcal{R}$). As A is a finite local $\mathbf{Q}_p$-algebra, this definition coincides with [KPX14, Def. 2.2.12].

If $\mathcal{D}_1$ and $\mathcal{D}_2$ are two (φ, Γ_K)-modules over $\mathcal{R}_A$. There is a (φ, Γ_K)-module $\mathrm{Hom}(\mathcal{D}_1, \mathcal{D}_2)$ defined over $\mathcal{R}_A$ whose underlying $\mathcal{R}_A$-module is the space of $\mathcal{R}_A$-linear maps from $\mathcal{D}_1$ to $\mathcal{D}_2$. Namely, $\mathcal{R}$ is a flat $\mathcal{R}$-module via ϕ, so that the canonical map $\mathcal{R} \otimes_{\mathcal{R},\phi} \mathrm{Hom}_{\mathcal{R}}(M_1, M_2) \to \mathrm{Hom}_{\mathcal{R}}(\mathcal{R} \otimes_{\mathcal{R},\phi} M_1, \mathcal{R} \otimes_{\mathcal{R},\phi} M_2)$ is an isomorphism, which isomorphism is used to define φ on $\mathrm{Hom}_{\mathcal{R}}(M_1, M_2)$.

For $i \geqslant 0$, the i-th cohomology group $H^i_{(\varphi,\Gamma_K)}(\mathcal{D})$ of a (φ, Γ_K)-module $\mathcal{D}$ is defined in [Liu07, 3.1]. If $\mathcal{D}_A$ is a (φ, Γ_K)-module over $\mathcal{R}_A$, it follows from [Liu07, Thm. 5.3] that $H^i_{(\varphi,\Gamma)}(\mathcal{D}_A)$ is of finite type over A and zero for $i > 2$.

For any continuous group homomorphism $\delta : K^\times \to L^\times$, we recall that we can construct a rank one (φ, Γ_K)-module $\mathcal{R}_L(\delta)$ over $\mathcal{R}_L$ such that the map $\delta \mapsto \mathcal{R}_L(\delta)$ induces a bijection between the set of continuous group homomorphisms $K^\times \to L^\times$ and the set of isomorphism classes of rank one (φ, Γ_K)-modules over $\mathcal{R}_L$ (see [KPX14, §6.1] for the precise construction of $\mathcal{R}_L(\delta)$).

By definition a (φ, Γ_K)-module over $\mathcal{R}[\frac{1}{t}]$ is a finite free $\mathcal{R}[\frac{1}{t}]$-module $\mathcal{M}$ with a ϕ-semilinear endomorphism φ and a semilinear action of Γ_K such that there exists a sub-$\mathcal{R}$-module $\mathcal{D}$ of $\mathcal{M}$ which is stable by φ and Γ_K, generates $\mathcal{M}$ as a $\mathcal{R}[\frac{1}{t}]$-module and is a (φ, Γ_K)-module over $\mathcal{R}$.

Lemma 3.1. *Let $\mathcal{M}$ be (φ, Γ_K)-module over $\mathcal{R}[\frac{1}{t}]$ and let $\mathcal{N}$ a sub-$\mathcal{R}[\frac{1}{t}]$-module of $\mathcal{M}$ which a direct factor as $\mathcal{R}[\frac{1}{t}]$-module and stable under φ and Γ_K. Then $\mathcal{N}$ is a (φ, Γ_K)-module over $\mathcal{R}[\frac{1}{t}]$.*

Proof. Let $\mathcal{D}$ be a sub-$\mathcal{R}$-module of $\mathcal{M}$, which is a (φ, Γ_K)-module and generates $\mathcal{M}$ as a $\mathcal{R}[\frac{1}{t}]$-module. It is sufficient to prove that $\mathcal{D}' := \mathcal{D} \cap \mathcal{N}$ is a (φ, Γ_K)-module over $\mathcal{R}$. It follows from [Ber02, Lemme 4.13] that $\mathcal{D}'$ is a finite free $\mathcal{R}$-module. Consequently we have to prove that the $\mathcal{R}$-linear map $\mathcal{R} \otimes_{\mathcal{R},\phi} \mathcal{D}' \to \mathcal{D}'$ given by the restriction of φ is an isomorphism. The snake lemma applied to the following morphism of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{R} \otimes_{\mathcal{R},\phi} \mathcal{D}' & \longrightarrow & \mathcal{R} \otimes_{\mathcal{R},\phi} \mathcal{D} & \longrightarrow & \mathcal{R} \otimes_{\mathcal{R},\phi} (\mathcal{D}/\mathcal{D}') & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \varphi_{\mathcal{D}'}} & & \downarrow{\scriptstyle \varphi_{\mathcal{D}}} & & \downarrow{\scriptstyle \varphi_{\mathcal{D}/\mathcal{D}'}} & & \\ 0 & \longrightarrow & \mathcal{D}' & \longrightarrow & \mathcal{D} & \longrightarrow & \mathcal{D}/\mathcal{D}' & \longrightarrow & 0 \end{array} /,$$

where $\varphi_{\mathcal{D}}$ is bijective by the very definition of a (φ, Γ_K)-module, shows that the map $\varphi_{\mathcal{D}/\mathcal{D}'}$ is surjective and that it is enough to check it is also injective to get that $\varphi_{\mathcal{D}'}$ is an isomorphism. Since $\mathcal{D}/\mathcal{D}'$ is $\mathcal{R}$-torsion free, its kernel is torsion free and a comparison of the ranks then shows that $\varphi_{\mathcal{D}/\mathcal{D}'}$ is injective as the ring $\mathcal{R}$ is Bezout. $\square$

If A is an object of $\mathcal{R}_A$, we define a (φ,Γ_K)-module over $\mathcal{R}_A[\frac{1}{t}]$ as being a (φ,Γ_K)-module $\mathcal{M}$ over $\mathcal{R}[\frac{1}{t}]$ together with a morphism of $\mathbf{Q}_p$-algebras from A into $\mathrm{End}_{\varphi,\Gamma_K}\mathcal{M}$ such that $\mathcal{M}$ is a finite free $\mathcal{R}_A[\frac{1}{t}]$-module.

3.2. **Filtered deformation functors.** Let A be an object of $\mathcal{C}$ and let $\mathcal{D}_A$ be a (φ,Γ_K)-module over $\mathcal{R}_A$. We define a filtration $\mathcal{F}$ of $\mathcal{D}_A$ as a sequence

$$0=\mathcal{F}_0\subset\mathcal{F}_1\subset\cdots\subset\mathcal{F}_m=\mathcal{D}_A$$

of sub-(φ,Γ_K)-modules of $\mathcal{D}_A$ such that each $\mathcal{F}_i$ is a direct factor of $\mathcal{D}_A$ as an $\mathcal{R}_A$-module. When $m=\mathrm{rk}_{\mathcal{R}_A}\mathcal{D}_A$ and each quotient $\mathcal{F}_i/\mathcal{F}_{i-1}$ is of rank 1 over $\mathcal{R}_A$, we say that $\mathcal{F}$ is a *triangulation* of $\mathcal{D}_A$.

Let $\mathcal{F}$ be a triangulation of a (φ,Γ_K)-module $\mathcal{D}$ over $\mathcal{R}_L$. For $1\leqslant i\leqslant d$, let δ_i the unique continuous morphism $K^\times\to L^\times$ such that $\mathcal{F}_i/\mathcal{F}_{i-1}\simeq\mathcal{R}_L(\delta_i)$. The character $\delta_1\otimes\cdots\otimes\delta_d$ from $(K^\times)^d$ to $L^\times$ depends only on $\mathcal{D}$ and $\mathcal{F}$ and is called the *parameter* of the triangulation $\mathcal{F}$.

If $\mathbf{k}=(k_\tau)_{\tau\in\Sigma}\in\mathbf{Z}^\Sigma$, we note $z^{\mathbf{k}}$ the character $z\mapsto\prod_{\tau\in\Sigma}\tau(z)^{k_\tau}$ from $K^\times$ into $L^\times$. A character $\delta_1\otimes\cdots\otimes\delta_d$ of $(K^\times)^d$ is called *regular* if, for all $i\neq j$, we have

$$\delta_i\delta_j^{-1}\notin\{z^{\mathbf{k}},z^{\mathbf{k}}\varepsilon_K;\ \mathbf{k}\in\mathbf{Z}^{[K:\mathbf{Q}_p]}\}. \tag{3.1}$$

From now we fix $\mathcal{D}$ a (φ,Γ_K)-module over $\mathcal{R}_L$ and $\mathcal{F}$ a filtration of $\mathcal{D}$. If A is an object of $\mathcal{C}$, we define $\mathfrak{X}_{\mathcal{D},\mathcal{F}}(A)$ as the set of isomorphism classes of triples $(\mathcal{D}_A,\pi,\mathcal{F}_A)$ where $\mathcal{D}_A$ is a (φ,Γ_K)-module over $\mathcal{R}_A$, π is an $\mathcal{R}_A$-linear map from $\mathcal{D}_A$ to $\mathcal{D}$ commuting to φ and Γ_K, inducing an isomorphism $\mathcal{D}_A\otimes_A L\xrightarrow{\sim}\mathcal{D}$, and $\mathcal{F}_A=(\mathcal{F}_{A,i})_{0\leqslant i\leqslant m}$ is a filtration of $\mathcal{D}_A$ such that $\pi(\mathcal{F}_{A,i})=\mathcal{F}_i$ for all $0\leqslant i\leqslant m$. This construction can be promoted naturally into a functor from $\mathcal{C}$ to the category of sets. In the case $K=\mathbf{Q}_p$, the functor $\mathfrak{X}_{\mathcal{D},\mathcal{F}}$ was defined by Chenevier in [Che11]. A direct adaptation of [Che11, Prop. 3.4] to our context shows that the functor $\mathfrak{X}_{\mathcal{D},\mathcal{F}}$ admits a versal deformation L-algebra, i.e. a complete noetherian local L-algebra R such that

$$\mathrm{Hom}_{\mathrm{pro}-\mathcal{C}}(R,-)\simeq\mathfrak{X}_{\mathcal{D},\mathcal{F}}.$$

When $\mathcal{F}=(0\subset\mathcal{D})$, we simply write $\mathfrak{X}_{\mathcal{D}}$ for the functor $\mathfrak{X}_{\mathcal{D},\mathcal{F}}$, which then coincides with the deformation functor of $\mathcal{D}$. There is a natural map of functors $\mathfrak{X}_{\mathcal{D},\mathcal{F}}\to\mathfrak{X}_{\mathcal{D}}$ which is defined by $(\mathcal{D}_A,\pi,\mathcal{F}_A)\mapsto(\mathcal{D}_A,\pi)$. If we assume in addition that $\mathrm{Hom}_{(\varphi,\Gamma_K)}(\mathrm{gr}_i(\mathcal{D}),\mathcal{D}/\mathcal{F}_i)=0$ for all i, then, adapting the argument of [Che11, Prop. 3.6] to the more general setting that we consider here shows that the map of functors $\mathfrak{X}_{\mathcal{D},\mathcal{F}}\to\mathfrak{X}_{\mathcal{D}}$ is injective and therefore we can identify $\mathfrak{X}_{\mathcal{D},\mathcal{F}}$ with a subfunctor of $\mathfrak{X}_{\mathcal{D}}$. In the particular case that $\mathcal{F}$ is a triangulation, the functor $\mathfrak{X}_{\mathcal{D},\mathcal{F}}$ was introduced in [BC09, Def. 2.3.2]; then the map of functors $\mathfrak{X}_{\mathcal{D},\mathcal{F}}\to\mathfrak{X}_{\mathcal{D}}$ is relatively representable if we assume $\mathrm{Hom}_{(\varphi,\Gamma_K)}(\mathrm{gr}_i(\mathcal{D}),\mathcal{D}/\mathcal{F}_i)=0$ for all i (see [BC09, Prop. 2.3.9]), an assumption that is satisfied if $\mathcal{F}$ is a triangulation of $\mathcal{D}$ with regular parameter as it follows from [Liu07, Prop. 3.10.(1)]. If $\mathcal{F}$ is a filtration of $\mathcal{D}$, let $\operatorname{End}_{\mathcal{F}}\mathcal{D}$ be the sub-$\mathcal{R}_L$-module of $\operatorname{End}\mathcal{D}$ whose elements are $\mathcal{R}_L$-linear maps respecting $\mathcal{F}$. It is a sub-(φ,Γ_K)-module of $\operatorname{End}\mathcal{D}$. A direct generalization of [Che11, Prop. 3.6] shows that if $H^2_{(\varphi,\Gamma_K)}(\operatorname{End}_{\mathcal{F}}\mathcal{D}) = 0$, the functor $\mathfrak{X}_{\mathcal{D},\mathcal{F}}$ is formally smooth. In particular if $H^2_{(\varphi,\Gamma_K)}(\operatorname{End}\mathcal{D}) = 0$, the functor $\mathfrak{X}_{\mathcal{D}}$ is formally smooth, which implies that a versal deformation ring for $\mathfrak{X}_{\mathcal{D}}$ is a formally smooth complete noetherian local L-algebra with residual field isomorphic to L, i.e. of the form $L[\![X_1,\dots,X_d]\!]$ for some non-negative integer d.

For the purpose of this paper we need an other kind of deformation problem. Let A be an object of $\mathcal{C}$ and $\mathcal{D}_A$ a (φ,Γ_K)-module over $\mathcal{R}_A$. The properties of the element $t\in\mathcal{R}$ show that the endomorphism ϕ and the action of Γ_K extends canonically to the ring $\mathcal{R}_A[\frac{1}{t}]$ and, if $\mathcal{D}$ is a (φ,Γ_K)-module over $\mathcal{R}_A$, there are canonical semilinear extensions of φ and of the action of Γ_K to $\mathcal{D}[\frac{1}{t}]$. A filtration of $\mathcal{D}_A[\frac{1}{t}]$ is a sequence

$$\mathcal{M} = \left(0\subset\mathcal{M}_1\subset\cdots\mathcal{M}_m = \mathcal{D}_A\left[\tfrac{1}{t}\right]\right)$$

by sub-$\mathcal{R}_A[\frac{1}{t}]$-modules which are direct factors and are stable under φ and Γ_K.

Remark 3.2. If $\mathcal{F} = (\mathcal{F}_i)_{0\leqslant i\leqslant m}$ is a filtration of $\mathcal{D}_A$, the family $\mathcal{F}[\frac{1}{t}] \coloneqq (\mathcal{F}_i[\frac{1}{t}])$ is a filtration of $\mathcal{D}_A[\frac{1}{t}]$. However, if $(\mathcal{M}_i)_{0\leqslant i\leqslant m}$ is a filtration of $\mathcal{D}_A[\frac{1}{t}]$, the family $(\mathcal{M}_i\cap\mathcal{D}_A)_{0\leqslant i\leqslant m}$ need not be a filtration of $\mathcal{D}_A$ since the $\mathcal{R}_A$-modules $\mathcal{M}_i\cap\mathcal{D}_A$ may fail to be projective.

A family of the form $(\mathcal{M}_i\cap\mathcal{D}_A)_{0\leqslant i\leqslant m}$, for a given filtration $\mathcal{M}_i$ of $\mathcal{D}_A[\frac{1}{t}]$, is what we call loosely an unsaturated filtration of $\mathcal{D}_A$. When $A = L$, the fact that $\mathcal{R}$ is a Bezout domain implies that the family $\mathcal{M}\cap\mathcal{D}_A \coloneqq (\mathcal{M}_i\cap\mathcal{D}_A)_{0\leqslant i\leqslant m}$ is actually a filtration of $\mathcal{D}_A$ and the map $\mathcal{M}\mapsto\mathcal{M}\cap\mathcal{D}_A$ is a bijection from the set of filtrations of $\mathcal{D}_A[\frac{1}{t}]$ onto the set of filtrations of $\mathcal{D}_A$ whose inverse is $\mathcal{F}\mapsto\mathcal{F}[\frac{1}{t}]$.

Let $\mathcal{D}$ be a (φ,Γ_K)-module over $\mathcal{R}_L$ and let $\mathcal{M}$ be a filtration of $\mathcal{D}[\frac{1}{t}]$. If A is an object of $\mathcal{C}$, we define $\mathfrak{X}_{\mathcal{D},\mathcal{M}}(A)$ as the set of isomorphism classes of triples $(\mathcal{D}_A,\pi,\mathcal{M}_A)$ where $\mathcal{D}_A$ is a (φ,Γ_K)-module over $\mathcal{R}_A$, π_A is a (φ,Γ_K)-module morphism $\mathcal{D}_A\to\mathcal{D}$ inducing an isomorphism $L\otimes_A\mathcal{D}_A\xrightarrow{\sim}\mathcal{D}$ and $\mathcal{M}_A$ is a filtration of $\mathcal{D}_A[\frac{1}{t}]$ such that $\pi(\mathcal{M}_{A,i}) = \mathcal{M}_i$. The construction $A\mapsto\mathfrak{X}_{\mathcal{D},\mathcal{M}}(A)$ can be promoted into a functor from $\mathcal{C}$ to the category of sets. When $\mathcal{F}\coloneqq\mathcal{M}\cap\mathcal{D}$ we can check that the map $(\mathcal{D}_A,\pi_A,\mathcal{F}_A)\mapsto(\mathcal{D}_A,\pi_A,\mathcal{F}_A[\frac{1}{t}])$ induces an injection of functors $\mathfrak{X}_{\mathcal{D},\mathcal{F}}\hookrightarrow\mathfrak{X}_{\mathcal{D},\mathcal{M}}$ and we use it to identify $\mathfrak{X}_{\mathcal{D},\mathcal{F}}$ with a subfunctor of $\mathfrak{X}_{\mathcal{D},\mathcal{M}}$.

Remark 3.3. When $\mathcal{M}$ is a triangulation of $\mathcal{D}$, the functor $\mathfrak{X}_{\mathcal{D},\mathcal{M}}$ coincides with the functor of isomorphism classes of the groupoid $X_{\mathcal{D},\mathcal{M}}$ introduced in [BHS, § 3.5].

The following statement is a direct consequence of the definitions; we state it for the sake of completeness and comfort of reading.

Scholium 3.4. *Let $\pi : \mathfrak{X} \to \mathfrak{Y}$ be a relatively representable morphism between functors from $\mathcal{C}$ to the category of sets. If $\mathfrak{Y}$ admits a versal deformation L-algebra, then $\mathfrak{X}$ admits a versal deformation L-algebra. More precisely if* $\operatorname{Spf} R \to \mathfrak{Y}$ *is a hull for $\mathfrak{Y}$ then the functor* $\operatorname{Spf} R \times_{\mathfrak{Y}} \mathfrak{X}$ *is pro-representable by a local complete noetherian L-algebra S and* $\operatorname{Spf} S \to \mathfrak{X}$ *is a hull for $\mathfrak{X}$.*

We will also need the following fact which is a direct consequence of [BHS, Prop. 3.4.6].

Proposition 3.5. *Let $\mathcal{D}$ be a (φ, Γ_K)-module and let $\mathcal{F}$ be a triangulation of $\mathcal{D}$ whose parameter is regular in the sense of* (3.1). *Let $\mathcal{M} := \mathcal{F}[\frac{1}{t}]$. Then the forgetful map $\mathfrak{X}_{\mathcal{D},\mathcal{M}} \to \mathfrak{X}_{\mathcal{D}}$ is injective and relatively representable. This implies that $\mathfrak{X}_{\mathcal{D},\mathcal{M}}$ admits a versal deformation L-algebra.*

This means that the map $\mathfrak{X}_{\mathcal{D},\mathcal{M}} \to \mathfrak{X}_{\mathcal{D}}$ is injective and that for all objects A in $\mathcal{C}$ and all $x \in \mathfrak{X}_{\mathcal{D}}(A)$, there is a unique quotient A_x of A such that for any map $A \to B$ in $\mathcal{C}$, the image of x in $\mathfrak{X}_{\mathcal{D}}(B)$ is in $\mathfrak{X}_{\mathcal{D},\mathcal{M}}(B)$ if and only if the map $A \to B$ factors through A_x.

3.3. **Crystalline (φ, Γ_K)-modules.** Let $K_0 = W(k_K)[\frac{1}{p}]$, let σ the absolute Frobenius automorphism of K_0 and $f = [k_K : \mathbf{F}_p] = [K_0 : \mathbf{Q}_p]$. If A is an object of $\mathcal{C}$, an *isocrystal* over k_K with coefficients in A is a pair (V, φ) where V is a finite projective $A \otimes_{\mathbf{Q}_p} K_0$-module and φ is an $\mathrm{Id}_A \otimes \sigma$-semilinear automorphism of V. Actually these conditions automatically imply that V is a finite free $A \otimes_{\mathbf{Q}_p} K_0$-module. Its rank is by definition its rank as a $A \otimes_{\mathbf{Q}_p} K_0$-module. If (V, φ) is an isocrystal over k_K with coefficients in A, we define $\chi(V, \varphi)$ as the characteristic polynomial of the $A \otimes_{\mathbf{Q}_p} K_0$-linear endomorphism φ^f. This polynomial is invariant under $\mathrm{Id}_A \otimes \sigma$, to the effect that $\chi(V, \varphi)$ lies in $A[X]$. Assume now that $A = L$. If $\chi(V, \varphi) = PQ$ where P and Q are coprime elements in $L[X]$, then there exists a unique φ-stable $L \otimes_{\mathbf{Q}_p} K_0$-submodule $W \subset V$ such that $\chi(W, \varphi|_W) = P$. Actually we have explicitely $W = \ker Q(\varphi^f)$.

Recall that there exists a left exact functor D_{cris} from the category of (φ, Γ_K)-modules over $\mathcal{R}_L$ to the category of isocrystals over k_K with coefficients in L. It is defined by $D_{\mathrm{cris}}(\mathcal{D}) := \mathcal{D}[\frac{1}{t}]^{\Gamma_K}$ (see [BC09, 2.2.7.] for the case where $K = \mathbf{Q}_p$). We say that a (φ, Γ_K)-module $\mathcal{D}$ over $\mathcal{R}_L$ is *crystalline* if $\mathrm{rk}_{K_0} D_{\mathrm{cris}}(\mathcal{D}) = \mathrm{rk}_{\mathcal{R}} \mathcal{D}$. Let $\mathcal{D}$ be a crystalline (φ, Γ_K)-module over $\mathcal{R}_L$. Arguing as in [BC09, 2.4.2.], there exists a bijection between sub-(φ, Γ_K)-modules of $\mathcal{D}$ which are direct summands as $\mathcal{R}_L$-module and φ-stable sub-$L \otimes_{\mathbf{Q}_p} K_0$-modules of $D_{\mathrm{cris}}(\mathcal{D})$.

A *refinement* of a rank d isocrystal (D, φ) over k is a filtration $F = (F_i)_{0 \leqslant i \leqslant d}$ of D

$$F_0 = 0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_d = D$$

such that each F_i is a $L \otimes_{\mathbf{Q}_p} K_0$-submodule stable under φ. Note that each F_i is necessarily free over $L \otimes_{\mathbf{Q}_p} K_0$ and consequently of rank i.

If $\mathcal{D}$ is a crystalline (φ, Γ_K)-module over $\mathcal{R}_L$, there is consequently a bijection $\mathcal{F} = (\mathcal{F}_i)_{0 \leqslant i \leqslant d} \mapsto D_{\mathrm{cris}}(\mathcal{F}) \coloneqq (D_{\mathrm{cris}}(\mathcal{F}_i))_{0 \leqslant i \leqslant d}$ between the set of triangulations of $\mathcal{D}$ and the set of refinements of $D_{\mathrm{cris}}(\mathcal{D})$. A refinement F of a k-isocrystal (D, φ) gives rise to a decomposition of $\chi(V, \varphi)$ as a product of polynomials of degree one

$$\chi(D, \varphi) = \prod_{i=1}^{n} \chi(F_i / F_{i-1}, \varphi).$$

In particular $\chi(D, \varphi)$ is split over L and the triangulation defines an ordering $(\phi_1, \dots, \phi_d)$ of the roots of $\chi(D, \varphi)$ such that $\chi(F_i, \varphi) = \prod_{j=1}^{i}(X - \phi_j)$. We define δ_F to be the unramified character $T(K) \to L^\times$ given by the formula

$$(a_1, \dots, a_d) \longmapsto \prod_{i=1}^{d} \phi_i^{v_K(a_i)}.$$

If $(D, \varphi) = D_{\mathrm{cris}}(\mathcal{D})$ for a crystalline (φ, Γ_K)-module $\mathcal{D}$ over $\mathcal{R}_L$ and if $\mathcal{F}$ is a triangulation of $\mathcal{D}$, we define $\delta_{\mathcal{F}} \coloneqq \delta_{D_{\mathrm{cris}}(\mathcal{F})}$. It follows from the classification of sub-(φ, Γ_K)-modules of rank one (φ, Γ_K)-modules ([KPX14, Prop. 6.2.8.(1)]) that the parameter of the triangulation $\mathcal{F}$ is the product of $\delta_{\mathcal{F}}$ with an algebraic character of $(K^\times)^d$.

Conversely if the polynomial $\chi(D, \varphi)$ is separable and split in $L[X]$, each ordering $(\phi_1, \dots, \phi_d)$ of its roots comes from a unique refinement of D. In this case, the character δ_F completely determines the refinement F.

We say that a crystalline (φ, Γ_K)-module $\mathcal{D}$ over $\mathcal{R}_L$ is *φ-generic* if the polynomial $\chi(D_{\mathrm{cris}}(\mathcal{D}))$ is separable split over L with pairwise distinct roots $(\phi_1, \dots, \phi_d)$ such that $\phi_i \phi_j^{-1} \neq p^f$ for $i \neq j$. This property in particular implies that for each triangulation $\mathcal{F}$ of $\mathcal{D}$, the parameter of $\mathcal{F}$ is regular so that the assumption on the triangulation $\mathcal{F}$ in Proposition 3.5 is satisfied. As a consequence we have the following relatively representable inclusions

$$\mathfrak{X}_{\mathcal{D}, \mathcal{F}} \subset \mathfrak{X}_{\mathcal{D}, \mathcal{F}[\frac{1}{t}]} \subset \mathfrak{X}_{\mathcal{D}},$$

where both $\mathfrak{X}_{\mathcal{D}}$ and $\mathfrak{X}_{\mathcal{D}, \mathcal{F}}$ are formally smooth, unlike the functor $\mathfrak{X}_{\mathcal{D}, \mathcal{F}[\frac{1}{t}]}$ which does not share the property in full generality.

Let $\mathcal{D}$ be a crystalline (φ, Γ_K)-module over $\mathcal{R}_L$. For an object A of $\mathcal{C}$, let $\mathfrak{X}_{\mathcal{D}}^{\mathrm{cris}}(A)$ the subset of $\mathfrak{X}_{\mathcal{D}}(A)$ of isomorphism classes of pairs $(\mathcal{D}_A, \pi_A)$ with $\mathcal{D}_A$ a crystalline (φ, Γ_K)-module. The subfunctor $A \mapsto \mathfrak{X}_{\mathcal{D}}^{\mathrm{cris}}(A)$ of $\mathfrak{X}_{\mathcal{D}}$ is simply denoted $\mathfrak{X}_{\mathcal{D}}^{\mathrm{cris}}$. If $\mathcal{D}_A$ is crystalline, the $A \otimes_{\mathbf{Q}_p} K_0$-module $D_{\mathrm{cris}}(\mathcal{D}_A)$ is finite free of rank $\mathrm{rk}_{\mathcal{R}_L} \mathcal{D}$. Assume moreover that $\mathcal{D}$ is φ-generic crystalline. Let $\mathcal{F}$ be a triangulation of $\mathcal{D}$ with associated refinement $F = D_{\mathrm{cris}}(\mathcal{F})$ and set $\mathcal{M} = \mathcal{F}[\frac{1}{t}]$.

Lemma 3.6. *Let A be an object of $\mathcal{C}$ and let $(\mathcal{D}_A,\pi_A)\in\mathfrak{X}_{\mathcal{D}}^{\mathrm{cris}}(A)$. There exists a unique complete flag F_A of $A\otimes_{\mathbf{Q}_p}K_0$-submodules of $D_{\mathrm{cris}}(\mathcal{D}_A)$ which is stable under φ and reduces to F modulo $\mathfrak{m}_A$.*

Proof. By assumption, the polynomial $\chi(D_{\mathrm{cris}}(\mathcal{D}))$ is separable split in $L[X]$ so that we can write

$$\chi(D_{\mathrm{cris}}(\mathcal{D}))=\prod_{i=1}^{n}(X-x_i)$$

and assume that the filtration F is given by $F_i=\ker\prod_{j=1}^{i}(\varphi^f-x_j)$.

Let $\chi_A(D_{\mathrm{cris}}(\mathcal{D}_A))\in A[X]$ be the characteristic polynomial of the $A\otimes_{\mathbf{Q}_p}K_0$-linear endomorphism φ^f of $D_{\mathrm{cris}}(\mathcal{D}_A)$. The reduction modulo $\mathfrak{m}_A$ of $\chi_A(D_{\mathrm{cris}}(\mathcal{D}_A))$ is the polynomial $\chi(D_{\mathrm{cris}}(\mathcal{D}))\in L[X]$ which is separable split in $L[X]$. Thus there exists a unique $(\tilde{x}_1,\dots,\tilde{x}_n)\in A^n$ such that $\chi_A(D_{\mathrm{cris}}(\mathcal{D}_A))=\prod_{i=1}^{n}(X-\tilde{x}_i)$ and, for $1\leqslant i\leqslant n$, $\tilde{x}_i\equiv x_i \mod \mathfrak{m}_A$. Considering the characteristic polynomials of the $\varphi^f|_{F_{A,i}}$ we can check that $F_{A,i}=\ker\prod_{j=1}^{i}(\varphi^f-\tilde{x}_j)$ defines the desired filtration. On the other hand any complete flag with the desired properties must fulfill this condition. □

Let

$$\mathcal{M}_A:=\left(\mathcal{R}_A\left[\tfrac{1}{t}\right]\otimes_{K_0\otimes_{\mathbf{Q}_p}A}F_{A,1}\subsetneq\cdots\subsetneq\mathcal{R}_A\left[\tfrac{1}{t}\right]\otimes_{K_0\otimes_{\mathbf{Q}_p}A}F_{A,n}=\mathcal{D}_A\left[\tfrac{1}{t}\right]\right)$$

where we used the canonical isomorphism ([Ber02, Thm. 0.2])

$$\mathcal{R}_A\left[\tfrac{1}{t}\right]\otimes_{K_0\otimes_{\mathbf{Q}_p}A}D_{\mathrm{cris}}(\mathcal{D}_A)\simeq\mathcal{D}_A\left[\tfrac{1}{t}\right]$$

and F_A is the filtration whose existence is proved in Lemma 3.6. Then $(\mathcal{D}_A,\pi_A,\mathcal{M}_A)$ is an element of $\mathfrak{X}_{\mathcal{D},\mathcal{M}}(A)$. This implies that we have a sequence of inclusions

$$\mathfrak{X}_{\mathcal{D}}^{\mathrm{cris}}\subset\mathfrak{X}_{\mathcal{D},\mathcal{F}[\frac{1}{t}]}\subset\mathfrak{X}_{\mathcal{D}}.$$

Remark 3.7. We point out that in general $\mathfrak{X}_{\mathcal{D}}^{\mathrm{cris}}$ does not embed into $\mathfrak{X}_{\mathcal{D},\mathcal{F}}$. This is only true if we impose some conditions on the relative position of $\mathcal{F}$ with respect to the Hodge filtration (see below).

3.4. **$\mathbf{B}_{\mathrm{dR}}^+$-representations.** If $\mathcal{D}$ is a (φ,Γ_K)-module over $\mathcal{R}$, we note $W_{\mathrm{dR}}^+(\mathcal{D})$ the $\mathbf{B}_{\mathrm{dR}}^+$-representation of $\mathcal{G}_K$ constructed in [Ber08, Prop. 2.2.6.]. The rank of the $\mathbf{B}_{\mathrm{dR}}^+$-module $W_{\mathrm{dR}}^+(\mathcal{D})$ is equal to the rank of the $\mathcal{R}$-module $\mathcal{D}$. We obtain an exact functor W_{dR}^+ from the category of (φ,Γ_K)-modules over $\mathcal{R}$ to the category of $\mathbf{B}_{\mathrm{dR}}^+$-representations of $\mathcal{G}_K$.

If A is an object of $\mathcal{C}$, an $A\otimes_{\mathbf{Q}_p}\mathbf{B}_{\mathrm{dR}}^+$-representation of $\mathcal{G}_K$ is a finite free $A\otimes_{\mathbf{Q}_p}\mathbf{B}_{\mathrm{dR}}^+$-module with a continuous semilinear action of $\mathcal{G}_K$. If $\mathcal{D}_A$ is a (φ,Γ_K)-module over $\mathcal{R}_A$, the $L\otimes_{\mathbf{Q}_p}\mathbf{B}_{\mathrm{dR}}^+$-representation $W_{\mathrm{dR}}^+(\mathcal{D}_A)$ is actually an $A\otimes_{\mathbf{Q}_p}\mathbf{B}_{\mathrm{dR}}^+$-module with a continuous semilinear action of $\mathcal{G}_K$. It follows from [BHS, Lemma 3.3.5.(i)] that $W^+_{\mathrm{dR}}(\mathcal{D}_A)$ is a finite free $A\otimes_{\mathbf{Q}_p}\mathbf{B}^+_{\mathrm{dR}}$-module and consequently an $A\otimes_{\mathbf{Q}_p}\mathbf{B}^+_{\mathrm{dR}}$-representation of $\mathcal{G}_K$. If W is an $L\otimes_{\mathbf{Q}_p}\mathbf{B}^+_{\mathrm{dR}}$-representation of $\mathcal{G}_K$ we define $\mathfrak{X}_W$ the functor from $\mathcal{C}$ to the category of sets such that $\mathfrak{X}_W(A)$ is the set of equivalence classes of pairs (W_A,π_A) where W_A is a $A\otimes_{\mathbf{Q}_p}\mathbf{B}^+_{\mathrm{dR}}$-representation of $\mathcal{G}_K$ and π_A is an $A\otimes_{\mathbf{Q}_p}\mathbf{B}^+_{\mathrm{dR}}$-linear and $\mathcal{G}_K$-equivariant morphism from W_A to W inducing an isomorphism $L\otimes_A W_A\xrightarrow{\sim}W$. If $\mathcal{D}$ is a (φ,Γ_K)-module over $\mathcal{R}_L$, the functor W^+_{dR} induces a map from $\mathfrak{X}_{\mathcal{D}}$ to $\mathfrak{X}_{W^+_{\mathrm{dR}}(\mathcal{D})}$.

Let $\mathcal{D}$ be a (φ,Γ_K)-module over $\mathcal{R}$. Let $D_{\mathrm{dR}}(\mathcal{D}) := (W^+_{\mathrm{dR}}(\mathcal{D})\otimes_{\mathbf{B}^+_{\mathrm{dR}}}\mathbf{B}_{\mathrm{dR}})^{\mathcal{G}_K}$. The *de Rham filtration* on $D_{\mathrm{dR}}(\mathcal{F})$ is defined by

$$\mathrm{Fil}^i_{\mathrm{dR}}(D_{\mathrm{dR}}(\mathcal{D})) := (t^iW^+_{\mathrm{dR}}(\mathcal{D}))^{\mathcal{G}_K}\subset D_{\mathrm{dR}}(\mathcal{D})$$

We say that $\mathcal{D}$ is *de Rham* if $\dim_K D_{\mathrm{dR}}(\mathcal{D})=\mathrm{rk}_{\mathcal{R}}\mathcal{D}$. If A is an object of $\mathcal{C}$ and if $\mathcal{D}_A$ is a (φ,Γ_K)-module over $\mathcal{R}_A$, then $D_{\mathrm{dR}}(\mathcal{D}_A)$ is a $A\otimes_{\mathbf{Q}_p}K$-module. If we assume that $\mathcal{D}_A$ is de Rham, then it is finite free over $A\otimes_{\mathbf{Q}_p}K$, and each $\mathrm{Fil}^i_{\mathrm{dR}}D_{\mathrm{dR}}(\mathcal{D})$ is a sub-$A\otimes_{\mathbf{Q}_p}K$-module.

A *filtered* $L\otimes_{\mathbf{Q}_p}K$-module is a finite free $L\otimes_{\mathbf{Q}_p}K$-module with a separated and exhaustive filtration by sub-$L\otimes_{\mathbf{Q}_p}K$-modules. The functor D_{dR} is a left exact functor from the category of (φ,Γ_K)-modules over L to the category of filtered $L\otimes_{\mathbf{Q}_p}K$-modules. The restriction of the functor D_{dR} to the subcategory of de Rham (φ,Γ_K)-modules is exact and a crystalline (φ,Γ_K)-module over $\mathcal{R}_L$ is de Rham. Moreover there is a canonical isomorphism of $L\otimes_{\mathbf{Q}_p}K$-modules $D_{\mathrm{cris}}(\mathcal{D})\otimes_{K_0}K\simeq D_{\mathrm{dR}}(\mathcal{D})$ (see for example [Ber08, Prop. 2.3.3]).

Let $\mathcal{D}$ be a crystalline (φ,Γ_K)-module over $\mathcal{R}_L$. For all $\tau\in\Sigma$, we define

$$D_{\mathrm{dR},\tau}(\mathcal{D}) := L\otimes_{K\otimes_{\mathbf{Q}_p}L,\tau}D_{\mathrm{dR}}(\mathcal{D})$$

It is a direct factor of $D_{\mathrm{dR}}(\mathcal{D})$ and we define a separated and exhaustive filtration on $D_{\mathrm{dR},\tau}(\mathcal{D})$ by

$$\mathrm{Fil}^i_{\mathrm{dR},\tau}D_{\mathrm{dR},\tau}(\mathcal{D}) := L\otimes_{K\otimes_{\mathbf{Q}_p}L,\tau}(\mathrm{Fil}^i_{\mathrm{dR}}D_{\mathrm{dR}}(\mathcal{D}))$$

A *Hodge-Tate type* is an element $\mathbf{k}=(\mathbf{k}_\tau)_{\tau\in\Sigma}\in(\mathbf{Z}^n)^{[K:\mathbf{Q}_p]}$ where each $\mathbf{k}_\tau$ is an increasing sequence of integers. We say that the Hodge-Tate type is *regular* if all these sequences of integers are *strictly* increasing. If $\mathcal{D}$ is a de Rham (φ,Γ_K)-module, *its Hodge-Tate type* is by definition $(k_{1,\tau}\leqslant\cdots\leqslant k_{n,\tau})_{\tau\in\Sigma}$ where the $k_{i,\tau}$ are the integers m such that $\mathrm{gr}^{-m}D_{\mathrm{dR},\tau}(\mathcal{D})\neq 0$, counted with multiplicity, where the multiplicity of m is defined as the dimension $\dim_L\mathrm{gr}^{-m}D_{\mathrm{dR},\tau}(\mathcal{D})$.

Let $\mathcal{D}$ be a crystalline (φ,Γ_K)-module over $\mathcal{R}_L$ and let $\mathcal{F}$ be a triangulation of $\mathcal{D}$. We say that $\mathcal{F}$ is *non critical*, if for all $1\leqslant i\leqslant\mathrm{rk}_{\mathcal{R}_L}\mathcal{D}$ and for all $\tau\in\Sigma$, there exists some $m\in\mathbf{Z}$ such that

$$(L\otimes_{K_0\otimes_{\mathbf{Q}_p}L,\tau|_{K_0}}D_{\mathrm{cris}}(\mathcal{F}_i))\oplus\mathrm{Fil}^m_{\mathrm{dR},\tau}=D_{\mathrm{dR},\tau}(\mathcal{D}).$$

In this case we obviously have $m+i = \mathrm{rk}\,\mathcal{D}$. With the help of the functor W_{dR}^+, we can easily construct an exact functor W_{dR} from the category of (φ,Γ_K)-modules over $\mathcal{R}[\frac{1}{t}]$ to the category of $\mathbf{B}_{\mathrm{dR}}$-representations of $\mathcal{G}_K$ such that, for $\mathcal{D}$ a (φ,Γ_K)-module over $\mathcal{R}$, we have

$$W_{\mathrm{dR}}\left(\mathcal{D}\left[\tfrac{1}{t}\right]\right) = W_{\mathrm{dR}}^+(\mathcal{D}) \otimes_{\mathbf{B}_{\mathrm{dR}}^+} \mathbf{B}_{\mathrm{dR}}.$$

If A is an object of $\mathcal{C}$, the image by W_{dR} of a (φ,Γ_K)-module over $\mathcal{R}_A[\frac{1}{t}]$ is a finite free as $A\otimes_{\mathbf{Q}_p}\mathbf{B}_{\mathrm{dR}}$-module and consequently an $A\otimes_{\mathbf{Q}_p}\mathbf{B}_{\mathrm{dR}}$-representation of $\mathcal{G}_K$ (see [BHS, Lemma 3.3.5.(ii)]).

If A is an object of $\mathcal{C}$ and W_A is an $A\otimes_{\mathbf{Q}_p}\mathbf{B}_{\mathrm{dR}}$-representation of $\mathcal{G}_K$ of rank n, we define a *complete flag* of W to be a filtration $(F_i)_{0\leqslant i\leqslant n}$ of W by sub-$A\otimes_{\mathbf{Q}_p}\mathbf{B}_{\mathrm{dR}}$-modules stable under $\mathcal{G}_K$ such that F_i is a free $A\otimes_{\mathbf{Q}_p}\mathbf{B}_{\mathrm{dR}}$-module of rank i.

Let W be a $L\otimes_{\mathbf{Q}_p}\mathbf{B}_{\mathrm{dR}}^+$-representation of $\mathcal{G}_K$ and let F be a complete flag of $W\otimes_{\mathbf{B}_{\mathrm{dR}}^+}\mathbf{B}_{\mathrm{dR}}$ stable under the action of $\mathcal{G}_K$. Let A be an object of the category $\mathcal{C}$. A *deformation of the pair* (W,F) over A is an element (W_A,π_A,F_A) where W_A is a $A\otimes_{\mathbf{Q}_p}\mathbf{B}_{\mathrm{dR}}^+$-representation of $\mathcal{G}_K$, π_A a $\mathcal{G}_K$-equivariant isomorphism from $W_A\otimes_A L$ to W and F_A a complete flag of $W_A\otimes_{\mathbf{B}_{\mathrm{dR}}^+}\mathbf{B}_{\mathrm{dR}}$ such that $F = (\pi_A\otimes \mathrm{Id}_{\mathbf{B}_{\mathrm{dR}}})(F_A)$. We denote by $\mathfrak{X}_{W,F}$ the functor from the category $\mathcal{C}$ to the category of sets, that maps an object A of $\mathcal{C}$ to the isomorphism class of deformations of (W,F).

Let $\mathcal{D}_A$ be a (φ,Γ_K)-module over $\mathcal{R}_A$ and $\mathcal{M}_A$ a triangulation of $\mathcal{D}_A[\frac{1}{t}]$. It follows from Lemma 3.1 that each $\mathcal{M}_{A,i}$ is a (φ,Γ_K)-module over $\mathcal{R}_A[\frac{1}{t}]$. Thus

$$W_{\mathrm{dR}}(\mathcal{M}_A) := \left(W_{\mathrm{dR}}(\mathcal{M}_{A,0}) \subset \cdots \subset W_{\mathrm{dR}}(\mathcal{M}_{A,n}) = W_{\mathrm{dR}}\left(\mathcal{D}\left[\tfrac{1}{t}\right]\right)\right)$$

is a complete flag of $W_{\mathrm{dR}}^+(\mathcal{D}_A)\otimes_{\mathbf{B}_{\mathrm{dR}}^+}\mathbf{B}_{\mathrm{dR}}$. For $\mathcal{D}$ a (φ,Γ_K)-module over $\mathcal{R}_L$ and $\mathcal{F}$ a triangulation of $\mathcal{D}$, we deduce from this fact that the functor W_{dR}^+ extends to a map of functors

$$\mathfrak{X}_{\mathcal{D},\mathcal{F}[\frac{1}{t}]} \longrightarrow \mathfrak{X}_{W_{\mathrm{dR}}^+(\mathcal{D}),W_{\mathrm{dR}}(\mathcal{F}[\frac{1}{t}])} \tag{3.2}$$

The following proposition is essentially [BHS, Cor. 3.5.6.].

Proposition 3.8. *If $\mathcal{D}$ is a φ-generic crystalline (φ,Γ_K)-module over $\mathcal{R}_L$ with regular Hodge-Tate type and $\mathcal{F}$ is a triangulation of $\mathcal{D}$, then the map* (3.2) *is formally smooth.*

Proposition 3.9. *Let $\mathcal{D}$ be a φ-generic crystalline (φ,Γ_K)-module over $\mathcal{R}_L$ of regular Hodge-Tate type. Then, for all objects A of $\mathcal{C}$, the preimage of the trivial $A\otimes_{\mathbf{Q}_p}\mathbf{B}_{\mathrm{dR}}^+$-representation of $\mathcal{G}_K$ under the map $W_{\mathrm{dR}}^+ : \mathfrak{X}_{\mathcal{D}}(A) \to \mathfrak{X}_{W_{dR}^+(\mathcal{D})}(A)$ identifies with $\mathfrak{X}_{\mathcal{D}}^{\mathrm{cris}}(A)$. Consequently the following sequence of L-vector spaces is exact*

$$0 \to T\mathfrak{X}_{\mathcal{D}}^{\mathrm{cris}} \to T\mathfrak{X}_{\mathcal{D}} \to T\mathfrak{X}_{W_{dR}^+(\mathcal{D})}.$$

Proof. An object $(\mathcal{D}_A, \pi_A)$ has a trivial image by W_{dR}^+ if and only if $\mathcal{D}_A$ is a de Rham (φ, Γ_K)-module. We can conclude as in the proof of [HS16, Cor. 2.7.(i)]. Namely it follows from the p-adic monodromy theorem ([Ber08, Thm. 2.3.5.(1)]) that $\mathcal{D}_A$ is a potentially semistable (φ, Γ_K)-module. Being an extension of finitely many cristalline representations, it is actually semistable. It follows from the φ-genericity assumption on $\mathcal{D}$ that no quotient of eigenvalues of φ^f on $D_{\mathrm{st}}(\mathcal{D}_A)$ can be equal to p^f, so that the monodromy operator of $D_{\mathrm{st}}(\mathcal{D}_A)$ is trivial. Hence $\mathcal{D}_A$ is crystalline. □

Corollary 3.10. *Let $\mathcal{D}$ be a φ-generic crystalline (φ, Γ_K)-module over $\mathcal{R}_L$ of regular Hodge-Tate type and $\mathcal{F}$ a triangulation of $\mathcal{D}$. Then, for all objects A of $\mathcal{C}$, the preimage of the trivial $A \otimes_{\mathbf{Q}_p} \mathbf{B}_{\mathrm{dR}}^+$-representation of $\mathcal{G}_K$ under the map* (3.2) *identifies with $\mathfrak{X}_{\mathcal{D}}^{\mathrm{cris}}(A)$. Moreover the following sequence is exact*

$$0 \longrightarrow T\mathfrak{X}_{\mathcal{D}}^{\mathrm{cris}} \longrightarrow T\mathfrak{X}_{\mathcal{D},\mathcal{F}[\frac{1}{t}]} \longrightarrow T\mathfrak{X}_{W_{\mathrm{dR}}^+(\mathcal{D}),W_{\mathrm{dR}}(\mathcal{F}[\frac{1}{t}])} \longrightarrow 0$$

Proof. The first assertion is a direct consequence of Proposition 3.9. The second assertion follows after evaluation at $L[\varepsilon]$. The exactness on the right is a consequence of Proposition 3.8. □

3.5. **Main theorem: the local version.** Let $\mathcal{D}$ be a φ-generic crystalline (φ, Γ_K)-module over $\mathcal{R}_L$. We write $\mathrm{Tri}(\mathcal{D})$ for the set of triangulations of $\mathcal{D}$, which is in bijection with the set of refinements of $D_{\mathrm{cris}}(\mathcal{D})$. The local theorem states as follows

Theorem 3.11. *Let $\mathcal{D}$ be a φ-generic crystalline (φ, Γ_K)-module over $\mathcal{R}_L$ of regular Hodge-Tate type. Let $\mathrm{Tri}(\mathcal{D})$ be the set of triangulations of $\mathcal{D}$. Then the L-linear map*

$$\bigoplus_{\mathcal{F} \in \mathrm{Tri}(\mathcal{D})} T\mathfrak{X}_{\mathcal{D},\mathcal{F}[\frac{1}{t}]} \longrightarrow T\mathfrak{X}_{\mathcal{D}}$$

is surjective.

Remark 3.12. The special case of the result where all refinements of D are assumed non critical is a theorem due to G. Chenevier for $K = \mathbf{Q}_p$ ([Che11, Thm. 3.19]) and to K. Nakamura for an arbitrary extension K of $\mathbf{Q}_p$ ([Nak13, Thm. 2.62.]).

Before giving the proof of Theorem 3.11, let us recall some constructions and results from [BHS], to which we refer the reader for relevent definitions if needed.

Let W be an almost de Rham $L \otimes_{\mathbf{Q}_p} \mathbf{B}_{\mathrm{dR}}^+$-representation of $\mathcal{G}_K$ (see [BHS, 3.1]). Let i be a $L \otimes_{\mathbf{Q}_p} K$-linear isomorphism $(L \otimes_{\mathbf{Q}_p} K)^n \xrightarrow{\sim} D_{\mathrm{pdR}}(W)$. Let $\mathfrak{X}_W^{\square}$ be the functor from $\mathcal{C}$ to the category of sets such that $\mathfrak{X}_W^{\square}(W)$ is the set of isomorphism classes of triples (W_A, π_A, i_A) where W_A is some $A \otimes_{\mathbf{Q}_p} \mathbf{B}_{\mathrm{dR}}^+$-representation of $\mathcal{G}_K$, π_A is map from W_A to W inducing an isomorphism from $L \otimes_A W_A$ to W and i_A is an isomorphism between $(A \otimes_{\mathbf{Q}_p} K)^n$ and $D_{\mathrm{pdR}}(W_A)$ compatible with π_A

and i. Let $\mathfrak{g}$ be the Lie algebra of the algebraic group $\mathrm{GL}_{n,K}$ and let $\tilde{\mathfrak{g}} \to \mathfrak{g}$ be Grothendieck's simultaneous resolution of singularities. As in section 2.2 we consider the scheme $X \coloneqq \tilde{\mathfrak{g}} \times_{\mathfrak{g}} \tilde{\mathfrak{g}}$. It follows from [BHS, Lem. 3.2.2] that the forgetful map $\mathfrak{X}_W^\square \to \mathfrak{X}_W$ is formally smooth, and [BHS, Thm. 3.2.5] states that the functor $\mathfrak{X}_W^\square$ is pro-representable by the completion of $\tilde{\mathfrak{g}}_{K/\mathbf{Q}_p,L}$ at the point $x = (0, i^{-1}(\mathrm{Fil}_{\mathrm{dR}}))$.

Let F be a complete flag of $W \otimes_{\mathbf{B}_{\mathrm{dR}}^+} \mathbf{B}_{\mathrm{dR}}$ stable under $\mathcal{G}_K$. We can define $\mathfrak{X}_{W,F}$ as in section 3.4 and $\mathfrak{X}_{W,F}^\square$ by adding a framing of $D_{\mathrm{pdR}}(W_A)$ for $(W_A, \pi_A, F_A) \in \mathfrak{X}_{W,F}(A)$. The forgetful map $\mathfrak{X}_{W,F}^\square \to \mathfrak{X}_{W,F}$ is then formally smooth and the functor $\mathfrak{X}_{W,F}^\square$ is pro-representable by the completion of $X_{K/\mathbf{Q}_p,L}$ at the point $x_F = (F_1, 0, F_2)$ where $F_1 = i^{-1}(D_{\mathrm{pdR}}(F))$ and $F_2 = i^{-1}(\mathrm{Fil}_{\mathrm{dR}})$ ([BHS, Cor. 3.5.8.(i)]). Moreover, the following diagram is commutative

$$\begin{array}{ccc} \mathfrak{X}_{W,F}^\square & \xrightarrow{\text{forget}} & \mathfrak{X}_W^\square \\ \downarrow \wr & & \downarrow \wr \\ \widehat{X_{K/\mathbf{Q}_p,L}}_{x_F} & \xrightarrow{\pi_2} & \widehat{\tilde{\mathfrak{g}}_{K/\mathbf{Q}_p,L}}_x \end{array} \tag{3.3}$$

where the upper horizontal map is the forgetful map and the lower horizontal map is induced by the second projection of X on $\tilde{\mathfrak{g}}$. When $W = W_{\mathrm{dR}}^+(\mathcal{D})$ for a (φ, Γ_K)-module $\mathcal{D}$ and $F = W_{\mathrm{dR}}(\mathcal{M})$ for $\mathcal{M}$ a triangulation of $\mathcal{D}[\frac{1}{t}]$ we will use the shorter notation $x_{\mathcal{M}}$ in place of $x_{W_{\mathrm{dR}}(\mathcal{M})}$.

Proof of Theorem 3.11. Let $W \coloneqq W_{\mathrm{dR}}^+(\mathcal{D})$. In a first step we prove that the L-linear map

$$\bigoplus_{\mathcal{F} \in \mathrm{Tri}(\mathcal{D})} T\mathfrak{X}_{W, W_{\mathrm{dR}}(\mathcal{F}[\frac{1}{t}])} \longrightarrow T\mathfrak{X}_W$$

is surjective. Let's consider the commutative diagram

$$\begin{array}{ccc} \bigoplus_{\mathcal{F} \in \mathrm{Tri}(\mathcal{D})} T\mathfrak{X}_{W, W_{\mathrm{dR}}(\mathcal{F}[\frac{1}{t}])}^\square & \longrightarrow & T\mathfrak{X}_W^\square \\ \downarrow & & \downarrow \\ \bigoplus_{\mathcal{F} \in \mathrm{Tri}(\mathcal{D})} T\mathfrak{X}_{W, W_{\mathrm{dR}}(\mathcal{F}[\frac{1}{t}])} & \longrightarrow & T\mathfrak{X}_W \end{array}$$

As the forgetful map $\mathfrak{X}_W^\square \to \mathfrak{X}_W$ is formally smooth, it induces a surjection on the tangent spaces. Consequently it is sufficient to prove that the upper horizontal map is surjective.

Because of the commutative diagram (3.3) this is equivalent to the surjectivity of the map

$$\pi_2 : \sum_{\mathcal{F} \in \mathrm{Tri}(\mathcal{D})} T_{x_{\mathcal{F}[\frac{1}{t}]}} X_{K/\mathbf{Q}_p,L} \longrightarrow T_x \tilde{\mathfrak{g}}_{K/\mathbf{Q}_p,L}$$

induced by the second projection. Let α be the morphism of K-schemes $X = \widetilde{\mathfrak{g}} \times_{\mathfrak{g}} \widetilde{\mathfrak{g}} \to \widetilde{\mathfrak{g}}$ given by the second projection. Let $\alpha_{K/\mathbf{Q}_p}$ its image by the Weil restriction functor from K to $\mathbf{Q}_p$ and let $\alpha_{K/\mathbf{Q}_p,L}$ be the base change of $\alpha_{K/\mathbf{Q}_p}$ to L. For each $\tau \in \Sigma$ we write α_τ for the base change of α by $\tau : K \to L$. Then we have the following decompositions

$$\begin{aligned} X_{K/\mathbf{Q}_p} \times_{\mathbf{Q}_p} L &\simeq \prod_{\tau\in\Sigma} X_\tau, \\ \widetilde{\mathfrak{g}}_{K/\mathbf{Q}_p,L} &\simeq \prod_{\tau\in\Sigma} \widetilde{\mathfrak{g}}_\tau, \\ \alpha_{K/\mathbf{Q}_p,L} &= (\alpha_\tau)_{\tau\in\Sigma}. \end{aligned}$$

Therefore it only remains to prove that for each $\tau \in \Sigma$, the L-linear map

$$d\alpha_\tau : \bigoplus_{\mathcal{F}\in \mathrm{Tri}(\mathcal{D})} T_{x_{\mathcal{F}[\frac{1}{t}],\tau}} X_\tau \longrightarrow T_{x_\tau}\widetilde{\mathfrak{g}}_\tau \tag{3.4}$$

is surjective.

The $L \otimes_{\mathbf{Q}_p} K_0$-linear endomorphism $\Phi := \varphi^f$ of $D_{\mathrm{cris}}(\mathcal{D})$ induces an L-linear endomorphism Φ_τ of $D_{\mathrm{dR},\tau}(\mathcal{D}) = D_{\mathrm{cris},\tau|_{K_0}}(\mathcal{D})$ for all $\tau \in \Sigma$. This endomorphism is killed by the polynomial $\chi(D_{\mathrm{cris}}(\mathcal{D}), \varphi) \in L[X]$ which, by assumption, is separable and split. This implies that Φ_τ is contained in a unique maximal split torus T_τ of $\mathrm{GL}(D_{\mathrm{dR},\tau}(\mathcal{D}))$ or equivalently that the Zariski closure in $\mathrm{GL}(D_{\mathrm{dR},\tau}(\mathcal{D}))$ of the group $\Phi_\tau^{\mathbf{Z}}$ is a maximal split torus T_τ. If $\mathcal{F}$ is a triangulation of $\mathcal{D}$, the complete flag $D_{\mathrm{cris}}(\mathcal{F})$ of $D_{\mathrm{cris}}(\mathcal{D})$ is stable under φ, as is the complete flag $D_{\mathrm{dR},\tau}(\mathcal{F}[\frac{1}{t}])$ under Φ_τ, and thus also T_τ. However the maximal split torus T_τ fixes exactly $n!$ complete flags of $D_{\mathrm{dR},\tau}(\mathcal{D})$. As $\mathcal{D}$ has exactly $n!$ triangulations we conclude that the set

$$\{x_{\mathcal{F}[\frac{1}{t}],\tau},\ \mathcal{F} \in \mathrm{Tri}(\mathcal{D})\}$$

is exactly the set of points $(F, 0, i^{-1}(\mathrm{Fil}_{\mathrm{dR},\tau})) \in X_\tau(L)$ such that F is fixed by the maximal split torus T_τ.

The surjectivity of the map (3.4) is thus a direct consequence of Theorem 2.6. This concludes the first step of the proof.

Now consider the commutative diagram with exact lines and columns

$$\begin{array}{ccccc}
0 & & 0 & & \\
\downarrow & & \downarrow & & \\
\bigoplus_{\mathcal{F}\in \mathrm{Tri}(\mathcal{D})} T\mathfrak{X}_{\mathcal{D}}^{\mathrm{cris}} & \xrightarrow{\ \sum\ } & T\mathfrak{X}_{\mathcal{D}}^{\mathrm{cris}} & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \\
\bigoplus_{\mathcal{F}\in \mathrm{Tri}(\mathcal{D})} T\mathfrak{X}_{\mathcal{D},\mathcal{F}[\frac{1}{t}]} & \xrightarrow{\ \beta\ } & T\mathfrak{X}_{\mathcal{D}} & & \\
\downarrow & & \downarrow{\scriptstyle W_{\mathrm{dR}}^{+}} & & \\
\bigoplus_{\mathcal{F}\in \mathrm{Tri}(\mathcal{D})} T\mathfrak{X}_{W_{\mathrm{dR}}^{+}(\mathcal{D}),W_{\mathrm{dR}}^{+}(\mathcal{F}[\frac{1}{t}])} & \longrightarrow & T\mathfrak{X}_{W_{\mathrm{dR}}^{+}(\mathcal{D})} & \longrightarrow & 0 \\
\downarrow & & & & \\
0 & & & &
\end{array}$$

The exactness of the vertical lines are a consequences of Proposition 3.9 and Corollary 3.10. The surjectivity of $\sum$ is trivial and the surjectivity of the lower horizontal map is what we proved as a first step. We can deduce from this that the map $W_{\mathrm{dR}}^{+}\circ\beta$ is surjective and call upon the "five" Lemma to conclude that the map β itself is surjective, which closes the proof of Theorem 3.11. $\square$

3.6. **The case of Galois representations.** If (ρ, V) is a continuous representation of the group $\mathcal{G}_K$ on some finite dimensional L-vector space V, let $\mathfrak{X}_{(\rho,V)}$ be the deformation functor over $\mathcal{C}$ of (ρ, V). According to [Ber02] there exists a functor D_{rig} from the category of continuous representation of the group $\mathcal{G}_K$ on finite dimensional L-vector spaces to the category of (φ, Γ_K)-modules over $\mathcal{R}_L$, which is proved fully faithful by [Col08, Cor. 1.5]. Then, [BC09, Lem. 2.2.7] shows that if A is an object of $\mathcal{C}$ and (ρ, V) is a continuous representation of $\mathcal{G}_K$ with some L-algebras morphism $A \to \mathrm{End}_{\mathcal{G}_K} V$, then V is a finite free A-module if and only if $D_{\mathrm{rig}}(V)$ is a finite free $\mathcal{R}_A$-module. The essential image of the functor D_{rig} is moreover stable under extensions. From these facts we conclude that if (ρ, V) is a continuous representation of $\mathcal{G}_K$ on some finite dimensional L-vector space, then the functor D_{rig} induces an isomorphism of functors

$$\mathfrak{X}_{(\rho,V)} \xrightarrow{D_{\mathrm{rig}}} \mathfrak{X}_{D_{\mathrm{rig}}(V)}. \tag{3.5}$$

If moreover $\mathcal{F}$ is a triangulation of $D_{\mathrm{rig}}(V)$ we define $\mathfrak{X}_{(\rho,V),\mathcal{F}[\frac{1}{t}]}$ as the functor from $\mathcal{C}$ to the category of sets sending an object A to the set of isomorphism classes of tuples $(\rho_A, V_A, \pi_A, \mathcal{M}_A)$ where (ρ_A, V_A) is a continuous representation of $\mathcal{G}_K$ on some finite free A-module V_A, π_A is $\mathcal{G}_K$ equivariant A-linear map $V_A \to V$ inducing an isomorphism $L \otimes_A V_A \xrightarrow{\sim} V$ and $\mathcal{M}_A$ is a triangulation of $D_{\mathrm{rig}}(V_A)[\frac{1}{t}]$ such that $D_{\mathrm{rig}}(\pi_A)(\mathcal{M}_A) = \mathcal{F}[\frac{1}{t}]$.

Introducing the commutative diagram that, with the help of the functor D_{rig}, relates the isomorphism β of Theorem 3.11 to the linear map

$$\bigoplus_{\mathcal{F}\in\mathrm{Tri}(D_{\mathrm{rig}}(V))} T\mathfrak{X}_{(\rho,V),\mathcal{F}[\frac{1}{t}]} \longrightarrow T\mathfrak{X}_{(\rho,V)}$$

induced by the forgetful functors, we derive the following rephrasing of Theorem 3.11

Corollary 3.13. *Let (ρ, V) be a L-linear φ-generic crystalline representation of the group $\mathcal{G}_K$ with regular Hodge-Tate type. The forgetful functors induce a surjective L-linear map*

$$\bigoplus_{\mathcal{F}\in\mathrm{Tri}(D_{\mathrm{rig}}(V))} T\mathfrak{X}_{(\rho,V),\mathcal{F}[\frac{1}{t}]} \longrightarrow T\mathfrak{X}_{(\rho,V)}.$$

3.7. **Components of the non saturated deformation ring.** This section contains some complements about the geometry of the formal scheme $\mathfrak{X}_{(\rho,V),\mathcal{F}[\frac{1}{t}]}$ that will be useful in the next chapter.

Let (ρ, V) be some n-dimensional L-linear φ-generic crystalline representation of $\mathcal{G}_K$ with regular Hodge-Tate type. Let $\mathcal{D} = D_{\mathrm{rig}}(V)$ and let $\mathcal{F}$ be a triangulation of $\mathcal{D}$. We note $\mathcal{M} := \mathcal{F}[\frac{1}{t}]$. Let $W_{\mathrm{dR}}(\mathcal{D}) := \mathbf{B}_{\mathrm{dR}} \otimes_{\mathbf{B}_{\mathrm{dR}}^+} W_{\mathrm{dR}}^+(\mathcal{D})$.

We fix a basis of V, i.e. an L-linear isomorphism $\iota : L^n \simeq V$ so that ρ can be identified with a group homomorphism $\rho : \mathcal{G}_K \to \mathrm{GL}_n(L)$. Let $\mathfrak{X}_\rho$ be the deformation functor of the pair (ρ, ι). Using the identification (3.5) we can define the deformation functors $\mathfrak{X}_{\rho,\mathcal{F}}$, $\mathfrak{X}_{\rho,\mathcal{M}}$, $\mathfrak{X}_\rho^{\mathrm{cris}}$. These are the obvious variants of the above functors in the context of (φ, Γ_K)-modules with the corresponding decorations.

Let $F := D_{\mathrm{cris}}(\mathcal{F})$ the complete flag of $D_{\mathrm{cris}}(\mathcal{D})$ associated to the triangulation $\mathcal{F}$. For each $\tau \in \Sigma$, let F_τ be the complete flag of $D_{\mathrm{dR},\tau}(D)$ image of F under the functor $(-)\otimes_{L\otimes_{\mathbf{Q}_p}K_0,\tau} L$. The stabilizer $B_{F,\tau}$ of F_τ is a Borel subgroup of $\mathrm{GL}(D_{\mathrm{dR},\tau}(\mathcal{D}))$ and there exists a unique $w_{F,\tau} \in \mathfrak{S}_n$ such that $\mathrm{Fil}_{\mathrm{dR},\tau} \in B_{F,\tau} w_{F,\tau}(F_\tau)$. We define $w_{\mathcal{F}} := (w_{F,\tau})_{\tau\in\Sigma} \in W = (\mathfrak{S}_n)^{[K:\mathbf{Q}_p]}$.

It follows from [BHS, Thm. 3.6.2.(ii)] and [BHS, Prop. 3.6.4.] that the deformation functor $\mathfrak{X}_{\rho,\mathcal{M}}$ is pro-represented by some complete noetherian L-algebra $R_{\rho,\mathcal{M}}$ which is reduced, Cohen-Macaulay and equidimensional of dimension

$$n^2 + [K : \mathbf{Q}_p]\frac{n(n+1)}{2}.$$

Its minimal primes are indexed by the set $\{w \in W,\ w \geqslant w_{\mathcal{F}}\}$ where the ordering on W is the Bruhat ordering.

Let $R_{\rho,\mathcal{M}}^w$ be the quotient of $R_{\rho,\mathcal{M}}$ by the minimal prime with index w. By [BHS, Thm. 3.6.2.(ii)], it is Cohen-Macaulay and normal. Let $\mathfrak{X}_{(\rho,V),\mathcal{M}}^w$ be the subfunctor of $\mathfrak{X}_{(\rho,V),\mathcal{M}}$ defined as the image of

$$\text{Spf}\, R^w_{\rho,\mathcal{M}} \subset \text{Spf}\, R_{\rho,\mathcal{M}} \longrightarrow \mathfrak{X}_{\rho,\mathcal{M}}. \tag{3.6}$$

It can be easily checked that the inclusion $\mathfrak{X}^w_{(\rho,V),\mathcal{M}} \subset \mathfrak{X}_{(\rho,V),\mathcal{M}}$ is relatively representable. Note that the definition of $\mathfrak{X}^w_{\rho,\mathcal{M}}$ does not depend on the choice of the basis of the L-vector space V.

Let $\mathfrak{t}$ be the diagonal torus of $\mathfrak{g} = \mathfrak{gl}_{n,K}$. We recall that there is a canonical map $\tilde{\mathfrak{g}} \to \mathfrak{t}$ mapping $(A, gB) \in \tilde{\mathfrak{g}}$ to the class of $\mathrm{Ad}(g^{-1})A$ in $\mathfrak{b}/\mathfrak{u}$. Here $\mathfrak{u} \subset \mathfrak{b}$ is the sub-Lie-algebra of nilpotent upper triangular matrices, and the quotient $\mathfrak{b}/\mathfrak{u}$ is canonically identified with $\mathfrak{t}$.

This projection induces a canonical map Θ from $\mathfrak{X}_{(\rho,V),\mathcal{M}}$ to the completion at $(0,0)$ of the L-scheme $\mathfrak{t}_{K/\mathbf{Q}_p,L} \times_{\mathfrak{t}_{K/\mathbf{Q}_p,L}/W} \mathfrak{t}_{K/\mathbf{Q}_p,L}$. The irreducible components of this scheme are in bijection with the group W. Let $\mathfrak{t}_w$ be the component defined by $\{(t, \mathrm{Ad}(w^{-1})t),\, t \in \mathfrak{t}_{K/\mathbf{Q}_p,L}\}$ and $\widehat{\mathfrak{t}_w}$ its completion at $(0,0)$. We recall the ensuing characterization of $\mathfrak{X}^w_{(\rho,V),\mathcal{M}}$ which follows the precise definition of Θ, as discussed in [BHS, Cor. 3.5.12].

Proposition 3.14. *Let w_1 and w_2 two elements of $\{w \in W,\, w \geqslant w_{\mathcal{F}}\}$. We have $\Theta(\mathfrak{X}^{w_1}_{(\rho,\mathcal{F}),\mathcal{M}}) \subset \widehat{\mathfrak{t}_{w_2}}$ if and only if $w_1 = w_2$.*

Finally we recall that the functor $\mathfrak{X}^{\text{cris}}_\rho$ is pro-representable by a formally smooth L-algebra of dimension $n^2 + [K : \mathbf{Q}_p]\frac{n(n-1)}{2}$ ([Kis08]). It follow from the proof of [BHS, Theorem 4.2.3] that we have

$$\mathfrak{X}^{\text{cris}}_{(\rho,V)} \subset \mathfrak{X}^{w_0}_{(\rho,V),\mathcal{M}}$$

as subfunctors of $\mathfrak{X}_{(\rho,V)}$ where, consistently with a well established notation introduced in Chapter 2 above, w_0 stands for the longest element of W.

4. Global deformation rings

Let F be a totally real field and E a totally imaginary quadratic extension that we assume to be unramified over F and such that all places v dividing p are split in E. Let G be a unitary group in n variables defined over F such that $G \times_F E$ is an inner form of $\mathrm{GL}_{n,E}$. We assume moreover that $G(F \otimes_{\mathbf{Q}} \mathbf{R})$ is compact and that the group G is quasi-split over all finite places of F. This implies that n is odd or that $4|n[F : \mathbf{Q}]$. If v is a place of F which splits in E, the group G splits at v. We fix a place $\tilde{v}$ of E dividing v and an isomorphism $G \times_F E_{\tilde{v}} \cong \mathrm{GL}_{n,E_{\tilde{v}}}$ which induces an isomorphism $G \times_F F_v \simeq \mathrm{GL}_{n,F_v}$. Let $B_v \subset G(F_v)$ be the subgroup corresponding to the Borel subgroup of upper triangular matrices of $\mathrm{GL}_n(F_v)$ under this isomorphism and $T_v \subset B_v$ the subgroup corresponding to the subgroup of diagonal matrices in $\mathrm{GL}_n(F_v)$. We write $T = \prod_{v|p} T_v$ and $B_p = \prod_{v|p} B_v$. Moreover we define $U_v \subset G(F_v)$ the maximal compact subgroup of $G(F_v)$ corresponding to $\mathrm{GL}_n(\mathcal{O}_{F_v})$ under this isomorphism.

Let U^p be a compact open subgroup of $G(\mathbb{A}^{p,\infty})$ of the form $\prod_{v\nmid p} U_v$ with U_v a compact open subgroup of $G(F_v)$ which is assumed to be hyperspecial when v is a place of F which is inert in E. Let S_p denote the set of places of F that divide p and let S be a finite set of places of F containing S_p and the finite set of places of F for which U_v is not hyperspecial. Finally we write $U = U^p \times U_p$, where $U_p = \prod_{v|p} U_v$ is a maximal compact subgroup of $G(F \otimes_{\mathbf{Q}} \mathbf{Q}_p)$.

We write E_S for the maximal extension of E that is unramified outside all places of E above the places in S and denote by $\mathcal{G}_{E,S} = \mathrm{Gal}(E_S/E)$ the corresponding Galois group. Let A be a $\mathbf{Z}_p$-algebra and ρ_A a representation of $\mathcal{G}_{E,S}$ on some finite free A-module V_A of rank n. We write ρ_A^c for the representation $g \mapsto \rho_A(cgc)$, where $c \in \mathrm{Gal}(\overline{F}/F)$ is a complex conjugation. The representation (ρ_A, V_A) is called *polarizable*, if there exists an isomorphism

$$(\rho^{\vee,c}, V_A^\vee) \cong (\rho \otimes \varepsilon^{n-1}, V_A),$$

where ε is the cyclotomic character. Such an isomorphism is called *polarization*.

We fix L a finite extension of $\mathbf{Q}_p$ and $(\overline{\rho}, \overline{V})$ a continuous polarized representation $\mathcal{G}_{E,S} \to \mathrm{GL}_n(k_L)$ which is absolutely irreducible so that it has a unique polarization up to scalar multiplication. We denote by $R_{\overline{\rho},S}$ the universal polarized deformation $\mathcal{O}_L$-algebra of $\overline{\rho}$. That is, the complete local $\mathcal{O}_L$-algebra pro-representing the functor of isomorphism classes of triples (ρ_A, V_A, ι_A), with V_A a finite free A-module with a continuous polarized action ρ_A of $\mathcal{G}_{E,S}$ and an isomorphism $\iota_A : V_A/\mathfrak{m}_A V_A \cong \overline{V}$ of $\mathcal{G}_{E,S}$-representations, on the category of local Artinian $\mathcal{O}_L$-algebras A with residue field k_L. The existence of the $\mathcal{O}_L$-algebra $R_{\overline{\rho},S}$ follows from [Che11, §1.1].

Let $\mathcal{X}_{\overline{\rho},S} = (\mathrm{Spf}\, R_{\overline{\rho},S})^{\mathrm{rig}}$ be the rigid analytic generic fiber of the formal scheme $\mathrm{Spf}\, R_{\overline{\rho},S}$. As $(\overline{\rho}, \overline{V})$ is absolutely irreducible, the L-points of $\mathcal{X}_{\overline{\rho},S}$ are in bijection with the set of isomorphism classes of continuous representations (ρ, V) of $\mathcal{G}_{E,S}$ on L-vector spaces such that $\rho^{\vee,c} \simeq \rho \otimes \varepsilon^{n-1}$ and such that there exists a $\mathcal{G}_{E,S}$-stable $\mathcal{O}_L$-lattice $V^\circ \subset V$ and a $\mathcal{G}_{E,S}$-equivariant isomorphism $V^\circ/\varpi_L V^\circ \simeq \overline{V}$. Given a point $x \in \mathcal{X}_{\overline{\rho}}$, we denote by (ρ_x, V_x) the associated representation of $\mathcal{G}_{E,S}$.

Fix an isomorphism $\iota : \overline{\mathbf{Q}_p} \simeq \mathbf{C}$. Recall that, if π is a (cuspidal) automorphic representation of G, there exists a unique polarized n-dimensional $\overline{\mathbf{Q}_p}$-representation (ρ_π, V_π) of $\mathrm{Gal}(\overline{E}/E)$ associated to π. If $(\pi^{p,\infty})^{U^p} \neq 0$ then this representation factors through $\mathcal{G}_{E,S}$. The existence of this Galois representation is a consequence of base change ([Lab11, Cor. 5.3]) and of the construction of Galois representations associated to some automorphic representation of $\mathrm{GL}_{n,E}$ (see [CH13]). We say that a point $x \in \mathcal{X}_{\overline{\rho},S}(L)$ is (G, U^p)-automorphic (resp. (G, U)-automorphic) if there exists a (cuspidal) automorphic representation π of G such that $(\pi^{p,\infty})^{U^p} \neq 0$ (resp. such that $(\pi^\infty)^U \neq 0$) and such that there is an isomorphism $(\rho_x, V_x \otimes_L \overline{\mathbf{Q}_p}) \simeq (\rho_\pi, V_\pi)$.

Moreover we say that $(\overline{\rho},\overline{V})$ is (G,U)-automorphic over L if there exists a (G,U)-automorphic point $x\in\mathcal{X}_{\overline{\rho},S}(L)$. Let $\mathcal{X}_{\overline{\rho},S}^{\mathrm{aut}}$ be the Zariski closure of the set of (G,U)-automorphic points in $\mathcal{X}_{\overline{\rho},S}^{\mathrm{aut}}$. The aim of this section is to prove the following theorem:

Theorem 4.1. *Assume that $p>2$, that all places of S are split in E and that the group $\overline{\rho}(\mathcal{G}_{E(\zeta_p)})$ is adequate in the sense of* [Tho12, Definition 2.3]. *Then the inclusion $\mathcal{X}_{\overline{\rho},S}^{\mathrm{aut}}\subset\mathcal{X}_{\overline{\rho},S}$ is the inclusion of a union of irreducible components (possibly empty if $(\overline{\rho},\overline{V})$ is not (G,U)-automorphic).*

From now on we assume $p>2$, the places of S split in E, $\overline{\rho}(\mathcal{G}_{E(\zeta_p)})$ adequate and that $(\overline{\rho},\overline{V})$ is (G,U)-automorphic.

Recall that, for a place $v\in S$, we fix a place $\tilde{v}$ of E dividing v. We write $\mathcal{G}_{E_{\tilde{v}}}$ for the choice of a decomposition group at $\tilde{v}$. Given a representation ρ of $\mathcal{G}_{E,S}$ we write $\rho_{\tilde{v}}$ for the restriction of ρ to $\mathcal{G}_{E_{\tilde{v}}}$.

Finally we assume that U^p is sufficiently small so that the compact open sub group $U\coloneqq\prod_v U_v$ is such that

$$\forall g\in G(\mathbb{A}_F^\infty),\ G(F)\cap gUg^{-1}=\{1\}. \tag{4.1}$$

4.1. **Recollections about eigenvarieties and patching.** Attached to the data G,U^p and $\overline{\rho}$ there is a so-called *eigenvariety*. For a place v dividing p let us write $\hat{T}_v$ for the rigid analytic space of continuous characters of T_v and similarly $\hat{T}_v^0$ for the space of continuous characters of the maximal compact subgroup $T_v^0\subset T_v$. Further let

$$\hat{T}=\prod_{v|p}\hat{T}_v \qquad\text{and}\qquad \hat{T}^0=\prod_{v|p}\hat{T}_v^0.$$

The eigenvariety associated to G,U^p and $\overline{\rho}$ is by definition the Zariski-closed rigid analytic subspace $Y(U^p,\overline{\rho})\subset\mathcal{X}_{\overline{\rho},S}\times\hat{T}_L$ that is the (scheme-theoretic) support of the locally analytic Jacquet-module $J_{B_p}(\hat{S}(U^p,L)_{\mathfrak{m}}^{\mathrm{an}})$ of the locally analytic representation underlying the $G(F\otimes_{\mathbf{Q}}\mathbf{Q}_p)$-representation on the space $\hat{S}(U^p,L)_{\mathfrak{m}}$ of p-adic automorphic forms of tame level U^p. Here $\mathfrak{m}$ is a certain maximal ideal of a Hecke-algebra corresponding to the residual Galois representation $\overline{\rho}$. We refer to [BHS17a, 3.1] for details of this construction.

We recall the notion of a *classical point* on $Y(U^p,\overline{\rho})$: We write $X^*(T)$ for the space of algebraic characters of the product of the diagonal tori in

$$(\mathrm{Res}_{F/\mathbf{Q}}G)_{\mathbf{C}}\cong\prod_{\tau:F\hookrightarrow\mathbf{C}}\mathrm{GL}_{n,\mathbf{C}}. \tag{4.2}$$

This space comes equipped with an action of the Weyl group W of $(\mathrm{Res}_{F/\mathbf{Q}}G)_{\mathbf{C}}$. As usual we write $w\cdot\lambda$ for the shifted *dot*-action of W on $X^*(T)$. We write w_0 for the longest element of W.

The isomorphism $\overline{\mathbf{Q}_p} \cong \mathbf{C}$ identifies $\lambda \in X^*(T)$ with a character $T_p \to \overline{\mathbf{Q}_p}^\times$ that we denote by z^λ. If L is a finite extension of $\mathbf{Q}_p$ such that $\mathrm{Res}_{F/\mathbf{Q}} G$ splits over L, then z^λ takes values in L and we may view it as an L-valued point of $\hat{T}$.

Given a representation π_∞ of $G(F \otimes_{\mathbf{Q}} \mathbf{R})$ we say that π_∞ is of weight $\lambda \in X^*(T)$ if it is the restriction to $G(F \otimes_{\mathbf{Q}} \mathbf{R})$ of the irreducible algebraic representation of $(\mathrm{Res}_{F/\mathbf{Q}} G)_{\mathbf{C}}$ of highest weight λ.

Let $\pi = \pi_\infty \otimes_{\mathbf{C}} \pi^{p,\infty} \otimes_{\mathbf{C}} \pi_p$ be an automorphic representation of G such that $(\pi^{p,\infty})^{U^p} \neq 0$ and such that π_∞ is of weight λ. Moreover we assume that, for all $v \in S_p$, the representation π_v is an unramified quotient of the smooth induced representation $(\mathrm{Ind}_{B_v}^{G_v} \delta_{\mathrm{sm},v}\delta_v)^{\mathrm{sm}}$ for some unramified character $\delta_{\mathrm{sm},v}$ of T_v with values in $L^\times$ and where δ_v is the smooth character

$$\delta_v = (1 \otimes |\cdot|_v \otimes \cdots \otimes |\cdot|_v^{n-1}).$$

Let $\delta_{\mathrm{sm}} := \bigotimes_{v \in S_p} \delta_{\mathrm{sm},v}$. The associated Galois-representation ρ_π is (G,U)-automorphic by definition and we have

$$(\rho_\pi, \delta_{\mathrm{sm}} z^\lambda) \in Y(U^p, \overline{\rho})(L) \subset \mathcal{X}_{\overline{\rho},S}(L) \times \hat{T}(L),$$

see [BHS17a, Proposition 3.4] for example. The point $x = (\rho_\pi, \delta_{\mathrm{sm}} z^\lambda)$ is called the classical point associated with $(\pi, \delta_{\mathrm{sm}})$.

It follows from [CH13] that, for $v \in S_p$, the representation $\rho_{\tilde{v}}$ is crystalline and that the character $\delta_{\mathrm{sm},v}$ is of the form $\delta_{\mathcal{F}_{\tilde{v}}}$ for $\mathcal{F}_{\tilde{v}}$ a triangulation of $\rho_{\tilde{v}}$ (see §3.3 for the definition of $\delta_{\mathcal{F}_{\tilde{v}}}$). We say that ρ is crystalline φ-generic if $\rho_{\tilde{v}}$ is crystalline φ-generic for all places v dividing p.

Assuming that ρ is crystalline φ-generic, it follows from the classification of intwertinning operators between principal series that

$$\mathcal{F}_v \longmapsto \delta_{\mathcal{F}_v}$$

induces a bijection between the set of smooth characters δ_{sm} such that $\pi_v \cong (\mathrm{Ind}_{B_v}^{G(F_v)} \delta_{\mathrm{sm}}\delta_v)^{\mathrm{sm}}$ and the triangulations of $\rho_{\tilde{v}}$.

Similarly, given a tuple $\underline{\mathcal{F}} = (\mathcal{F}_v)_{v \in S_p}$ of refinements we write $\delta_{\underline{\mathcal{F}}} = (\delta_{\mathcal{F}_v})_v$ for the corresponding unramified character of T_p. In this case $x_{\underline{\mathcal{F}}} := (\rho, z^\lambda \delta_{\underline{\mathcal{F}}})$ is a classical point of $Y(U^p, \overline{\rho})$ by construction, associated to the pair $(\pi, \delta_{\underline{\mathcal{F}}})$.

Fix an embedding $\tau : F_v \hookrightarrow \bar{\mathbf{Q}}_p$. Via the identification $\overline{\mathbf{Q}_p} \cong \mathbf{C}$ this embedding defines an embedding $\mathbf{Q} \hookrightarrow \mathbf{C}$ and we write W_τ for the factor of the Weyl group W corresponding to this embedding via the decomposition (4.2).

The relative position of the τ-part of the Hodge Filtration

$$\begin{aligned}\mathrm{Fil}_{\mathrm{dR},\tau} := \mathrm{Fil}_{\mathrm{dR}} \otimes_{F_v \otimes_{\mathbf{Q}_p} L, \tau \otimes \mathrm{id}} \bar{\mathbf{Q}}_p \subset D_{\mathrm{dR}}(\rho_v) \otimes_{F_v \otimes_{\mathbf{Q}_p} L, \tau \otimes \mathrm{id}} \bar{\mathbf{Q}}_p \\ = D_{\mathrm{cris}}(\rho_v) \otimes_{F_{v,0} \otimes_{\mathbf{Q}_p} L, \tau|_{F_{v,0}} \otimes \mathrm{id}} \bar{\mathbf{Q}}_p\end{aligned}$$

with respect to $\mathcal{F}_v \otimes_{F_{v,0}\otimes_{\mathbf{Q}_p} L, \tau\otimes \mathrm{id}} \bar{\mathbf{Q}}_p$ defines an element of the Weyl group $w_{\mathcal{F}_\tau} \in W_\tau$. We write $w_{\underline{\mathcal{F}}} \in W$ for the Weyl group element defined by the tuple $\underline{\mathcal{F}}$.

The following proposition summarizes the properties of the eigenvariety needed for the proof of the main theorem:

Proposition 4.2. *(i) The eigenvariety $Y(U^p, \overline{\rho})$ is reduced and equi-dimensional of dimension*

$$\dim Y(U^p, \overline{\rho}) = \dim \hat{T}^0 = n[F : \mathbf{Q}].$$

(ii) The set of classical points as defined above is Zariski-dense and has the accumulation property, i.e. for every classical point x and every open connected neighborhood U of x the classical, crystalline φ-generic points are Zariski-dense in U.

(iii) Let $x = x_{\underline{\mathcal{F}}}$ be a classical, crystalline φ-generic point associated to $(\pi, \delta_{\underline{\mathcal{F}}})$ as above. For a weight $\mu \in X^*(T)$, one has

$$(\rho, z^\mu \delta_{\underline{\mathcal{F}}}) \in Y(U^p, \overline{\rho}) \Longleftrightarrow \mu = ww_0 \cdot \lambda \text{ with } w \in W,\ w_{\underline{\mathcal{F}}} \preceq w.$$

For $w_{\underline{\mathcal{F}}} \preceq w$, we define

$$x_{\underline{\mathcal{F}},w} := (\rho, z^{ww_0\cdot\lambda}\delta_{\underline{\mathcal{F}}}).$$

(iv) Let x be as in (iii). Then the projection $\omega : Y(U^p, \overline{\rho}) \to \hat{T}^0$ is flat at the points $x_{\underline{\mathcal{F}},w}$.

(v) The projection $Y(U^p, \overline{\rho}) \to \mathcal{X}_{\overline{\rho},S}$ is locally on the source and the target a finite morphism.

Proof. Points (i) and (ii) are contained in [Che05, 3.8]. The statements can be obtained as well along the lines of Corollaire 3.12, Théorème 3.19 and Corollaire 3.20 of [BHS17b]. See Definition 3.2 and Proposition 3.4 of [BHS17a] for a comparison of the (a priori different) notions of classical points.
(iii) This is a direct consequence of [BHS, Theorem 5.3.3].
(iv) This is contained in [BHS, Theorem 5.4.2].
(v) The map $Y(U^p, \overline{\rho}) \to \mathcal{X}_{\overline{\rho},S}$ is the composite of the closed embedding $Y(U^p, \overline{\rho}) \subset \mathcal{X}_{\overline{\rho},S} \times \hat{T}$ with the projection $\mathcal{X}_{\overline{\rho},S} \times \hat{T} \to \mathcal{X}_{\overline{\rho},S}$. These two maps are locally of finite type so that the map $Y(U^p, \overline{\rho}) \to \mathcal{X}_{\overline{\rho}}$ is locally of finite type. We claim that the fibers are discrete and hence the morphism is locally quasi-finite. The Proposition then follows from [Hub96, Prop. 1.5.4.(c)].

Indeed, consider the morphism $\mathcal{X}_{\overline{\rho},S} \to \mathbb{A}^{n[F:\mathbf{Q}]}/W$ given by mapping ρ to the set of Hodge-Tate weights of the ρ_v, for $v|p$. Fix a point ρ and write $\mathrm{HT}(\rho)$ for its image in $\mathbb{A}^{n[F:\mathbf{Q}]}/W$ for the moment. The composition

$$q : \hat{T}^0 \longrightarrow \mathbb{A}^{n[F:\mathbf{Q}]} \longrightarrow \mathbb{A}^{n[F:\mathbf{Q}]}/W$$

of the logarithm with the projection map is obviously quasi-finite and hence $q^{-1}(\mathrm{HT}(\rho))$ is a discrete set. Finally the weight map $Y(U^p,\overline{\rho}) \to \hat{T}^0$ is quasi-finite by the usual argument using special coverings of Fredholm hypersurfaces (see e.g. [BHS17b, Proposition 3.11]). And hence the preimage of $q^{-1}(\mathrm{HT}(\rho))$ under the weight map is still a discrete set. As this set contains the fiber of $Y(U^p,\overline{\rho}) \to \mathcal{X}_{\overline{\rho},S}$ over ρ the claim follows. □

We further recall the *patched eigenvariety* $X_p(\overline{\rho})$ and its relation to the global object $Y(U^p,\overline{\rho})$. In [BHS17b, 3] we have carried out the following construction: Let us write

$$R_{\overline{\rho}_p} = \widehat{\bigotimes}_{v\in S_p} R_{\overline{\rho}_{\tilde{v}}} \text{ and } R_{\overline{\rho}^p} = \widehat{\bigotimes}_{v\in S\setminus S_p} R_{\overline{\rho}_{\tilde{v}}}$$

for the completed tensor products of the maximal reduced and $\mathbf{Z}_p$-flat quotients $R_{\overline{\rho}_v}$ of the universal framed deformation rings $R'_{\overline{\rho}_{\tilde{v}}}$ of $\overline{\rho}_{\tilde{v}}$. Let

$$R_{\overline{\rho},\mathcal{S}} := R_{\overline{\rho},S} \otimes_{\left(\widehat{\bigotimes}_{v\in S} R'_{\overline{\rho}_{\tilde{v}}}\right)} \left(\widehat{\bigotimes}_{v\in S} R_{\overline{\rho}_{\tilde{v}}}\right).$$

There exists an integer $g \geqslant 1$ and a commutative diagram with maps of local $\mathcal{O}_L$-algebras

$$\begin{array}{ccc} S_\infty := \mathcal{O}_L[\![\mathbf{Z}_p^q]\!] & \longrightarrow & R_\infty := \left(R_{\overline{\rho}_p} \widehat{\otimes}_{\mathcal{O}_L} R_{\overline{\rho}^p}\right)[\![y_1,\dots,y_g]\!] \\ \downarrow & & \downarrow \\ R_\infty \otimes_{S_\infty} \mathcal{O}_L & \longrightarrow & R_{\overline{\rho},\mathcal{S}} \end{array} \tag{4.3}$$

where the left vertical map is induced by the augmentation map $S_\infty \to \mathcal{O}_L$ and where $q = g + [F:\mathbf{Q}]\frac{n(n-1)}{2} + n^2|S|$.

We write $\mathcal{X}_\infty$ and $\mathcal{X}_{\overline{\rho}_p}$ for the rigid analytic generic fibers of $\operatorname{Spf} R_\infty$ and $\operatorname{Spf} R_{\overline{\rho}_p}$. Moreover we denote by $X_p(\overline{\rho}) \subset \mathcal{X}_\infty \times \hat{T}$ the patched eigenvariety constructed in [BHS17b]. Then there is a canonical embedding

$$Y(U^p,\overline{\rho}) \hookrightarrow X_p(\overline{\rho}) \times_{(\operatorname{Spf} S_\infty)^{\mathrm{rig}}} \operatorname{Sp} L \subset \mathcal{X}_\infty \times \hat{T}, \tag{4.4}$$

see [BHS17b, 4.1] or [BHS, (5.34)]. Let us abbreviate $X_p(\overline{\rho}) \times_{(\operatorname{Spf} S_\infty)^{\mathrm{rig}}} \operatorname{Sp} L$ by $Y_p(\overline{\rho})$ for the moment. The precise relation of the local geometry of the patched eigenvariety and the (global) eigenvariety is given by the following proposition:

Proposition 4.3. *Assume that $x = x_{\mathcal{F}}$ is a classical, crystalline φ-generic point. For each $w \in W$ such that $x_{\underline{\mathcal{F}},w} \in Y(U^p,\overline{\rho})$ the morphism (4.4) induces an isomorphism of complete local rings*

$$\hat{\mathcal{O}}_{Y_p(\overline{\rho}),x_{\underline{\mathcal{F}},w}} \cong \hat{\mathcal{O}}_{Y(U^p,\overline{\rho}),x_{\mathcal{F},w}}.$$

Proof. This is [BHS, Proposition 5.4.1]. □

Finally we recall the relation of the patched eigenvariety with the space of trianguline representations, see [BHS17b]. Let $X_{\mathrm{tri}}(\overline{\rho}) = \prod_{v\in S_p} X_{\mathrm{tri}}(\overline{\rho}_v) \subset \mathcal{X}_{\overline{\rho}_p} \times \hat{T}$. Then there is a commutative diagram

$$\begin{array}{ccc} X_p(\overline{\rho}) & \xrightarrow{\iota} & X_{\mathrm{tri}}(\overline{\rho}) \times \mathcal{X}_{\overline{\rho}^p} \times \mathbb{U}^g \\ \downarrow & & \downarrow \\ \hat{T}^0 & \xrightarrow{\cong} & \hat{T}^0, \end{array} \tag{4.5}$$

where ι is a closed embedding that identifies $X_p(\overline{\rho})$ with a union of irreducible components of the target. Here $\mathbb{U}^g = (\mathrm{Spf}\, \mathcal{O}_L[\![y_1, \dots, y_g]\!])^{\mathrm{rig}}$, see [BHS17b, Theorem 3.21].

4.2. **A characterization of the tangent space.** We fix a (G, U)-automorphic representation $\rho \in \mathcal{X}_{\overline{\rho},S} \subset \mathcal{X}_\infty$ that is crystalline φ-generic. For the reminder of this subsection we introduce the following notations:

Let R be the complete local ring of $\mathcal{X}_\infty$ at ρ so that $\widehat{(\mathcal{X}_\infty)}_\rho = \mathrm{Spf}\, R$ and, for a given refinement $\underline{\mathcal{F}} = (\mathcal{F}_v)_{v\in S_p}$ of ρ and $w \in W$ such that $w_{\underline{\mathcal{F}}} \preceq w$ let $R_{\underline{\mathcal{F}},w}$ be the complete local ring of $X_p(\overline{\rho})$ at the point $x_{\underline{\mathcal{F}},w}$, so that $\widehat{X_p(\overline{\rho})}_{x_{\underline{\mathcal{F}},w}} = \mathrm{Spf}\, R_{\underline{\mathcal{F}},w}$. By [BHS, Lemma 4.3.3], the canonical map $R \to R_{\underline{\mathcal{F}},w}$ is a surjection. Similarly we define S as the complete local ring of $\mathcal{X}_{\overline{\rho},S}$ at ρ and $S_{\underline{\mathcal{F}},w}$ the complete local ring of $Y_p(\overline{\rho})$ at $x_{\underline{\mathcal{F}},w}$. Then we have a canonical surjection $R \twoheadrightarrow S$ and, by Proposition 4.3 an identification $S_{\underline{\mathcal{F}},w} = S \otimes_R R_{\underline{\mathcal{F}},w} = R_{\underline{\mathcal{F}},w} \otimes_{S_\infty} \mathcal{O}_L$.

Obviously the ring $R_{\underline{\mathcal{F}},w}$ decomposes as a tensor product

$$R_{\underline{\mathcal{F}},w} = \widehat{\bigotimes}_{v|p} R_{\mathcal{F}_v,w_v} \hat{\otimes} \widehat{\mathcal{O}}_{\mathcal{X}_{\overline{\rho}^p} \times \mathbb{U}^g, \rho^p},$$

where $R_{\mathcal{F}_v,w_v}$ is the complete local ring of $X_{\mathrm{tri}}(\overline{\rho}_v)$ at $x_{\mathcal{F}_v,w_v} = (\rho_v, z^{w_v w_0\cdot\lambda_v}\delta_{\mathcal{F}_v})$; and where ρ^p is the image of ρ in $\mathcal{X}_{\overline{\rho}^p} \times \mathbb{U}^g$.

By [BHS, Cor. 3.7.8] and [BHS, Thm. 3.6.2.(ii)] the quotient $R_{\mathcal{F}_v,w_v}$ of $R_{\overline{\rho}_{\tilde{v}}}$ coincides with the quotient $R^{w_v}_{\overline{\rho}_{\tilde{v}},\mathcal{F}_v}$ of $R_{\overline{\rho}_{\tilde{v}},\mathcal{F}_v}$ defined in section 3.7. We conclude that the map $R_\infty \to R_{\underline{\mathcal{F}},w}$ induces an isomorphism

$$R_\infty \otimes_{\left(\widehat{\bigotimes}_{v|p} R_{\overline{\rho}_{\tilde{v}}}\right)} \left(\widehat{\bigotimes}_{v|p} R^{w_v}_{\overline{\rho}_{\tilde{v}},\mathcal{F}_v}\right) \xrightarrow{\sim} R_{\underline{\mathcal{F}},w}. \tag{4.6}$$

Let us write $\mathrm{Spec}\, R_{\underline{\mathcal{F}}}$ for the scheme theoretic image of the canonical morphism

$$\coprod_{w_{\underline{\mathcal{F}}}\preceq w} \mathrm{Spec}\, R_{\underline{\mathcal{F}},w} \longrightarrow \mathrm{Spec}\, R$$

and $\mathrm{Spec}\, S_{\underline{\mathcal{F}}}$ for the scheme theoretic image of the canonical morphism

$$\coprod_{w_{\underline{\mathcal{F}}}\preceq w} \mathrm{Spec}\, S_{\underline{\mathcal{F}},w} \longrightarrow \mathrm{Spec}\, S.$$

Lemma 4.4. *(i) The scheme* $\operatorname{Spec} R_{\underline{\mathcal{F}}}$ *is reduced and Cohen-Macaulay of dimension*

$$g + [F : \mathbf{Q}]\frac{n(n+1)}{2} + n^2|S| = q + n[F : \mathbf{Q}].$$

(ii) The scheme $\operatorname{Spec} S_{\underline{\mathcal{F}},w}$ *is reduced and equi-dimensional of dimension* $n[F : \mathbf{Q}]$*. The same holds true for* $\operatorname{Spec} S_{\underline{\mathcal{F}}}$

(iii) Let $\eta \in \operatorname{Spec} S_{\underline{\mathcal{F}},w}$ *be a generic point, then* $\eta \notin \operatorname{Spec} R_{\underline{\mathcal{F}},w'}$ *for* $w' \neq w$*.*

Proof. (i) As ρ is automorphic the space $\mathcal{X}_{\overline{\rho}^p}$ is smooth of dimension $n^2|S\backslash S_p|$ at (the image of) ρ, by [Car12, Theorem 1.2] and [BLGGT14, Lemma 1.3.2 (1)]. The claim now follows from diagram (4.5), isomorphism (4.6) and the fact that $R^{w_v}_{\rho_{\tilde{v}},\mathcal{F}_v}$ is reduced, normal and Cohen-Macaulay of dimension $n^2 + [F_v : \mathbf{Q}_p]\frac{n(n+1)}{2}$ (see section 3.7).
(ii) As $S_{\underline{\mathcal{F}},w}$ is the complete local ring at some point of the eigenvariety $Y(U^p, \overline{\rho})$, the claims follow from the corresponding statements on $Y(U^p, \overline{\rho})$ in Proposition 4.2. The claim on $\operatorname{Spec} S_{\underline{\mathcal{F}}}$ then is a direct consequence.
(iii) Let us write $\operatorname{Spf} A$ for the formal completion of $\hat{T}^0$ at $\omega(\iota(x))$. Thus A is just the completed tensor-product of power series rings of dimension $n[F_v : \mathbf{Q}_p]$ indexed by $v \in S_p$. We identify $\operatorname{Spf} A$ with the formal completion of $\mathfrak{t}_{K/\mathbf{Q}_p,L}$ at the origin via $\kappa \mapsto \kappa - \omega(\iota(x))$.

Taking products over all $v \in S_p$ and the product with the formally smooth contribution from $\mathcal{X}_{\overline{\rho}^p} \times \mathbb{U}^g$ we obtain a commutative diagram as in [BHS, 2.5]:

$$\begin{array}{ccccc}
\operatorname{Spf} S_{\underline{\mathcal{F}},w} & \hookrightarrow & \operatorname{Spf} R_{\underline{\mathcal{F}},w} & \hookrightarrow & \operatorname{Spf} R_{\underline{\mathcal{F}}} \\
& \searrow & \downarrow & & \downarrow \underline{\Theta} \\
& & \operatorname{Spf} A & \xrightarrow{\psi} & \prod_{v|p}(\mathfrak{t}_{F_v/\mathbf{Q}_p,L} \times_{\mathfrak{t}_{F_v/\mathbf{Q}_p,L}/W_v} \mathfrak{t}_{F_v/\mathbf{Q}_p,L})^{\wedge}_0,
\end{array}$$

where $\underline{\Theta}$ is the product of all the maps Θ for all $v|p$, as defined before Proposition 3.14.

Recall that $\mathfrak{t}_{F_v/\mathbf{Q}_p,L} \times_{\mathfrak{t}_{F_v/\mathbf{Q}_p,L}/W_v} \mathfrak{t}_{F_v/\mathbf{Q}_p,L}$ decomposes as a product $\prod_{w'_v \in W_v} \mathfrak{t}_{w'_v}$. It follows from [BHS, Cor. 3.5.12] that the morphism ψ identifies $\operatorname{Spf} A$ with the product $\hat{\mathfrak{t}}_w = \prod_{v|p} \hat{\mathfrak{t}}_{w_v,0}$. Let us write $\operatorname{Spf} B$ for the formal scheme in the lower left corner of the diagram, and $\operatorname{Spf} B_{w'} = \hat{\mathfrak{t}}_{w'} = \prod_{v|p} \hat{\mathfrak{t}}_{w'_v,0}$.

Passing to rings and applying $\operatorname{Spec}(-)$ the above diagram becomes:

$$\begin{array}{ccccccc}
\operatorname{Spec} S_{\underline{\mathcal{F}},w} & \hookrightarrow & \operatorname{Spec} R_{\underline{\mathcal{F}},w} & & \hookrightarrow & & \operatorname{Spec} R_{\underline{\mathcal{F}}} \\
& \overset{\omega}{\searrow} & \downarrow & & & & \downarrow \underline{\Theta} \\
& & \operatorname{Spec} A & \xrightarrow{\cong} & \operatorname{Spec} B_w & \hookrightarrow & \operatorname{Spec} B,
\end{array}$$

where we use the same latters for the maps by abuse of notation. Now, by Proposition 4.2, the map ω is flat and hence dominant when restricted to each irreducible component of $\operatorname{Spec} S_{\underline{\mathcal{F}},w}$. On the other hand $(\underline{\Theta})(\operatorname{Spec} R_{\underline{\mathcal{F}},w'}) = \operatorname{Spec} B_{w'}$ by [BHS, Cor. 3.5.12] (compare also Proposition 3.14), and $\operatorname{Spec} B_{w'}$ does not contain the generic point of $\operatorname{Spec} B_w$ for $w \neq w'$. The claim follows from this. □

Lemma 4.5. *The canonical maps* $R_{\underline{\mathcal{F}}} \otimes_{S_\infty} \mathcal{O}_L \to R_{\underline{\mathcal{F}}} \otimes_R S$ *and* $R_{\underline{\mathcal{F}}} \otimes_R S \to S_{\underline{\mathcal{F}}}$ *are isomorphisms, i.e.*

$$\operatorname{Spec} S_{\underline{\mathcal{F}}} = \operatorname{Spec} R_{\underline{\mathcal{F}}} \cap \operatorname{Spec} S = \operatorname{Spec} R_{\underline{\mathcal{F}}} \times_{\operatorname{Spec} R} \operatorname{Spec} S \tag{4.7}$$

as subschemes of $\operatorname{Spec} R$. *In particular* $\operatorname{Spec} S_{\underline{\mathcal{F}}} \subset \operatorname{Spec} R_{\underline{\mathcal{F}}}$ *is a closed subscheme that is cut out by* q *equations.*

Proof. From (4.3), we deduce a sequence of surjective maps

$$R_{\underline{\mathcal{F}}} \otimes_{S_\infty} \mathcal{O}_L \twoheadrightarrow R_{\underline{\mathcal{F}}} \otimes_R S \twoheadrightarrow S_{\mathcal{F}}.$$

First note that $\operatorname{Spec} R_{\underline{\mathcal{F}}} \times_{\operatorname{Spec} S_\infty} \operatorname{Spec} \mathcal{O}_L$ is cut out by q equations in $\operatorname{Spec} R_{\underline{\mathcal{F}}}$.

By definition $\operatorname{Spec} R_{\underline{\mathcal{F}}} = \bigcup_{w \geqslant w_{\underline{\mathcal{F}}}} \operatorname{Spec} R_{\underline{\mathcal{F}},w}$ and $\operatorname{Spec} S_{\underline{\mathcal{F}}} = \bigcup_{w \geqslant w_{\underline{\mathcal{F}}}} \operatorname{Spec} S_{\underline{\mathcal{F}},w}$ as topological spaces. Consequently we have an equality of sets

$$\begin{aligned}
\operatorname{Spec} R_{\underline{\mathcal{F}}} \times_{\operatorname{Spec} S_\infty} \operatorname{Spec} \mathcal{O}_L &= \bigcup_{w \geqslant w_{\underline{\mathcal{F}}}} (\operatorname{Spec} R_{\underline{\mathcal{F}},w} \times_{\operatorname{Spec} S_\infty} \operatorname{Spec} \mathcal{O}_L) \\
&= \bigcup_{w \geqslant w_{\underline{\mathcal{F}}}} (\operatorname{Spec} R_{\underline{\mathcal{F}},w} \times_{\operatorname{Spec} R} \operatorname{Spec} S) \\
&= \operatorname{Spec} S
\end{aligned}$$

Indeed $\operatorname{Spec} S_{\underline{\mathcal{F}},w} = \operatorname{Spec} R_{\underline{\mathcal{F}},w} \times_{\operatorname{Spec} R} \operatorname{Spec} S$ by Proposition 4.3. Hence (4.7) is true on the level of topological spaces. As $\operatorname{Spec} S_{\underline{\mathcal{F}}}$ is reduced it remains to show that $\operatorname{Spec} R_{\underline{\mathcal{F}}} \times_{\operatorname{Spec} S_\infty} \operatorname{Spec} \mathcal{O}_L$ is reduced as well.

We have

$$\dim(\operatorname{Spec} R_{\underline{\mathcal{F}}} \times_{\operatorname{Spec} S_\infty} \operatorname{Spec} \mathcal{O}_L) = \dim \operatorname{Spec} S_{\underline{\mathcal{F}}} = \dim \operatorname{Spec} R_{\underline{\mathcal{F}}} - q$$

and consequently $\operatorname{Spec} R_{\underline{\mathcal{F}}} \times_{\operatorname{Spec} S_\infty} \operatorname{Spec} \mathcal{O}_L$ is Cohen-Macaulay as $\operatorname{Spec} R_{\underline{\mathcal{F}}}$ is (see e.g. [Gro64, Cor. 16.5.6]).
By [Gro65, Proposition 5.8.5], it remains to prove that $\operatorname{Spec} R_{\underline{\mathcal{F}}} \times_{\operatorname{Spec} S_\infty} \operatorname{Spec} \mathcal{O}_L$ is generically reduced. Let $\eta \in \operatorname{Spec} R_{\underline{\mathcal{F}}} \times_{\operatorname{Spec} S_\infty} \operatorname{Spec} \mathcal{O}_L$ be the generic point of some irreducible component and write $\mathfrak{q}$ for the corresponding prime ideal of R. Then η is the generic point of an irreducible component in $\operatorname{Spec} S_{\underline{\mathcal{F}},w}$ for some w and it follows that

$$(\mathcal{O}_L \otimes_{S_\infty} R_{\underline{\mathcal{F}}})_{\mathfrak{q}} = \mathcal{O}_L \otimes_{S_\infty} (R_{\underline{\mathcal{F}}})_{\mathfrak{q}} = \mathcal{O}_L \otimes_{S_\infty} (R_{\underline{\mathcal{F}},w})_{\mathfrak{q}} = (S_{\underline{\mathcal{F}},w})_{\mathfrak{q}},$$

where the second equality is a consequence of Lemma 4.4 (iii) and the last equality is Proposition 4.3. As $(S_{\underline{\mathcal{F}},w})_{\mathfrak{q}}$ is reduced so is $(\mathcal{O}_L \otimes_{S_\infty} R_{\underline{\mathcal{F}}})_{\mathfrak{q}}$ and the claim follows. □

Let us write R_{cris} for the quotient of R such that for all $v \in S_p$ and any morphism $R \to A$ (for some finite dimensional L-algebra A) the $\mathcal{G}_{E_{\tilde{v}}}$ representation on A^n induced by $R_{\bar{\rho}_v} \to R \to A$ is crystalline. This quotient exists by the main result of [Kis08].

If A is a complete noetherian local ring, we write t_A for the tangent space of the functor $\operatorname{Spf} A$, ie $t_A \coloneqq T \operatorname{Spf} A$.

Lemma 4.6. *(i) The local ring R_{cris} is formally smooth of dimension q.*
(ii) The tangent spaces of R_{cris} and S intersect trivially inside t_R, i.e.

$$t_{R_{\mathrm{cris}}} \cap t_S = 0.$$

(iii) There is an inclusion $t_{R_{\mathrm{cris}}} \subset t_{R_{\underline{\mathcal{F}}}}$.

Proof. (i) This follows from the smoothness of $\mathcal{X}_{\bar{\rho}^p}$ at the image of ρ (see above) and the fact that the generic fiber of a crystalline deformation rings is smooth of dimension $n^2 + [F_v : \mathbf{Q}_p]\frac{n(n-1)}{2}$ by [Kis08] and the definition of q.
(ii) With the notation introduced here this is the statement of [All16, Theorem A.1].
(iii) This is a direct consequence of $\operatorname{Spec} R_{\mathrm{cris}} \subset \operatorname{Spec} R_{\underline{\mathcal{F}},w_0} \subset \operatorname{Spec} R_{\underline{\mathcal{F}}}$, which follows from section 3.7. □

Corollary 4.7. *There is a direct sum decomposition*

$$t_{R_{\mathrm{cris}}} \oplus t_{S_{\underline{\mathcal{F}}}} = t_{R_{\underline{\mathcal{F}}}}$$

of subspaces of t_R.

Proof. As $\operatorname{Spec} S_{\underline{\mathcal{F}}} \subset \operatorname{Spec} S$ we have $t_{S_{\underline{\mathcal{F}}}} \subset t_S$ and hence $t_{S_{\underline{\mathcal{F}}}} \cap t_{R_{\mathrm{cris}}} = 0$ by Lemma 4.6 (ii). Moreover $t_{R_{\mathrm{cris}}} \subset t_{R_{\underline{\mathcal{F}}}}$ and $t_{R_{\mathrm{cris}}}$ has dimension q. The claim now follows from the fact that $\operatorname{Spec} S_{\underline{\mathcal{F}}} \subset \operatorname{Spec} R_{\underline{\mathcal{F}}}$ is cut out by q equations by Lemma 4.5 and hence

$$\operatorname{codim}(t_{S_{\underline{\mathcal{F}}}}, t_{R_{\underline{\mathcal{F}}}}) \leqslant q.$$

□

Corollary 4.8. *The canonical map of tangent spaces*

$$\bigoplus_{\underline{\mathcal{F}}} t_{S_{\underline{\mathcal{F}}}} \longrightarrow t_S$$

is a surjection. Here the sum is taken over all tuples $\underline{\mathcal{F}} = (\mathcal{F}_v)_{v \in S_p}$ of Frobenius stable flags $\mathcal{F}_v$ of $D_{\mathrm{cris}}(\rho_v)$.

Proof. Consider the commutative diagram

$$\begin{array}{ccc} \bigoplus_{\underline{\mathcal{F}}} t_{R_{\underline{\mathcal{F}}}} & \twoheadrightarrow & t_R \\ \downarrow & & \downarrow \\ \bigoplus_{\underline{\mathcal{F}}} t_{R_{\underline{\mathcal{F}}}}/t_{R_{\mathrm{cris}}} & \longrightarrow & t_R/t_{R_{\mathrm{cris}}} \\ {\scriptstyle\alpha}\uparrow & & {\scriptstyle\beta}\uparrow \\ \bigoplus_{\underline{\mathcal{F}}} t_{S_{\underline{\mathcal{F}}}} & \xrightarrow{\gamma} & t_S. \end{array}$$

It follows from Corollary 4.7 that α is an isomorphism, and from Lemma 4.6 (ii) that β is injective. Moreover the upper horizontal arrow is surjective by Corollary 3.13. It follows from an obvious diagram chaise that γ is a surjection. □

Remark 4.9. We point out that it is a direct consequence of the proof of Corollary 4.8 that the map

$$t_S \longrightarrow t_R/t_{R_{\mathrm{cris}}}$$

is an isomorphism.

4.3. **Proof of Theorem 4.1.** We now prove the main result, Theorem 4.1. With the above preparation, the final argument just follows the original method of Gouvea-Mazur [Maz97] and Chenevier [Che11] in the case of modular forms (i.e. in the case $n = 2$), resp. in the case $n = 3$.

Let us write $\mathcal{X}_{\overline{\rho},S}^{\mathrm{aut,sm}} \subset \mathcal{X}_{\overline{\rho},S}^{\mathrm{aut}}$ for the smooth locus which is Zariski-open and dense in $\mathcal{X}_{\overline{\rho},S}^{\mathrm{aut}}$. Let us fix an irreducible component C of $\mathcal{X}_{\overline{\rho},S}^{\mathrm{aut}}$. We need to show that C is an irreducible component of $\mathcal{X}_{\overline{\rho},S}$. By slight abuse of notations we write $C^{\mathrm{sm}} = C \cap \mathcal{X}_{\overline{\rho},S}^{\mathrm{aut,sm}}$.

As by definition $\mathcal{X}_{\overline{\rho},S}^{\mathrm{aut}}$ is the Zariski-closure of the (G, U)-automorphic points in $\mathcal{X}_{\overline{\rho},S}$, it follows that C^{sm} contains a (G, U)-automorphic point ρ. By the construction preceding Proposition 4.2 there is a point $y = (\rho, \delta) \in Y(\overline{\rho}, U^p)$ and by Proposition 4.2 (ii) the classical, crystalline φ-generic points accumulate at y. It follows that there is a classical, crystalline φ-generic point $y' = (\rho', \delta') \in Y(U^p, \overline{\rho})$ such that $\rho' \in C^{\mathrm{sm}}$. We may replace ρ by ρ' (and y by y') and hence assume that ρ is (G, U)-automorphic and crystalline φ-generic.

Proposition 4.10. *There is an equality of tangent spaces*

$$T_\rho \mathcal{X}_{\overline{\rho},S}^{\mathrm{aut}} = T_\rho \mathcal{X}_{\overline{\rho},S}.$$

Proof. The inclusion $T_\rho \mathcal{X}_{\overline{\rho},S}^{\mathrm{aut}} \subset T_\rho \mathcal{X}_{\overline{\rho},S}$ is obvious and we need to prove the converse inclusion.

After enlarging L if necessary, we may assume that the point ρ is an L-valued point of $\mathcal{X}_{\overline{\rho},S}$ and that L contains all eigenvalues of the crystalline Frobenius on the Weil-Deligne representation $\mathrm{WD}(D_{\mathrm{cris}}(\rho_v))$ associated to $D_{\mathrm{cris}}(\rho_v)$ for all $v \in S_p$.

For each choice of a tuple $\underline{\mathcal{F}}$ of complete Frobenius stable flags $\mathcal{F}_v$ in $D_{\mathrm{cris}}(\rho_v)$ and each Weyl group element $w \preceq w_{\underline{\mathcal{F}}}$ we have constructed points $x_{\underline{\mathcal{F}},w} \in Y(U^p, \overline{\rho})$ that map to ρ under the canonical projection $f : Y(U^p, \overline{\rho}) \to \mathcal{X}_{\overline{\rho},S}$. As f is locally on the source and the target a finite morphism by Proposition 4.2 (v), and as the induced map

$$\hat{\mathcal{O}}_{\mathcal{X}_{\overline{\rho},S},\rho} \longrightarrow \hat{\mathcal{O}}_{Y(U^p,\overline{\rho}),x_{\underline{\mathcal{F}},w}}$$

is a surjection (as a consequence of [BHS, Lemma 4.3.3] for example) we find an open neighborhood U of ρ in $\mathcal{X}_{\overline{\rho}}$ and for all $w_{\underline{\mathcal{F}}} \preceq w$ open neighborhoods $V_{\underline{\mathcal{F}},w} \subset Y(U^p, \overline{\rho})$ of $x_{\underline{\mathcal{F}},w}$ such that the restriction of f is a closed immersion $V_{\underline{\mathcal{F}},w} \hookrightarrow U$. As the classical points are Zariski-dense in $V_{\underline{\mathcal{F}},w}$ by Proposition 4.2 (ii), we find that $\bigcup_w V_{\underline{\mathcal{F}},w} \subset U \cap \mathcal{X}^{\mathrm{aut}}_{\overline{\rho},S}$.

The formation of scheme-theoretic images commutes with flat base change, hence in particular with passing to the complete local ring at ρ. It follows (using the notation from subsection 4.2) that

$$\operatorname{Spec} S_{\underline{\mathcal{F}}} \subset \operatorname{Spec} \hat{\mathcal{O}}_{\mathcal{X}^{\mathrm{aut}}_{\overline{\rho},S},\rho} \subset \operatorname{Spec} \hat{\mathcal{O}}_{\mathcal{X}_{\overline{\rho},S},\rho} = \operatorname{Spec} S.$$

We deduce that

$$t_{S_{\underline{\mathcal{F}}}} \subset t_{\hat{\mathcal{O}}_{\mathcal{X}_{\overline{\rho},S},\rho}} = T_\rho \mathcal{X}^{\mathrm{aut}}_{\overline{\rho}}.$$

As this conclusion holds true for each choice of $\underline{\mathcal{F}}$, Corollary 4.8 implies the claimed inclusion $T_\rho \mathcal{X}_{\overline{\rho}} = t_S \subset T_\rho \mathcal{X}^{\mathrm{aut}}_{\overline{\rho}}$. □

We can conclude the proof of Theorem 4.1.

Proof of Theorem 4.1. Given a rigid analytic space Z and a point $z \in Z$ we write $\dim_z Z$ for the dimension of Z at the point z, i.e. for the dimension of the local ring $\mathcal{O}_{Z,z}$ of Z at z.

Assume in the first place that the group U^p is sufficiently small to satisfy (4.1). We then have a chain of inequalities

$$\dim_\rho \mathcal{X}^{\mathrm{aut}}_{\overline{\rho},S} = \dim_\rho C \leqslant \dim_\rho \mathcal{X}_{\overline{\rho},S} \leqslant \dim T_\rho \mathcal{X}_{\overline{\rho},S} = \dim T_\rho \mathcal{X}^{\mathrm{aut}}_{\overline{\rho},S} = \dim_\rho \mathcal{X}^{\mathrm{aut}}_{\overline{\rho},S},$$

as ρ is (by assumption) a smooth point of $\mathcal{X}^{\mathrm{aut}}_{\overline{\rho},S}$. Here the equality $\dim T_\rho \mathcal{X}_{\overline{\rho},S} = \dim T_\rho \mathcal{X}^{\mathrm{aut}}_{\overline{\rho},S}$ is Proposition 4.10. It follows that equality holds and hence the (necessarily unique) irreducible component C of $\mathcal{X}^{\mathrm{aut}}_{\overline{\rho},S}$ containing ρ is an irreducible component of $\mathcal{X}_{\overline{\rho},S}$.

Now assume that U^p does not necessary satisfy (4.1). Then we can find a place $v_1 \notin S$ of F such that v_1 is split in E. Let $V_{v_1} \subset G(F_{v_1})$ sufficiently small so that the group $V^p := V_{v_1} \times \prod_{v \nmid p, v \neq v_1} U_v$ satisfies (4.1) and let $S' = S \cup \{v_1\}$. We have a closed immersion $\mathcal{X}_{\overline{\rho},S} \subset \mathcal{X}_{\overline{\rho},S'}$ and it follows from local-global compatibility theorems that a point of $\mathcal{X}_{\overline{\rho},S}$ is (G,V)-automorphic if and only if it is (G,U)-automorphic. Let x be a (G,U)-automorphic point and let Z be an irreducible component of $\mathcal{X}_{\overline{\rho},S}$ containing x. Let Z' be some irreducible component of $\mathcal{X}_{\overline{\rho},S'}$ containing Z'. As x is a (G,V)-automorphic point, it is a smooth point of Z' and $\dim Z' = [F:\mathbf{Q}]\frac{n(n+1)}{2}$ (see [All16, Thm. C]). On the other hand, we have $\dim Z \geqslant [F:\mathbf{Q}]\frac{n(n+1)}{2}$ so that we have $Z = Z'$. We have proved that (G,V)-automorphic points are Zariski-dense in $Z' = Z$. As these points are also (G,U)-automorphic we can conclude that (G,U)-automorphic points are Zariski-dense in Z. □

4.4. **Proof of Theorem 1.2.** We finally turn to the proof of the main theorem as stated in the introduction. Thie result follows from Theorem 4.1 using base change results for unitary groups.

Proof of Theorem 1.2. Let G be a unitary group over F^+ which is an outer form of $\mathrm{GL}_{n,\overline{F}}$, which is quasisplit at every finite place and such that $G(F^+ \otimes_{\mathbf{Q}} \mathbf{R})$ is compact. the existence of such a unitary group follows for example from the results of [Clo91, §2]. By assumption there exists some regular cohomological cuspidal automorphic representation π such that $\overline{\rho} \otimes_{\mathbf{F}} \overline{\mathbf{F}_p} \simeq \overline{\rho}_\pi$. It follows from [Lab11, Thm. 5.4] that the representation π is the weak base change of some automorphic, automatically cuspidal, representation σ of G. So that $\overline{\rho} \otimes_{\mathbf{F}} \overline{\mathbf{F}_p} \simeq \overline{\rho}_\sigma$. Let U^p be some compact open subgroup of $G(\mathbb{A}^{p,\infty})$ such that $\sigma^{U^p} \neq 0$. The representation π is unramified at finite places $v \notin S$. Consequently it follows from [Lab11, Thm. 5.9] that we can choose the group U^p spherical at places not in S and that U_v is spherical for all $v|p$. Then the representation $\overline{\rho}$ is (G,U)-automorphic. Consequently we can apply Theorem 4.1 to conclude that the Zariski closure of the (G,U)-automorphic points in $\mathcal{X}_{\overline{\rho},S}$ is a union of irreducible components. However it follows from Cor. 5.3, Thm. 5.4 and Thm. 5.9 in [Lab11] that a point of $\mathcal{X}_{\overline{\rho},S}$ is automorphic if and only if it is (G,U^p)-automorphic for some U^p as above. This concludes the proof. □

4.5. **Remarks on the existence of enough automorphic points.** We end by discussing that the main theorem conjecturally should imply density of automorphic points in $\mathcal{X}_{\overline{\rho},S}$. Let us write $\mathcal{X}_{\overline{\rho},S} = \bigcup C_i$ for the decomposition into irreducible components. Then, obviously Theorem 4.1 implies that the (G,U)-automorphic points are Zariski-dense in $\mathcal{X}_{\overline{\rho},S}$, if $C_i \setminus \bigcup_{j\neq i} C_j$ contains a (G,U)-automorphic point for each i.

A result like this is the main result of Allen [All]. As loc. cit. is formulated for Galois representations associated to automorphic representations of GL_n rather than for forms on a unitary group, we repeat the argument for the convenience of the reader. The argument however is taken from [All].

Given a set of Hodge-Tate weights $\mathbf{k}_v$ we write $R_{\overline{\rho}_v}^{\mathbf{k}_v-\mathrm{cris}}$ for the quotient of $R_{\overline{\rho}_v}$ parametrizing crystalline deformations with Hodge-Tate weights $\mathbf{k}_v$. For $\mathbf{k} = (\mathbf{k}_v)_{v\in S_p}$ we write $R_{\overline{\rho}_p}^{\mathbf{k}-\mathrm{cris}}$ for the completed tensor product of the local rings $R_{\overline{\rho}_v}^{\mathbf{k}_v-\mathrm{cris}}$.

Theorem 4.11. *Assume that*

(a) the representation $\overline{\rho}$ is adequate
(b) the group $H^0(\mathcal{G}_{E_v}, \mathrm{ad}^0(\overline{\rho}))$ vanishes for each $v \in S_p$
(c) there exists a lift ρ of $\overline{\rho}$ that is crystalline of Hodge-Tate weight $\mathbf{k}$ and for each irreducible component $\mathcal{X}^p$ of $\mathcal{X}_{\overline{\rho}^p}$ all irreducible components of $(\mathrm{Spf}\, R_{\overline{\rho}_p}^{\mathbf{k}-\mathrm{cris}})^{\mathrm{rig}}$ are $\mathcal{X}^p$-automorphic (compare [BHS17b, Conjecture 3.25]*).*

Then each irreducible component of $\mathcal{X}_{\overline{\rho}}$ contains a (G, U)-automorphic point in its interior.

Remark 4.12. The assumption (c) is true for example, if $\overline{\rho}$ has a potentially diagonalizable lift.

Proof. Note that the points of $\mathcal{X}_{\overline{\rho}}$ are in bijection to those of $\mathrm{Spec}\, R_{\overline{\rho}}[1/p]$ and the the irreducible components of $\mathcal{X}_{\overline{\rho}}$ are in bijection to those of $\mathrm{Spec}\, R_{\overline{\rho}}[1/p]$.

The assumption $H^0(\mathcal{G}_{E_v}, \mathrm{ad}(\overline{\rho})) = 0$ implies that the local deformation rings $R_{\overline{\rho}_v}$ are formally smooth.
We consider the patching set-up as above. Then it follows from [All, Lemma 1.1.2] that the fiber product

$$\Big(\mathrm{Spec}\, \widehat{\bigotimes}_{v \in S_p} R_{\overline{\rho}_v}^{\mathbf{k}_v-\mathrm{cris}}\Big) \times_{R_{\overline{\rho}_p}} \mathrm{Spec}\, R_{\overline{\rho}}[1/p] \tag{4.8}$$

meets each irreducible component of $\mathrm{Spec}\, R_{\overline{\rho}}[1/p]$. By the automorphy lifting conjecture, assumption (c), all points in the intersection (4.8) are (G, U)-automorphic.

It is left to show that such a point ρ lies on a unique irreducible component. The subspace $\mathcal{X}_{\overline{\rho}} \subset \mathcal{X}_\infty$ is cut out by q equations and $\mathcal{X}_\infty$ is formally smooth at ρ. It follows that each irreducible component C_i of $\mathcal{X}_{\overline{\rho}}$ has dimension larger or equal to $\dim \mathcal{X}_\infty - q$. On the other hand, as in the proof of Corollary 4.7, we deduce from Lemma 4.6 (ii) that

$$\mathrm{codim}(t_\rho \mathcal{X}_{\overline{\rho}}, t_\rho \mathcal{X}_\infty) \geqslant q.$$

It follows that ρ is a formally smooth point of $\mathcal{X}_{\overline{\rho}}$. ☐

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