Diplomarbeit

Perturbative Calculation around the Kink Interface in 2-loop Order

Feng Xu

Institut für Theoretische Physik
Westfälische Wilhelms-Universität Münster

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Im Vergleich zu der am 12. Juli 2013 im Prüfungsamt eingereichten Arbeit sind in dieser korrigierten Version dieser Arbeit einige kleine Fehler behoben worden.

To my parents

Contents

Preface

I was not very excited about the topic of this thesis as Mr. Münster assigned it to me at the beginning. The fact that I've made a long detour before I finally toddled off to the theoretical physics institute, which should've been done several years ago, dulled my motivation somewhat. I couldn't even choose a topic myself at that time, since I really had no idea what would be a reasonable start for me in theoretical physics, although, ironically, I've always wanted to do it.

Bearing in mind that I have to finish this thesis no matter what, because it would be my last shot to get my first degree title. I started to sniff for clues about what I was dealing with, and soon I became absorbed in the literature; this thesis, which touches the statistical field theory, turned out to be a perfect guidance for a novice like me.

For the understanding of the fundamental concepts in field theory I have mainly stuck to Michel Le Bellac and Anthony Zee's interpretation [1,2], then I combined and reformulated their explanation in my own words, and that basically put up chapter 2. I was not afraid to dig into the details here, since I wanted to make this part self-explanatory, in particular, the mechanism of Wick contractions is the key to understand Feynman diagrams, which are efficient ways to visualize the former, as we will see, those Feynman diagrams are crucial to this thesis. In Chapter 1 I briefly introduced the phenomenon of interface roughening near critical point and how statistical field theory is related to the critical phenomena in condensed matter physics [1,3,4], together with Chapter 2, they form the part I. Section 2.3 and 2.4 are about connected correlation functions [1] and proper vertices [1,5], which are actually not closely related to the core of this thesis, i.e. part II, yet belong to the perturbation theory.

In chapter 3 I went on to the physical model of the kink interface, which is set up near the critical point of a continuous phase transition. A perturbative calculation around the classic kink solution followed in chapter 4; all the Feynman diagrams up to 2-loop order was then carefully derived from the generating functional, there Hoppe's detailed handling in a similar situation [6] inspired me. As for the concrete calculations of selected diagrams, I adopted the same method used by Michael Köpf[7].

Acknowledgements

I would like to express my appreciation to Prof. Gernot Münster for his tolerance, trust and mentoring.

I very much thank Dr. Jochen Heitger for the time he has taken to review my thesis.

I have been fortunate in the pleasant working group at the theoretical physics institute, where I enjoyed interacting with other friendly members. Many have given me technical insights and inspiring hints, which have both directly and indirectly helped me in the course of writing this thesis. In this regard, I would like to thank in particular Dr. Georg Bergner, Tobias Berheide, Florian König[8], Sven Musberg, Umut Deniz Özugurel, Stefano Piemonte and Kai Sparenberg.

Special thanks go to Mr. and Mrs. Hüseyin for their generous help in extending my visa. Thanks as well go to Dr. Xiaoyong Chu, who has been encouraging me to finish my thesis.

I also owe my ongoing gratitude to my parents, to my cousin Dr. Xue Rui, for their constant support.

Feng Xu

Westfälische Wilhelms-Universität Münster
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Part I

Motivation and foundation

1

Introduction

I like pushing boundaries.

— Lady Gaga

Interface is a surface forming a common boundary separating two portions of matter. Many systems of statistical mechanics, such as liquid gas coexistence, binary liquid mixtures or anisotropic ferromagnetism, possess interesting interfaces.

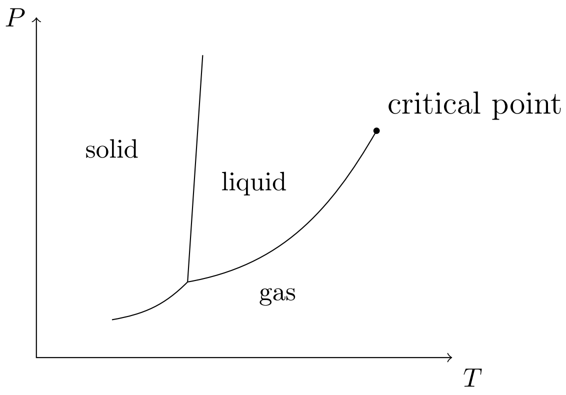


Figure 1.1 Phase diagram

Typical pure substances have phase diagrams of the classic form shown in Figure 1.1. As we can see, upon approaching a critical point the difference between liquid and gas gradually vanishes, in other words, the width of their otherwise sharp localized interface extends to all possible dimensions—we call this interface roughening; at the same time, the density fluctuations get larger, and the correlation length diverges. The fluid turns opaque once the linear dimensions of the fluctuations become comparable with the wavelength of the light. Analogous phenomena have also been observed in a binary fluid systems as well as at the ferromagnetic transitions. It turns out that these systems exhibit universal behaviour of certain quantities when reaching the critical point; they undergo second-order phase transitions, which are also called the critical phenomena.

The long range correlation caused by the second-order phase transitions could be troublesome as it plants divergency in the classic perturbation theory; but on the other hand, ignoring the complexity in such large-scale cooperative phenomena, we might expect that some of its properties would only depend on very general features (like order parameter) rather than details of the interactions. This aspect is called *universality.* In fact the critical phenomena of the systems mentioned above all belong to the same universality class as the Ising model[9], which was developed to describe ferromagnetic transition.

The Ising universality class can be alternatively accommodated in the framework of the Ginzburg-Landau theory[10], which is a Euclidean, massive and real φ^4-theory. Its Lagrangian possesses a double well potential, whose minima respectively correspond to, say, the two fluids A and B in a binary fluid system. When a spontaneous symmetry breaking occurs in the system, that is when the two components of the fluid begin to separate, we get the picture in Figure 1.2,

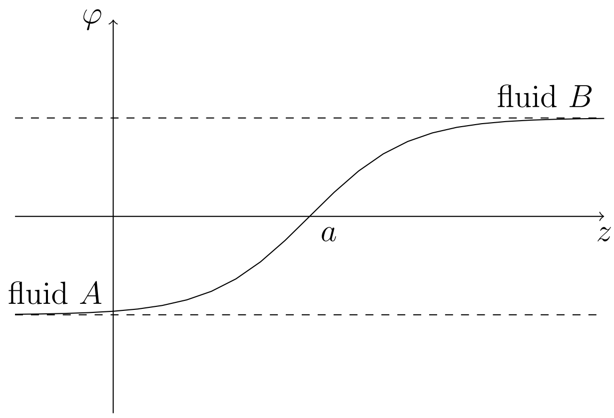


Figure 1.2 The kink solution in φ^4-theory

where φ is, according to the bold pick by Lev Davidovich Landau[11], the order parameter, in this special case it can be defined as the difference between the concentrations of the two fluids A and B; the interface is set perpendicular to the z-axis, along which we see the separation of the fluids. The kink interface, which was proposed back then by Johannes Diderik van der Waals [12], can then be explored in the field theory.

It is especially interesting to investigate the dependence of the interface width on the system size, as it characterizes the roughening of the interface. Earlier approaches involving capillary wave model [13] suggested that the interface diverges in width logarithmically as its size L increases. By employing the Ginzburg-Landau-Model Michael Köpf implemented the interfacial profile $\langle \varphi \rangle$ into a properly defined interface width w, the analytical results [14] based on the one-loop approximation of $\langle \varphi \rangle$ has shown encouraging agreement with capillary wave theory and Monte Carlo simulations [15–18] of the Ising model.

For a better quantitative comparison with Monte Carlo simulations, approximation of higher orders are required. For this purpose, a further exploration of the interfacial profile $\langle \varphi \rangle$ up to 2-loop order will be presented in this thesis.

2

Critical phenomena

Questions about the cause and mechanism of the phase transitions belong to the oldest problems in physics. Paul Ehrenfest [19] originally classified various kinds of phase transitions by judging the different thermodynamic quantities that go through discontinuous changes. In modern classification scheme the second-order phase transitions, or continuous phase transitions, are those in which basic thermodynamic quantities show sudden changes, but more gentle than a discontinuous jump in the basic variables.

2.1 Ferromagnetic transition

The ferromagnetic transition serves as the most familiar example of a second-order phase transition [1,3]. Once heated above the Curie-Temperature T_C the ferromagnet loses its magnetization, no preferred spatial direction of the ferromagnet can be identified, and the state is invariant under all rotations. Below T_C, by contrast, the ferromagnet possesses a spontaneous magnetization, since it is not due to any applied field; from the microscopic point of view this means that all the electrons in the incomplete inner shells of the atoms within the ferromagnet have their spins effectively aligned in one preferred direction, since every spin is associated with a magnetic moment, all of which add when aligned, thus producing the magnetization. The state of the ferromagnet is then only invariant under rotations around axes parallel to the magnetization direction. The phenomenon is known as *spontaneous symmetry breaking.*

In addition one introduces the *order parameter* of the transition, the macroscopic variables, which can only be meaningfully defined for the two involved phases of the transition. For the above example the magnetization, zero in the high-temperature phase and non-zero in the low-temperature phase, are the order parameters of the ferromagnetic transition.

2.2 Ising model

The first non-trivial model for ferromagnetism was suggested by Wilhelm Lenz to his student Ernest Ising as a thesis subject. The interaction between spins, which tends to align them, seems essential to ferromagnetism, it's thus reasonable to replace the atoms of the ferromagnet by the electrons, which are responsible for the ferromagnetism, in other words, the electrons are placed at the sites of the underlying crystal lattice. For further simplification only the interaction between the adjacent spins are considered, so the simplest Hamilton with a tendency to align the spins is

$$H = -J \sum_{\langle i,j \rangle} \boldsymbol{\sigma}_i \boldsymbol{\sigma}_j,$$

where J is a positive coupling constant, and the $\boldsymbol{\sigma}_i$ are Pauli matrices; the notion $\langle i, j \rangle$ indicates summation over nearest neighbours. The Hamilton defines the *quantum Heisenberg model*. Near the critical point, quantum fluctuations are dominated by statistical fluctuations, hence the Pauli matrices $\boldsymbol{\sigma}_i$ may be replaced by classical vectors $\boldsymbol{S}_i$ of length 1, this defines the *classical Heisenberg model*.

The resultant model is still too complicated, Lenz assumed another approximation, the vectors $\boldsymbol{S}_i$ are replaced by scalers S_i, which can only take two values, $S = +1$ or $S_i = -1$. Then the Hamilton can be written as

$$H = -J \sum_{\langle i,j \rangle} S_i S_j, \quad S_{i,j} = \pm 1,$$

and the partition function Z as

$$Z = \sum_{\{S_i\}} \mathrm{e}^{\frac{J}{kT} \sum_{\langle i,j \rangle} S_i S_j},$$

with the first sum running over all configurations

$$\sum_{\{S_i\}} = \sum_{S_1 = \pm 1} \sum_{S_2 = \pm 1} \cdots .$$

The whole strategy is to find the equation simple enough to solve, yet not to lose the essential features of the physics, but at that time there existed absolutely no methods for assessing the approximations.

Ising calculated the free energy for the model in one dimension, but didn't find the phase transition as predicted, nonetheless, the model was named after him. Until very much later, Rudolf Peierls [20] generalized the result and proved that in the absence of long-range interactions, no one-dimensional system can display a phase transition. The solutions to the two dimensional Ising model, especially Onsager's calculation [21], proved the existence of a ferromagnetic transition. A complete solution to the three-dimensional Ising model still doesn't exist until present, but the well known approximations by now are so convincing, that hardly any additional information could be expected from the precise analytical solution. Today we know that Ising model is qualitatively a good model of ferromagnetism, but quantitatively some of its predictions are poor.

2.3 Ginzburg-Landau Theory

As mentioned, Ising model is complicated, we needed approximation, and many kinds of mean field theories were developed before, following the work of Pierre Weiss [22]. However the approximations are not always reliable, and they neglect the effects of fluctuations. This problem leads us to the theory of Ginzburg and Landau, a statistical field theory, which is well adapted to deal with fluctuations.

In his previous work Landau associated each phase transition with a broken symmetry, and used the order parameter to describe the nature and extent of symmetry breaking, then he translated this idea into a mathematical theory, known as Landau theory, which is another classic approach to deal with second-oder phase transitions. Based on Landau theory the Ginzburg-Landau theory introduces a Hamiltonian $H[\varphi_i]$ depending on the order parameter, or a *field* φ_i, which is defined on the sites i of a lattice and plays the same role as the Ising spin, with the probability of the configuration φ_i being proportional to $\mathrm{e}^{-H[\varphi_i]}$. The connection between Ginzburg-Landau theory and the Ising model is not *a priori* clear, we merely assume they belong to the same universality class.

It should be stressed that the Ginzburg-Landau Hamiltonian $H[\varphi_i]$ is intended to describe the physical system *only in the vicinity of the critical point*, and it has the form

$$H[\varphi_i] = a^D \sum_i \left[\frac{1}{2}(\nabla\varphi_i)^2 + \frac{1}{2} r_0(T)\varphi_i^2 + \frac{1}{4!} u_0 \varphi_i^4 \right],$$

where a is the lattice spacing, the notations r_0 and u_0 are conventional in statistical physics. By analogy with Ising model, interactions between nearest neighbors are introduced by means of the discretized gradient $\nabla\varphi_i$ defined by

$$\nabla\varphi_i = \nabla\varphi(\boldsymbol{x}_i) = \frac{1}{a}[\varphi(\boldsymbol{x}_i + \boldsymbol{\mu}) - \varphi(\boldsymbol{x}_i)],$$

where the vector $\boldsymbol{\mu}$ links site $\boldsymbol{x_i}$ to one of the nearest neighbours. Instead of $\varphi(\boldsymbol{x}_i - \boldsymbol{\mu})\varphi(\boldsymbol{x}_i)$ the formulation $(\nabla\varphi_i)^2$ is adopted in the Hamiltonian, in fact, when comparing both, the latter one only yields extra terms φ_i^2, which amounts to a redefinition of r_0. Then the partition function is written as

$$Z = \int \prod_i \mathrm{d}\varphi_i \ \mathrm{e}^{-H[\varphi_i]}.$$

It's actually more convenient to calculate with a continuum formulation, where $\boldsymbol{x}_i$ varies continuously within the physical system, instead of insisting upon the lattice, then $\boldsymbol{x}_i \to \boldsymbol{x}$, and the field $\varphi(\boldsymbol{x}_i)$ becomes a function of the point $\boldsymbol{x}$, $\varphi(\boldsymbol{x}_i) \to \varphi(\boldsymbol{x})$.

In the critical region the important fluctuations have the length scale $\gg a$, so that the continuum formulation should be equivalent to the lattice formulation. In the continuum limit the Ginzburg-Landau Hamiltonian becomes

$$H[\varphi] = \int \mathrm{d}^D x \left[\frac{1}{2}(\nabla\varphi)^2 + \frac{1}{2}r_0(T)\varphi^2 + \frac{1}{4!}u_0\varphi^4\right].$$

The Hamiltonian is a *functional* of the field $\varphi(\boldsymbol{x})$. In order to construct the partition function we must integrate over all the configuration $\varphi(\boldsymbol{x})$, i.e. we must evaluate a *path integral*

$$Z = \int \mathcal{D}\varphi \ \mathrm{e}^{-H[\varphi]}. \tag{2.1}$$

The integration measure $\mathcal{D}\varphi$ is defined by

$$\mathcal{D}\varphi = \lim_{a\to 0} \mathcal{N}(a) \prod_i \mathrm{d}\varphi_i \ ,$$

where $\mathcal{N}(a)$ is a factor chosen to ensure that the limit exists, these multiplicative constants are unimportant, since they cancel between numerator and denominator during the calculation of correlation functions later. The existence of such an integration measure still needs mathematical proof, in tough cases we should go back to the lattice formulation.

3

Perturbation theory

In order to start calculating the fluctuations in the phase transition, we still need go through the basic concepts and techniques in field theory. From the mathematical point of view, statistical field theory is closely related to quantum field theory, they have similar formulations, so that we can borrow the vocabulary and the notations from the quantum field theory. Accordingly we could rewrite the Laundau-Ginzburg Hamiltonian as

$$H[\varphi] = \int \mathrm{d}^D x \left[\frac{1}{2}(\nabla\varphi)^2 + \frac{1}{2}m^2\varphi^2 + \frac{g}{4!}\varphi^4\right], \tag{3.1}$$

with $r_0 \to m^2$ and $u_0 \to g$, where m is a mass and g a coupling constant.

3.1 Wick's theorem and the generating functional

The essential aim is to evaluate the n-point correlation functions, which is the expectation value of the product of n fields

$$\langle\varphi(x_1)\varphi(x_2)\cdots\varphi(x_n)\rangle = \frac{1}{Z}\int \mathcal{D}\varphi\ \varphi(x_1)\varphi(x_2)\cdots\varphi(x_n)\mathrm{e}^{-H[\varphi]},$$

with the partition function Z given by (2.1). Before diving into the details of the calculation, I would like to demonstrate the simple cases first, then it should be easier to follow the general idea.

Notice that the probability distribution $\mathrm{e}^{-H[\varphi]}$ contains a *Gaussian* part. A significant fraction of the theoretical physics literature consists of varying and elaborating the basic Gaussian integral,[2] and here we start already with a *generating function* $Z(j)$ defined by

$$Z(j) = \int \mathrm{d}x\ p(x)\mathrm{e}^{jx} = \int \mathrm{d}x\ \mathrm{e}^{-\frac{1}{2}ax^2+jx}, \tag{3.2}$$

where $p(x) = \mathrm{e}^{-\frac{1}{2}ax^2}$ is a Gaussian probability distribution over a random variable, you see, the generating function is actually a variation of Gaussian integral. The *moments* $\langle x^n \rangle$, defined by

$$\langle x^n \rangle = \frac{\displaystyle\int \mathrm{d}x \; x^n p(x)}{\displaystyle\int \mathrm{d}x \; p(x)},$$

can be obtained by differentiating the generating function $Z(j)$

$$\langle x^n \rangle = \frac{1}{Z(0)} \left. \frac{\partial^n Z}{\partial j^n} \right|_{j=0};$$

together with the evaluation

$$Z(j) = \sqrt{\frac{2\pi}{a}} \mathrm{e}^{\frac{j^2}{2a}} = Z(0)\mathrm{e}^{\frac{j^2}{2a}},$$

we obtain

$$\begin{aligned} \langle x^{2n} \rangle &= \left. \frac{\partial^{2n}}{\partial j^{2n}} \mathrm{e}^{\frac{j^2}{2a}} \right|_{j=0} = \left. \frac{\partial^{2n}}{\partial j^{2n}} \sum_n \frac{1}{n!} \left(\frac{j^2}{2a} \right)^n \right|_{j=0} \\ &= \frac{\partial^{2n}}{\partial j^{2n}} \frac{1}{n!} \left(\frac{j^2}{2a} \right)^n = (2n-1)(2n-3) \cdots 5 \cdot 3 \cdot 1 \frac{1}{a^n}. \end{aligned} \tag{3.3}$$

Imagine $2n$ points and connect them in pairs, the first point can be connected to one of the rest $2n-1$ points, the second point can now be connected to the remaining $2n-3$ points, and so on, until all points are paired. We could interpret the factor $(2n-1)(2n-3)\cdots 5{\cdot}3{\cdot}1$ as the number of ways to connect $2n$ points in pairs, then to each pair a factor a^{-1} is assigned. Gian Carlo Wick made this clever observation, which further leads to Wick's theorem. A pattern of pairing all the points is known as *Wick contraction*.

To make this important theorem more clear, that is, to really distinguish the points, we promote the generating function (3.2) to the one with n variables

$$\begin{aligned} Z(\boldsymbol{j}) &= \int \prod_i \mathrm{d}x_i \; \mathrm{e}^{-\frac{1}{2}\boldsymbol{x}^T A \boldsymbol{x} + \boldsymbol{j}^T \boldsymbol{x}} \\ &= \left[\frac{(2\pi)^n}{\det A} \right]^{\frac{1}{2}} \mathrm{e}^{\frac{1}{2}\boldsymbol{j}^T A^{-1} \boldsymbol{j}}, \end{aligned} \tag{3.4}$$

where A is a real symmetric $n \times n$ matrix, $\boldsymbol{x}$ and $\boldsymbol{j}$ are both n-dimensional vectors, with $\boldsymbol{x}^T A \boldsymbol{x} = \sum_{i,j} x_i A_{ij} x_j$ and $\boldsymbol{j}^T \boldsymbol{x} = \sum_i j_i x_i$.

Accordingly the generalization of (3.3) yields a moment of order $2n$

$$\langle x_k x_l \cdots x_p x_q \rangle = \frac{\partial^{2n}}{\partial j_k \partial j_l \cdots \partial j_p \partial j_q} \, \mathrm{e}^{\frac{1}{2}\boldsymbol{j}^T A^{-1} \boldsymbol{j}} \Big|_{\boldsymbol{j}=0} = \sum_{\text{Wick}} A^{-1}_{ab} \cdots A^{-1}_{cd} \, ,$$

where the set of indices {a,b,...,c,d} represents a permutation of {k,l,..., q,p}, the way of pairing the indices under A^{-1} suggests a certain Wick contraction, and the sum is over all the possible Wick contractions. The calculation is straightforward, starting with a second moment

$$\langle x_m x_n \rangle = \frac{\partial^2}{\partial j_m \partial j_n} \left(\frac{1}{2} \sum_{k,l} j_k A^{-1}_{kl} j_l \right) = A^{-1}_{mn}, \tag{3.5}$$

until finally Wick's theorem is taking shape, *all the moments of a Gaussian distribution can be expressed as functions of the second moments alone.*[1]

Now we've made enough preparation to discuss about correlation functions. In the case of the field theory, similar to the generating function, we introduce the *generating functional*

$$Z(j) = \int \mathcal{D}\varphi \; \exp \left\{ -H[\varphi] + \int \mathrm{d}^D x \; j(x)\varphi(x) \right\}, \tag{3.6}$$

where the function $j(x)$ is called the *source of the field* φ. The $2n$-point correlation functions are nothing but the moments of order $2n$, obtained by differentiating the functional $Z(j)$

$$\langle \varphi(x_k)\varphi(x_l) \cdots \varphi(x_p)\varphi(x_q) \rangle = \frac{1}{Z(0)} \frac{\delta^{2n} Z}{\delta j(x_k) \delta j(x_l) \cdots \delta j(x_p) \delta j(x_q)} \Big|_{j=0} .$$

It's convenient to divide the Ginzburg-Landau Hamiltonian into a Gaussian part H_0

$$H_0 = \int \mathrm{d}^D x \left[\frac{1}{2} (\nabla\varphi)^2 + \frac{1}{2} m^2 \varphi^2 \right],$$

and a *interaction* part V

$$V = \int \mathrm{d}^D x \; \mathcal{V}[\varphi] = \frac{g}{4!} \int \mathrm{d}^D x \; \varphi^4(x),$$

then $Z(j)$ can be written as

$$Z(j) = \int \mathcal{D}\varphi \; \exp \left\{ -H_0[\varphi] - V[\varphi] + \int \mathrm{d}^D x \; j(x)\varphi(x) \right\}.$$

However, this formulation cannot be evaluated directly, that's why we need to expand it in a perturbation series.

3.2 The perturbation expansion and Feynman diagrams

In order to get familiar with the general rules for the perturbative calculation of correlation functions, we further adopt the Ginzburg-Landau Hamiltonian, or more exactly, the interaction φ^4, and the results can be easily generalized to other interactions.

Again we start with the simple case in one variable

$$Z(j) = \int \mathrm{d}x \; \mathrm{e}^{-\frac{1}{2}ax^2 - \frac{\lambda}{4!}x^4 + jx}.$$

It's easy enough to calculate the integral, if we expand $\mathrm{e}^{-\frac{\lambda}{4!}x^4}$, so that

$$\begin{aligned} Z(j) &= \int \mathrm{d}x \; \mathrm{e}^{-\frac{1}{2}ax^2 + jx} \left[1 - \frac{\lambda}{4!}x^4 + \frac{1}{2!}\left(\frac{\lambda}{4!}\right)^2 x^8 + \cdots \right] \\ &= \left[1 - \frac{\lambda}{4!}\left(\frac{\mathrm{d}}{\mathrm{d}j}\right)^4 + \frac{1}{2!}\left(\frac{\lambda}{4!}\right)^2 \left(\frac{\mathrm{d}}{\mathrm{d}j}\right)^8 + \cdots \right] \int \mathrm{d}x \; \mathrm{e}^{-\frac{1}{2}ax^2 + jx} \\ &= \mathrm{e}^{-\frac{\lambda}{4!}\left(\frac{\mathrm{d}}{\mathrm{d}j}\right)^4} \int \mathrm{d}x \; \mathrm{e}^{-\frac{1}{2}ax^2 + jx} = \sqrt{\frac{2\pi}{a}} \mathrm{e}^{-\frac{\lambda}{4!}\left(\frac{\mathrm{d}}{\mathrm{d}j}\right)^4} \mathrm{e}^{\frac{j^2}{2a}}. \end{aligned}$$

Here we've used the reformulation

$$f(x)\mathrm{e}^{jx} = f\left(\frac{\partial}{\partial j}\right) \mathrm{e}^{jx},$$

which can be proved through a Taylor expansion of $f\left(\frac{\partial}{\partial j}\right)$ near the origin. Remember we still need to evaluate the moment $\langle x^{2n} \rangle$ by

$$\langle x^{2n} \rangle = \frac{1}{Z(0)} \int \mathrm{d}x \; x^{2n} \mathrm{e}^{-\frac{1}{2}ax^2 - \frac{\lambda}{4!}x^4} = \frac{1}{Z(0)} \left. \frac{\partial^{2n} Z}{\partial j^{2n}} \right|_{j=0},$$

but first we want to take a closer look at the differentiation

$$\begin{aligned} \left. \frac{\partial^{2n} Z}{\partial j^{2n}} \right|_{j=0} &= \sqrt{\frac{2\pi}{a}} \frac{\partial^{2n}}{\partial j^{2n}} \left. \left[\mathrm{e}^{-\frac{\lambda}{4!}\left(\frac{\mathrm{d}}{\mathrm{d}j}\right)^4} \mathrm{e}^{\frac{j^2}{2a}} \right] \right|_{j=0} \\ &= \sqrt{\frac{2\pi}{a}} \frac{\partial^{2n}}{\partial j^{2n}} \left\{ \left[1 - \frac{\lambda}{4!}\left(\frac{\mathrm{d}}{\mathrm{d}j}\right)^4 + \frac{1}{2!}\left(\frac{\lambda}{4!}\right)^2 \left(\frac{\mathrm{d}}{\mathrm{d}j}\right)^8 + \cdots \right] \right. \\ &\quad \times \left. \left. \left[1 + \frac{j^2}{2a} + \frac{1}{2!}\left(\frac{j^2}{2a}\right)^2 + \frac{1}{3!}\left(\frac{j^2}{2a}\right)^3 + \cdots \right] \right\} \right|_{j=0}, \end{aligned} \tag{3.7}$$

which boils down to an expansion in powers of λ. To obtain the term of order λ, we obviously need to extract $-\frac{\lambda}{4!}\left(\frac{\mathrm{d}}{\mathrm{d}j}\right)^4$ from $\mathrm{e}^{-\frac{\lambda}{4!}\left(\frac{\mathrm{d}}{\mathrm{d}j}\right)^4}$. Notice that only the term $\frac{1}{(2+n)!}\left(\frac{j^2}{2a}\right)^{2+n}$ from the expansion of $e^{\frac{j^2}{2a}}$ survives after taking all the differentiations, both $\frac{\partial^{2n}}{\partial j^{2n}}$ and $-\frac{\lambda}{4!}\left(\frac{\mathrm{d}}{\mathrm{d}j}\right)^4$; applying $j=0$ in the end, then without considering the factor $\sqrt{\frac{2\pi}{a}}$ we get

$$-\frac{\lambda}{4!}\frac{(4+2n)!}{(2+n)!}\frac{1}{(2a)^{2+n}},$$

analogous

$$\frac{1}{2!}\left(-\frac{\lambda}{4!}\right)^2\frac{(8+2n)!}{(4+n)!}\frac{1}{(2a)^{4+n}}$$

can be deduced for the term of order λ^2. In general, the term of order λ^m is written as

$$\begin{aligned}&\frac{1}{m!}\left(-\frac{\lambda}{4!}\right)^m\frac{(4m+2n)!}{(2m+n)!}\frac{1}{(2a)^{2m+n}}\\ =&\frac{1}{m!}\left(-\frac{\lambda}{4!}\right)^m(4m+2n-1)(4m+2n-3)\cdots5\cdot3\cdot1\frac{1}{a^{2m+n}}.\end{aligned}$$

This expression looks familiar compared to (3.3), and indeed we can figure out a pattern here using Wick's theorem. For the term of order λ^m we can think of $4m+2n$ points, with $2n$ *external* points originated from $\frac{\partial^{2n}}{\partial j^{2n}}$ (or x^{2n}) relating to the moment of order $2n$ and $4m$ *internal* points from $\frac{1}{m!}\left(-\frac{\lambda}{4!}\right)^m\left(\frac{\mathrm{d}}{\mathrm{d}j}\right)^{4m}$ in the expansion of $\mathrm{e}^{-\frac{\lambda}{4!}\left(\frac{\mathrm{d}}{\mathrm{d}j}\right)^4}$ (or $\frac{1}{m!}\left(-\frac{\lambda}{4!}\right)^m x^{4m}$ in the expansion of $\mathrm{e}^{-\frac{\lambda}{4!}x^4}$), the rest, you already know, is to pair the points into Wick contractions.

Richard Feynman merged the $4m$ internal points into m *vertices*, with each vertex sending out 4 open connections from one point as in Figure 3.1, and presented each Wick contraction with lines drawn among the external points and the vertices, the resultant diagram is known as a *Feynman diagram*.

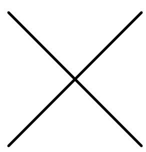


Figure 3.1 Vertex

The next step is to calculate $Z(0)$ with

$$
\begin{aligned}
Z(0) &= \int \mathrm{d}x \ \mathrm{e}^{-\frac{1}{2}ax^2 - \frac{\lambda}{4!}x^4 + jx}\Big|_{j=0} \\
&= \mathrm{e}^{-\frac{\lambda}{4!}\left(\frac{\mathrm{d}}{\mathrm{d}j}\right)^4} \int \mathrm{d}x \ \mathrm{e}^{-\frac{1}{2}ax^2 + jx}\Big|_{j=0} = \left(\frac{2\pi}{a}\right)^{\frac{1}{2}} \mathrm{e}^{-\frac{\lambda}{4!}\left(\frac{\mathrm{d}}{\mathrm{d}j}\right)^4} \mathrm{e}^{\frac{j^2}{2a}}\Big|_{j=0} \\
&= \left(\frac{2\pi}{a}\right)^{\frac{1}{2}} \left\{\left[1 - \frac{\lambda}{4!}\left(\frac{\mathrm{d}}{\mathrm{d}j}\right)^4 + \frac{1}{2!}\left(\frac{\lambda}{4!}\right)^2\left(\frac{\mathrm{d}}{\mathrm{d}j}\right)^8 + \cdots\right]\right. \\
&\quad \times \left.\left[1 + \frac{j^2}{2a} + \frac{1}{2!}\left(\frac{j^2}{2a}\right)^2 + \frac{1}{3!}\left(\frac{j^2}{2a}\right)^3 + \cdots\right]\right\}\Big|_{j=0}. \qquad (3.8)
\end{aligned}
$$

If you compare the expansion with (3.7), only the differentiation $\frac{\partial^{2n}}{\partial j^{2n}}$ is missing, which, by following the same idea in solving (3.7), simply means that the external points are excluded from the Wick contractions, and only the internal points, or the vertices are involved. The resultant Feynman diagrams are known as *vacuum fluctuations*.

To obtain the moment $\langle x^{2n}\rangle$, we must divide (3.7) by (3.8). Division by $Z(0)$ cancels all the diagrams containing vacuum fluctuations from (3.7). Instead of the explicit calculation we can prove this in a general way.

A diagram of order λ^m in (3.7), with $m = p + q$; possesses a factor $\frac{1}{m!}$ and m vertices, suppose p vertices of them form vacuum fluctuations, which are disconnected from the rest of the diagram containing the external points, and there are C_m^q ways of choosing p vertices out of m. Then (3.7) can be written as

$$
\begin{aligned}
&\sum_m \sum_{p+q=m} \frac{C_m^q}{m!}[\text{vacuum fluctuation}(p)][\text{rest}(q)] \\
&= \sum_{p,q} \frac{1}{p!q!}[\text{vacuum fluctuation}(p)][\text{rest}(q)] \\
&= \sum_p \frac{1}{p!}[\text{vacuum fluctuation}(p)] \sum_q \frac{1}{q!}[\text{rest}(q)]
\end{aligned}
$$

Notice that $\sum_p \frac{1}{p!}[\text{vacuum fluctuation}(p)]$ is just $Z(0)$, and the moment $\langle x^{2n}\rangle$ is then given by $\sum_q \frac{1}{q!}[\text{rest}(q)]$, which can be reconstructed using Wick's theorem.

In addition, instead of λ we can expand $Z(j)$ in powers of j as well

$$\begin{aligned} Z(j) &= \sum_s \frac{1}{s!} j^s \int \mathrm{d}x \; x^s \mathrm{e}^{-\frac{1}{2}ax^2 - \frac{\lambda}{4!}x^4} \\ &= Z(0) \sum_s \frac{1}{s!} j^s \langle x^s \rangle. \end{aligned} \tag{3.9}$$

All these features in the case of one variable are structurally the same as the corresponding features in field theory. For more details, we still want to add the case of n variables

$$\begin{aligned} Z(\boldsymbol{j}) &= \int \prod_q \mathrm{d}x_q \; \mathrm{e}^{-\frac{1}{2}\boldsymbol{x}^T A \boldsymbol{x} - \frac{\lambda}{4!}\boldsymbol{x}^4 + \boldsymbol{j}^T \boldsymbol{x}} \\ &= \left[\frac{(2\pi)^n}{\det A} \right]^{\frac{1}{2}} \mathrm{e}^{-\frac{\lambda}{4!}\sum_i \left(\frac{\partial}{\partial j_i}\right)^4} \mathrm{e}^{\frac{1}{2}\boldsymbol{j}^T A^{-1} \boldsymbol{j}}, \end{aligned}$$

where $\boldsymbol{x}^4 = \sum_i x_i^4$. Alternatively, as in (3.9) $Z(\boldsymbol{j})$ can be written as

$$\begin{aligned} Z(\boldsymbol{j}) &= \sum_s \sum_{i_1,\ldots,i_s} \frac{1}{s!} j_{i_1} \cdots j_{i_s} \int \prod_q \mathrm{d}x_q \; x_{i_1} \cdots x_{i_s} \mathrm{e}^{-\frac{1}{2}\boldsymbol{x}^T A \boldsymbol{x} - \frac{\lambda}{4!}\boldsymbol{x}^4} \\ &= Z(0) \sum_s \sum_{i_1,\ldots,i_s} \frac{1}{s!} j_{i_1} \cdots j_{i_s} \langle x_{i_1} \cdots x_{i_s} \rangle. \end{aligned}$$

Now with Wick's theorem we evaluate $\langle x_i x_j \rangle$ up to order λ^2

$$\begin{aligned} \langle x_i x_j \rangle = \frac{1}{Z(0)} \int \prod_q \mathrm{d}x_q \; x_i x_j \mathrm{e}^{-\frac{1}{2}\boldsymbol{x}^T A \boldsymbol{x}} \\ \times \left[1 - \frac{\lambda}{4!} \sum_n x_n^4 + \frac{1}{2!} \left(-\frac{\lambda}{4!} \right)^2 \sum_{m,n} x_m^4 x_n^4 + \mathcal{O}(\lambda^2) \right] \end{aligned}$$

The idea is to draw possible types of Feynman diagrams for each term, involving all the external and internal points with the latter treated as vertices; then the diagrams containing vacuum fluctuations are discarded, since they are canceled after the division by Z(0). In this case x_i and x_j are the external points, and the order in λ suggests the number of the vertices.

The term of order λ^0, without vertices, simply yields A_{ij}^{-1}, as already evaluated in (3.5). We can think of the index as labels for the sites on a lattice, and the matrix element A_{ij}^{-1}, known as a *propagator*, describes the propagation between i and j.

For the term of order λ there are two types of diagrams[1], shown in Figure 3.2.

Figure 3.2

The analytic expression corresponding to a diagram is evaluated according to the *Feynman rules*: To every line joining two points, a propagator is assigned; every vertex possesses a factor $-\lambda$, and over every vertex is summed; for every diagram a multiplicative *symmetry factor* is calculated. Then the first diagram in Figure 3.2 is written as

$$-\frac{\lambda}{4!} \cdot 12 \sum_n A^{-1}_{in} A^{-1}_{nn} A^{-1}_{jn},$$

The factor 12 suggests the number of all possible Wick contractions, the whole factor $\frac{12}{4!} = \frac{1}{2}$ yields the symmetry factor. The second diagram contains vacuum fluctuation and does not feature in the expansion.

As for the term of order λ^2 we find three types of diagrams containing no vacuum fluctuations[1], shown in Figure 3.3.

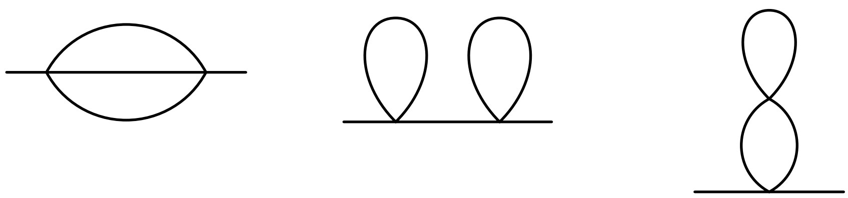


Figure 3.3

Thus the term is summarized as

$$\lambda^2 \sum_{m,n} \left[\frac{1}{6} \cdot A^{-1}_{im} (A^{-1}_{mn})^3 A^{-1}_{jn} + \frac{1}{4} \cdot A^{-1}_{im} A^{-1}_{mm} A^{-1}_{mn} A^{-1}_{nn} A^{-1}_{jn} \right.$$
$$\left. + \frac{1}{4} \cdot A^{-1}_{im} (A^{-1}_{mn})^2 A^{-1}_{nn} A^{-1}_{jn} \right],$$

with the summands corresponding to the order of the diagrams.

We should still pay attention that the vertices m and n can be permuted, which yields a multiplicative factor 2!, but this is exactly canceled by the factor $\frac{1}{2!}$ from the expansion of $\mathrm{e}^{-\frac{\lambda}{4!}\boldsymbol{x}^4}$. Analogues, for the term of order λ^n, the factor $n!$ from permuting the vertices is canceled by the factor $\frac{1}{n!}$ from the expansion of the exponential.

The discussion above can be almost directly carried over to the perturbative field theory. For the Landau-Ginzburg Hamiltonian

$$\int \mathrm{d}^D x \, \left[\frac{1}{2}(\nabla\varphi)^2 + \frac{1}{2}m^2\varphi^2 + \frac{g}{4!}\varphi^4\right]$$

the generating functional reads

$$\begin{aligned}
Z(j) &= \int \mathcal{D}\varphi \, \exp\left\{\int \mathrm{d}^D x \, \left(-\frac{1}{2}(\nabla\varphi)^2 - \frac{1}{2}m^2\varphi^2 - \frac{g}{4!}\varphi^4 + j\varphi\right)\right\} \\
&= \exp\left[-\frac{g}{4!}\int \mathrm{d}^D z \, \left(\frac{\delta}{\delta j(z)}\right)^4\right] \\
&\quad \cdot \int \mathcal{D}\varphi \, \exp\left[\int \mathrm{d}^D x \, \left(-\frac{1}{2}(\nabla\varphi)^2 - \frac{1}{2}m^2\varphi^2 + j\varphi\right)\right] \\
&= \mathcal{N}\exp\left[-\frac{g}{4!}\int \mathrm{d}^D z \, \left(\frac{\delta}{\delta j(z)}\right)^4\right] \\
&\quad \cdot \exp\left(\frac{1}{2}\int \mathrm{d}^D x \, \mathrm{d}^D y \; j(x)G_0(x-y)j(y)\right),
\end{aligned}$$

where $G_0(x-y)$ is the two-point correlation function of the Gaussian model [1,8], with

$$G_0(x-y) = \int \frac{\mathrm{d}^D k}{(2\pi)^D}\frac{\mathrm{e}^{-\mathrm{i}k(x-y)}}{k^2+m^2}, \tag{3.10}$$

it plays the same role of A_{ij}^{-1} as the propagator; $\mathcal{N}$ represents a normalization constant, it cancels during the calculation of correlation functions.

We can also expand $Z(j)$ in j

$$\begin{aligned}
Z(j) &= \sum_s \frac{1}{s!}\int \mathrm{d}x_1 \cdots \mathrm{d}x_s \; j(x_1)\cdots j(x_s) \int \mathcal{D}\varphi \; \varphi(x_1)\cdots\varphi(x_s)\mathrm{e}^{-H} \\
&= Z(0)\sum_s \frac{1}{s!}\int \mathrm{d}x_1 \cdots \mathrm{d}x_s \; j(x_1)\cdots j(x_s)G^{(s)}(x_1,\ldots,x_s),
\end{aligned}$$

where $G^{(s)}(x_1,\ldots,x_s)$ is known as the s-point correlation functions or s-point Green's functions, which are the analogues of the moments. In particular, the

2-point Green's function reads

$$G(x_i, x_j) = \frac{1}{Z(0)} \int \mathcal{D}\varphi\ \varphi(x_i)\varphi(x_j)\mathrm{e}^{-H},$$

which share exactly the same type of Feynman diagrams with $\langle x_i x_j \rangle$. In the corresponding Feynman rules, A_{ij}^{-1} is replaced by $G_0(x_i - x_j)$ and the summation over the vertices is replaced by integral.

We can take a further look at the 4-point Green's function $G(x_i, x_j, x_k, x_l)$ up to order g

$$\begin{aligned} &G(x_i, x_j, x_k, x_l) \\ =& \frac{1}{Z(0)} \int \mathcal{D}\varphi\ \varphi(x_i)\varphi(x_j)\varphi(x_k)\varphi(x_l)\mathrm{e}^{-H} \\ =& \frac{1}{Z(0)} \int \mathcal{D}\varphi\ \varphi(x_i)\varphi(x_j)\varphi(x_k)\varphi(x_l)\mathrm{e}^{-H_0} \\ &\cdot \left[1 - \frac{g}{4!} \int \mathrm{d}^D z\ \varphi(z)^4 + \mathcal{O}(g^2)\right] \end{aligned}$$

Again the evaluation follows Wick's theorem. The term of order g^0 is the sum of three *disconnected diagrams* [1] shown in Figure 3.4,

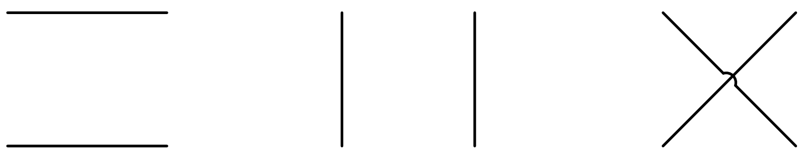


Figure 3.4

written as

$$G_0(x_i - x_j)G_0(x_k - x_l) + G_0(x_i - x_k)G_0(x_j - x_l) + G_0(x_i - x_l)G_0(x_j - x_k).$$

For the order g there are three types of diagrams[1], as in Figure 3.5.

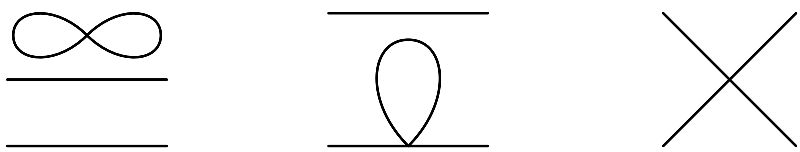


Figure 3.5

The first type of diagram (from left to right in Figure 3.5) contains vacuum fluctuations, and is eliminated on division by $Z(0)$; the second type represents disconnected diagrams, and can be written as product of two-point correlation functions, which have already been calculated; the third type, the most interesting one, is connected, and is expressed as

$$-\frac{g}{4!}\cdot 4!\int \mathrm{d}^D z\ G_0(x_i-z)G_0(x_j-z)G_0(x_k-z)G_0(x_l-z).$$

3.3 Connected correlation functions

As already observed in the example of $G(x_i, x_j, x_k, x_l)$, the disconnected diagrams can be separated into two or more disjoint parts. To ease the calculation we want to limit ourselves to the connected diagrams, which cannot be further decomposed without cutting at least one line.

In general the n-point correlation function $G^{(s)}$ can be subdivided into connected diagrams*, as in Figure 3.6.

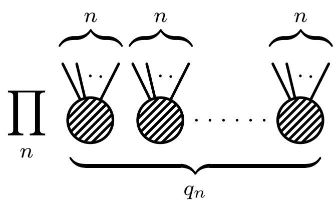


Figure 3.6

The subdiagrams are sorted out according to the number of their external points, we assume that there are q_n connected subdiagrams $G_c^{(n)}$ with n external points, so that $\sum_n nq_n = s$. The number of such disconnected diagrams is

$$\frac{s!}{\prod\limits_n q_n!(n!)^{q_n}}.$$

By permuting the s external points of $G^{(s)}$ we generate $s!$ candidates for the disconnected diagrams, but the permutation of the q_n subdiagrams $G_c^{(n)}$ or the n external points connected to any of them leads to equivalent diagrams,

*There might be print errors in the original proof given by Le Bellac [1], here I've redrawn the schema in Figure 3.6, and modified the proof accordingly, yet I also came down to the same conclusion, you may check it if you care.

whence the division by $\prod_n q_n!(n!)^{q_n}$. The generating function $Z(j)$ then reads

$$\begin{aligned}\frac{Z(j)}{Z(0)} &= \sum_s \frac{1}{s!}\int \mathrm{d}x_1\cdots\mathrm{d}x_s\; j(x_1)\cdots j(x_s)G^{(s)}(x_1,\ldots,x_s)\\ &= \sum_s \frac{1}{s!}\int \mathrm{d}x_1\cdots\mathrm{d}x_s\; j(x_1)\cdots j(x_s)\\ &\quad\times \sum_{\sum_n nq_n = s} G_c^{(1)}(x_1)\cdots G_c^{(n)}(\cdots x_s)\\ &= \sum_{q_1,\ldots,q_n}\prod_n \frac{1}{q_n!}\left[\frac{\int \mathrm{d}x_1\cdots\mathrm{d}x_n\; j(x_1)\cdots j(x_n)G_c^{(n)}(x_1,\ldots,x_n)}{n!}\right]^{q_n}\\ &= \exp\left(\sum_n \frac{1}{n!}\int \mathrm{d}x_1\cdots\mathrm{d}x_n\; j(x_1)\cdots j(x_n)G_c^{(n)}(x_1,\ldots,x_n)\right).\end{aligned}$$

We actually found the *generating functional $W(j)$ of the connected correlation function*

$$W(j) = \ln\frac{Z(j)}{Z(0)} = \sum_n \frac{1}{n!}\int \mathrm{d}x_1\cdots\mathrm{d}x_n\; j(x_1)\cdots j(x_n)G_c^{(n)}(x_1,\ldots,x_n).$$

3.4 Proper vertices

To further simplify the Feynman diagrams, we define the *proper vertex*, or *1-particle irreducible vertex* (1-PI vertex), which cannot be separated by cutting a single internal propagator, and from which *full propagators* corresponding to external lines are removed. A full propagator is just the 2-point Green's function, which can be written as

$$G^{(2)} = G_0 + G_0\Sigma G^{(2)} = G_0\left(1+\sum_n(\Sigma G_0)^n\right) = \frac{1}{G_0^{-1}-\Sigma},$$

where G_0 is the bare propagator, as in (3.5), Σ is called *self-engergy*, the sum of all 2-point 1-PI diagrams shorn of their external lines.

In the following we prove that *the generating function of proper vertices is the Legendre transform of $W(j)$* [1,5].

The Legendre transformation of $W(j)$ is written as Γ, given by

$$\Gamma[\varphi_c] = W(j) - \int \mathrm{d}^D x\; \varphi_c(x)j(x),\ \text{with}\ \frac{\delta\Gamma[\varphi_c]}{\delta\varphi_c(x)} = -j(x)$$

where

$$\varphi_c(x) = \frac{\delta W(j)}{\delta j(x)} = \frac{1}{Z(j)}\frac{\delta Z(j)}{\delta j(x)} = \frac{1}{Z(j)}\int \mathcal{D}\varphi\ \varphi(x)\mathrm{e}^{-H[\varphi]+j\varphi}$$

is the mean value of $\varphi(x)$ in the presence of $j(x)$, also known as the *classical field*, obviously, when $j = 0$, $\varphi_c(x)$ takes the mean value $\langle\varphi(x)\rangle$.

By differentiating the identity

$$\begin{aligned}\delta(x-y) = \frac{\delta j(x)}{\delta j(y)} &= -\frac{\delta^2\Gamma}{\delta\varphi_c(x)\delta j(y)} = -\int \mathrm{d}^D z\ \frac{\delta^2\Gamma}{\delta\varphi_c(x)\delta\varphi_c(z)}\frac{\delta\varphi_c(z)}{\delta j(y)} \\ &= -\int \mathrm{d}^D z\ \frac{\delta^2\Gamma}{\delta\varphi_c(x)\delta\varphi_c(z)}\frac{\delta^2 W}{\delta j(z)\delta j(y)} \end{aligned} \tag{3.11}$$

with respect to $j(w)$ we obtain

$$\begin{aligned}&\int \mathrm{d}^D z\ \frac{\delta^3\Gamma}{\delta\varphi_c(x)\delta\varphi_c(z)\delta j(w)}\frac{\delta^2 W}{\delta j(z)\delta j(y)} \\ &\qquad + \int \mathrm{d}^D z\ \frac{\delta^2\Gamma}{\delta\varphi_c(x)\delta\varphi_c(z)}\frac{\delta^3 W}{\delta j(z)\delta j(y)\delta j(w)} = 0,\end{aligned} \tag{3.12}$$

which, with the same manipulation in (3.11), can be written as

$$\begin{aligned}&\int \mathrm{d}^D z\,\mathrm{d}^D u\ \frac{\delta^3\Gamma}{\delta\varphi_c(x)\delta\varphi_c(z)\delta\varphi_c(u)}\frac{\delta^2 W}{\delta j(u)\delta j(w)}\frac{\delta^2 W}{\delta j(z)\delta j(y)} \\ &\qquad + \int \mathrm{d}^D z\ \frac{\delta^2\Gamma}{\delta\varphi_c(x)\delta\varphi_c(z)}\frac{\delta^3 W}{\delta j(z)\delta j(y)\delta j(w)} = 0.\end{aligned} \tag{3.13}$$

Multiply (3.13) with $\dfrac{\delta^2 W}{\delta j(x)\delta j(v)}$ and integrate over x, we find the relation

$$\begin{aligned}&\int \mathrm{d}^D z\,\mathrm{d}^D u\,\mathrm{d}^D x\ \frac{\delta^3\Gamma}{\delta\varphi_c(x)\delta\varphi_c(z)\delta\varphi_c(u)}\frac{\delta^2 W}{\delta j(u)\delta(w)}\frac{\delta^2 W}{\delta j(z)\delta(y)}\frac{\delta^2 W}{\delta j(x)\delta j(v)} \\ &= -\int \mathrm{d}^D z\,\mathrm{d}^D x\ \frac{\delta^2\Gamma}{\delta\varphi_c(x)\delta\varphi_c(z)}\frac{\delta^2 W}{\delta j(x)\delta j(v)}\frac{\delta^3 W}{\delta j(z)\delta j(y)\delta j(w)} \\ &= \frac{\delta^3 W}{\delta j(v)\delta j(y)\delta j(w)}.\end{aligned} \tag{3.14}$$

Now apply $j = 0$ in (3.14) and define

$$\Gamma^{(3)}(x,z,u) = \left.\frac{\Gamma^3}{\delta\varphi_c(x)\delta\varphi_c(z)\delta\varphi_c(u)}\right|_{\varphi_c=\langle\varphi\rangle},$$

we have

$$\int \mathrm{d}^D z\, \mathrm{d}^D u\, \mathrm{d}^D x\; \Gamma^{(3)}(x,z,u) G_c^{(2)}(u,w) G_c^{(2)}(z,y) G_c^{(2)}(x,v) \\ = G_c^{(3)}(v,y,w). \quad (3.15)$$

where $G_c^{(2)} = G^{(2)}$ is the full propagator, $G_c^{(3)}$ is the 3-point connected correlation function, so that $\Gamma^{(3)}(x,z,u)$ must be a 3-point proper vertex, with all the full propagators explicitly removed.

By every successive functional differentiation of (3.14) with respect to j, then applying $j=0$, we get on the left-hand side of the equation higher oders of the 1-PI irreducible terms including a n-point proper vertex

$$\Gamma^{(n)}(x_1 \cdots x_n) = \frac{\delta^n \Gamma}{\delta j x_1 \cdots \delta j x_n}$$

and other uninteresting 1-PI reducible terms, so the proof is complete.

3.5 The loop expansion

We can expand the generating functional $Z(j)$ not only in powers of the coupling constant g and the source $j(x)$, but also in number of *loops*. For this purpose we write $Z(j)$ as

$$\begin{aligned} Z(j) &= \int \mathcal{D}\varphi\; \exp\left\{ -\beta H[\varphi] + \int \mathrm{d}^D x\; j(x)\varphi(x) \right\} \\ &= \mathcal{N} \exp\left\{ -\beta \int \mathrm{d}^D x\; \mathcal{V}\left(\frac{\delta}{\delta j(x)} \right) \right\} \\ &\quad \cdot \exp\left\{ \frac{1}{2} \int \mathrm{d}^D x\, \mathrm{d}^D y\; j(x) \beta^{-1} G_0(x,y) j(y) \right\}, \end{aligned}$$

as a result, every interaction (or vertex) is multiplied by β, and every propagation (or line) is multiplied by β^{-1}. Therefore every diagram with V vertices, I internal lines and E external lines is multiplied by $\beta^{V-I-E} = \beta^{1-E-L}$, where $L = I - V + 1$ is the number of loops. We can interpret the loops this way: to connect all the vertices at least $V-1$ internal lines are needed, an extra internal line produces an extra loop, thus $I-(V-1)$ loops can be made.

Part II

The roughening of the interface

4

The system with kink interface

The aim of this thesis is to further investigate the interfacial profile $\langle\varphi\rangle$ up to 2-loop order in a Landau-Ginzburg-system with boundary conditions. Here we adopt the Landau-Ginzburg Hamiltonian

$$H[\varphi] = \int \mathrm{d}^D x\ \mathcal{H}[\varphi] = \int \mathrm{d}^D x\ \left[\frac{1}{2}(\nabla\varphi)^2 - \frac{1}{4}m_0^2\varphi^2 + \frac{g_0}{4!}\varphi^4 + \frac{3}{8}\frac{m_0^4}{g_0}\right], \quad (4.1)$$

notice that comparing to (3.1) the Hamiltonian density $\mathcal{H}[\varphi]$ in (4.1) now has two local minima due to a sign change of the φ^2 term, as we can see in Figure 4.1, suggesting the possibility of a broken symmetry.

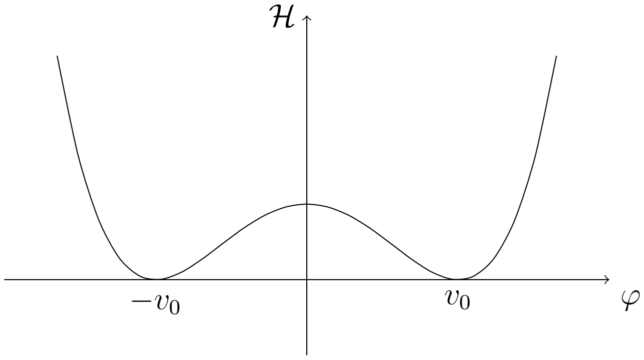


Figure 4.1 The broken symmetry

4.1 The interface and kink solutions

In order to describe the physical model we are about to work with, it would be convenient to set it up in an orthogonal coordinate system with one direction z extending to infinity, and the other directions $(x_1, \dots, x_{D-1}) = \vec{x}$ restricted in space $[0, L]^{D-1}$, so that any point within the system can be written as $(x_1, \dots, x_{D-1}, z)$. We want to ignore the surface effect in $(x_1, \dots, x_{D-1})$

due to the limited dimension, and assume the periodic boundary conditions $\varphi(x_1 + L, \ldots, x_{D-1}, z) = \cdots = \varphi(x_1, \ldots, x_{D-1} + L, z) = \varphi(x_1, \ldots, x_{D-1}, z)$. To further feature an interface in the system we add in the z-direction the boundary conditions

$$\varphi(x) = \begin{cases} v_0, & z \to +\infty \\ -v_0, & z \to -\infty, \end{cases} \tag{4.2}$$

which force the formation of a continuous transition from $-v_0$ to v_0 in the system, where $\varphi_0 = \pm v_0 = \pm\sqrt{3m_0^2/g_0}$ are the two local minima of the Hamiltonian density $\mathcal{H}[\varphi]$ with $\mathcal{H}[\pm v_0] = 0$, where $\pm v_0$ are the space independent solutions of the equation

$$\left.\frac{\delta \mathcal{H}}{\delta \varphi}\right|_{\varphi=\varphi_0} = \left(-\nabla^2 - \frac{m_0^2}{2}\right)\varphi_0 + \frac{g_0}{3!}\varphi_0^3 = 0. \tag{4.3}$$

Beside the pair of degenerate minima at $\varphi_0 = \pm v_0$ the equation (4.3) also possesses space dependent solutions, known as the *kink* solutions or Cahn-Hilliard-profiles[23]

$$\varphi_0^{(a)} = v_0 \tanh\left[\frac{m_0}{2}(z - a)\right], \tag{4.4}$$

which are constructed, in particular, to meet the boundary conditions (4.2). It has the profile that asymptotically approaches the two values $\pm v_0$ as $z \to \pm\infty$, as shown in figure 4.2.

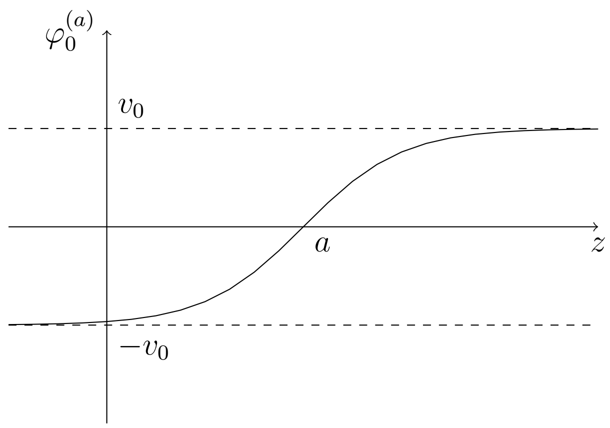


Figure 4.2 The Cahn-Hilliard-profile

The parameter $a \in \mathbb{R}$ can take any value, and it indicates the intersection of the kink with the z-axis.

Now the mean value $\langle\varphi\rangle$ near the critical point shall be evaluated by expanding it around one of the stable solutions $\varphi_0 := \varphi_0^{(0)}$, where the Hamiltonian has local minima. By defining the fluctuation $\eta := \varphi - \varphi_0$, $\langle\varphi\rangle$ can be written as

$$\begin{aligned}\langle\varphi\rangle = \langle\varphi_0+\eta\rangle = \frac{1}{Z_0}\int \mathcal{D}\eta\ (\varphi_0+\eta)\mathrm{e}^{-\beta H[\varphi_0+\eta]} = \varphi_0 + \frac{1}{Z_0}\int \mathcal{D}\eta\ \eta\ \mathrm{e}^{-\beta H[\varphi_0+\eta]} \\ = \varphi_0 + \langle\eta\rangle, \qquad (4.5)\end{aligned}$$

where

$$Z_0 = \int \mathcal{D}\eta\ \mathrm{e}^{-\beta H[\varphi_0+\eta]},$$

and $H[\varphi_0+\eta]$ can be expanded in Taylor series

$$\begin{aligned}H[\varphi_0+\eta] = H[\varphi_0] + \frac{1}{2!}\int \mathrm{d}^D x\ \eta(x)\left[-\nabla^2 - \frac{m_0^2}{2} + \frac{g_0}{2}\varphi_0^2(x)\right]\eta(x) \\ + \frac{g_0}{3!}\int \mathrm{d}^D x\ \varphi_0(x)\eta^3(x) + \frac{g_0}{4!}\int \mathrm{d}^D x\ \eta^4(x), \quad (4.6)\end{aligned}$$

where, with a few steps, the value of $H[\varphi_0]$ can be worked out [6,7] as

$$H[\varphi_0] = L^{D-1}\frac{2m_0^3}{g_0},$$

then the problem is reduced to the calculation of $\langle\eta\rangle$ in a loop expansion.

Clearly, the kink solutions are extrema of the Hamiltonian, by examining the sign of the second derivatives of H at kink solutions, which can be directly read in (4.6), given by the so-called *fluctuation operators* [7]

$$\mathbb{K} = -\nabla^2 - \frac{m_0^2}{2} + \frac{g_0}{2}v_0^2\tanh^2\left[\frac{m_0}{2}(z-a)\right],$$

we can see if the kink solutions are the local minima, and in fact, we are dealing with the eigenvalue problem

$$\mathbb{K}\eta(x) = \lambda\eta(x).$$

The eigenfunctions and eigenvalues of the fluctuation operator are analytically available, but we'll discuss about them in detail later. For now we just give the conclusion that all the eigenvalues of $\mathbb{K}$ are positive except one, which is zero, known as the *translation mode* or *zero mode*, thus the kink solutions are local minima, or classically stable in all modes except the translation mode.

4.2 The translation mode

The existence of the translation mode originates from the translational invariance symmetry of equation (4.3) under the boundary conditions [24], this results in a whole family of kink solutions in (4.4), all having the same shape, yet differing from one another by a translation in z-axis. All these kink solutions have the same energy $H[\varphi_0^{(a)}]$, since the integration of the Hamiltonian density goes through the whole space in $z \in \mathbb{R}$, the parameter a can be eliminated with a simple substitution, so that the kink solutions build a equipotential curve in the functional space. By defining a small displacement of the kink solution along the z-axis

$$\begin{aligned}\delta\varphi_0^{(a)} = \varphi_0^{(a+\delta a)} - \varphi_0^{(a)} &= \frac{\partial \varphi_0^{(a)}}{\partial a}\delta a + \mathcal{O}\left((\delta a)^2\right) \\ &= -\frac{\partial \varphi_0^{(a)}}{\partial z}\delta a + \mathcal{O}\left((\delta a)^2\right),\end{aligned}$$

we obtain

$$\begin{aligned}H[\varphi_0^{(a+\delta a)}] &= H[\varphi_0^{(a)} + \delta\varphi_0^{(a)}] \\ &= H[\varphi_0^{(a)}] + \frac{1}{2!}\int \mathrm{d}^D x\ \delta\varphi_0^{(a)}\left[-\nabla^2 - \frac{m_0^2}{2} + \frac{g}{2}\left(\varphi_0^{(a)}\right)^2\right]\delta\varphi_0^{(a)} + \cdots \\ &= H[\varphi_0^{(a)}] + \frac{1}{2!}(\delta a)^2\int \mathrm{d}^D x\ \frac{\partial\varphi_0^{(a)}}{\partial z}\mathbb{K}\frac{\partial\varphi_0^{(a)}}{\partial z} + \mathcal{O}\left((\delta a)^3\right) \qquad (4.7)\end{aligned}$$

notice that

$$H[\varphi_0^{(a+\delta a)}] = H[\varphi_0^{(a)}],$$

thus the term of order $(\delta a)^2$ (including all the other terms containing higher order of δa) in (4.7) must vanish, and this just suggests the translation mode. It's also possible to prove [6,7] that the normalized eigenfunction of the translation mode is

$$\eta_{\vec{0}\xi_0}^{(a)} = \frac{1}{\sqrt{H[\varphi_0]}}\frac{\partial\varphi_0^{(a)}}{\partial z}.$$

4.3 Collective coordinate methode

Unfortunately, the translation mode causes divergency of the Gaussian integral, to get around with this we propose the *collective coordinate method* [6,7], which, in our case, isolates the equipotential coordinate a, and thus separates the translation mode from the integral.

In this method we first define the expansion coefficient of the fluctuation $\eta^{(a)} = \varphi - \varphi_0^{(a)}$ corresponding to the translation mode as

$$c_0(a) := \int \mathrm{d}^D x \; \eta^{(a)} \frac{1}{\sqrt{H[\varphi_0]}} \frac{\partial \varphi_0^{(a)}}{\partial z},$$

and introduce the identity

$$1 = \int \mathrm{d}c_0 \; \delta(c_0) = \int \mathrm{d}a \; \frac{\mathrm{d}c_0}{\mathrm{d}a} \delta(c_0),$$

where

$$\begin{aligned}
\frac{\mathrm{d}c_0}{\mathrm{d}a} &= \frac{\mathrm{d}}{\mathrm{d}a} \left(\int \mathrm{d}^D x \; \eta^{(a)} \frac{1}{\sqrt{H[\varphi_0]}} \frac{\partial \varphi_0^{(a)}}{\partial z} \right) \\
&= \int \mathrm{d}^D x \left(\frac{\partial \eta^{(a)}}{\partial a} \frac{1}{\sqrt{H[\varphi_0]}} \frac{\partial \varphi_0^{(a)}}{\partial z} + \eta^{(a)} \frac{1}{\sqrt{H[\varphi_0]}} \frac{\partial^2 \varphi_0^{(a)}}{\partial z \partial a} \right) \\
&= \int \mathrm{d}^D x \left[\frac{1}{\sqrt{H[\varphi_0]}} \left(\frac{\partial \varphi_0^{(a)}}{\partial z} \right)^2 - \eta^{(a)} \frac{1}{\sqrt{H[\varphi_0]}} \frac{\partial^2 \varphi_0^{(a)}}{\partial z^2} \right] \\
&= \sqrt{H[\varphi_0]} \left(1 - \int \mathrm{d}^D x \; \eta^{(a)} \frac{1}{H[\varphi_0]} \frac{\partial^2 \varphi_0^{(a)}}{\partial z^2} \right).
\end{aligned}$$

Then by inserting the identity in the partition function

$$\int \mathcal{D}\eta^{(a)} \; \mathrm{e}^{-\beta H[\varphi^{(a)} + \eta^{(a)}]},$$

we obtain

$$\begin{aligned}
&\int \mathrm{d}a \int \mathcal{D}\eta^{(a)} \; \frac{\mathrm{d}c_0}{\mathrm{d}a} \delta(c_0) \mathrm{e}^{-\beta H[\varphi_0^{(a)} + \eta^{(a)}]} \\
&= \sqrt{H[\varphi_0]} \int \mathrm{d}a \int \mathcal{D}\eta^{(a)} \left\{ \left(1 - \int \mathrm{d}^D x \; \eta^{(a)} \frac{1}{H[\varphi_0]} \frac{\partial^2 \varphi_0^{(a)}}{\partial z^2} \right) \right. \\
&\qquad \left. \cdot \, \delta \left(\int \mathrm{d}^D x \; \eta^{(a)} \frac{1}{\sqrt{H[\varphi_0]}} \frac{\partial \varphi_0^{(a)}}{\partial z} \right) \mathrm{e}^{-\beta H[\varphi_0^{(a)} + \eta^{(a)}]} \right\}. \qquad (4.8)
\end{aligned}$$

Each kink solution, independent of a, contributes equally to the integral, hence we may replace $\varphi_0^{(a)}$ by φ_0, and integrate the parameter a only over the interval $[-\frac{T}{2}, \frac{T}{2}]$, known as a *regulation*, then (4.8) becomes

$$T\sqrt{H[\varphi_0]} \int \mathcal{D}\eta \left\{ \delta \left(\int \mathrm{d}^D x\ \eta \frac{1}{\sqrt{H[\varphi_0]}} \frac{\partial \varphi_0}{\partial z} \right) \cdot \left(1 - \int \mathrm{d}^D x\ \eta \frac{1}{H[\varphi_0]} \frac{\partial^2 \varphi_0}{\partial z^2} \right) \mathrm{e}^{-\beta H[\varphi_0 + \eta]} \right\}, \quad (4.9)$$

In order to see how the δ-function influences the functional integral, we expand the fluctuation η in the eigenfunctions η_i of the fluctuation operator $\mathbb{K}$ with

$$\eta = \sum_i c_i \eta_i,$$

where c_i are the expansion coefficients. Accordingly, the functional measure becomes

$$\mathcal{D}\eta \rightarrow \mathcal{N} \mathrm{d}c_0 \prod_n \mathrm{d}c_n,$$

where $\mathcal{N}$ is a constant, which ensures the equivalency of the transformation, then the functional integral in (4.9) can be written as

$$\int \mathcal{D}\eta\ \delta \left(\int \mathrm{d}^D x\ \eta \frac{1}{\sqrt{H[\varphi_0]}} \frac{\partial \varphi_0}{\partial z} \right) \cdots = \int \mathcal{D}\eta\ \delta \left(\int \mathrm{d}^D x\ \eta_0 \sum_i c_i \eta_i \right) \cdots \rightarrow \mathcal{N} \int \mathrm{d}c_0\ \delta(c_0) \int \prod_n \mathrm{d}c_n \cdots = \mathcal{N} \int \prod_n \mathrm{d}c_n \cdots, \quad (4.10)$$

where we used the orthonomality relation

$$\int \mathrm{d}^D x\ \eta_i \eta_j = \delta_{ij}.$$

The reformulation in (4.10) basically means that the δ-function filters out the translation mode. For this reason we can rewrite (4.9) as

$$T\sqrt{H[\varphi]} \int_{N^\perp} \mathcal{D}\eta \left(1 - \int \mathrm{d}^D x\ \eta \frac{1}{H[\varphi_0]} \frac{\partial^2 \varphi_0}{\partial z^2} \right) \mathrm{e}^{-\beta H[\varphi_0 + \eta]},$$

where the symbol $N^\perp$ indicates that we shall integrate over the subspace only containing fluctuations orthogonal to the translation mode, rather than the space made up of all the fluctuations. With an added source $j \in N^\perp$, the generating functional Z of $\langle\eta\rangle$ can be written as

$$\frac{Z(j)}{Z(0)} = \frac{1}{Z(0)} T\sqrt{H[\varphi_0]}e^{-\beta H[\varphi_0]}\frac{\mathcal{N}'}{\sqrt{\det \mathbb{K}'}}\Bigg\{\left(1 - \int \mathrm{d}^D x \; \frac{1}{H[\varphi_0]}\frac{\partial^2\varphi_0}{\partial z^2}\frac{\delta}{\delta j}\right) \cdot \exp\left(-\frac{g_0}{3!}\beta\int \mathrm{d}^D x \; \varphi_0\frac{\delta^3}{\delta j^3} - \frac{g_0}{4!}\beta\int \mathrm{d}^D x \; \frac{\delta^4}{\delta j^4}\right)\Bigg\} \cdot \exp\left(\frac{1}{2}\int \mathrm{d}^D x \, \mathrm{d}^D x' \; j(x)\beta^{-1}\mathbb{K}'^{-1}_{xx'}j(x')\right), \quad (4.11)$$

where $\mathcal{N}'$ is an unimportant renormalization constant; the new fluctuation operator

$$\mathbb{K}' = \mathbb{K}|_{N^\perp}$$

does not possess the translation mode and is thus invertible.

4.4 The spectrum of the fluctuation operator $\mathbb{K}(\varphi_0)$

To derive the propagator $\mathbb{K}'^{-1}$, which clearly plays a important part in the functional integral, our first step is to analyze the spectrum of the fluctuation operator $\mathbb{K}(\varphi_0)$, i.e. to solve the eigenvalue equation

$$\begin{aligned} &\mathbb{K}(\varphi_0)\eta(x) = \lambda\eta(x) \\ \text{i.e.} \quad &\left[-\nabla^2 - \frac{m_0^2}{2} + \frac{3m_0^2}{2}\tanh^2\left(\frac{m_0}{2}z\right)\right]\eta(x) = \lambda\eta(x), \end{aligned} \quad (4.12)$$

which has already been thoroughly studied[6,24]. In the following we would just like to summarize the result. To ease the problem, we divide the $\mathbb{K}(\varphi_0)$ in two parts, i.e.

$$\mathbb{K}(\varphi_0) = -\Delta^{(D-1)} + \tilde{\mathbb{K}},$$

where $\tilde{\mathbb{K}}$ only concerns the z-component of x

$$\tilde{\mathbb{K}} = -\partial_z^2 - \frac{m_0^2}{2} + \frac{3m_0^2}{2}\tanh\left(\frac{m_0}{2}z\right).$$

With the separation of the variables method, we establish two eigenvalue equations

$$-\Delta^{(D-1)}\eta'_{\vec{n}}(\vec{x}) = \zeta_{\vec{n}}\eta'_{\vec{n}}(\vec{x}) \quad (4.13)$$

and

$$\tilde{\mathbb{K}}\tilde{\eta}_\xi(z) = \xi\tilde{\eta}_\xi(z), \quad (4.14)$$

then (4.12) is solved by

$$\eta_{\vec{n}\xi}(x) = \eta'_{\vec{n}}(\vec{x})\tilde{\eta}_\xi(z),$$

with the eigenvalue

$$\lambda_{\vec{n}\xi} = \zeta_{\vec{n}} + \xi.$$

Remember the periodic boundary conditions for $\vec{x} \in [0, L]^{D-1}$, we obtain for (4.13) the discrete spectrum

$$\zeta_{\vec{n}} = \frac{4\pi^2}{L^2}\vec{n}^2, \;\; \vec{n} = \mathbb{Z}^{D-1},$$

with the normalized eigenfunctions

$$\eta'_{\vec{n}}(\vec{x}) = L^{\frac{1-D}{2}} \exp\left(\mathrm{i}\frac{2\pi}{L}\vec{n}\cdot\vec{x}\right), \;\; \vec{x} \in [0, L]^{D-1}.$$

The eigenvalue problem (4.14), reformulated as

$$\left\{-\frac{\hbar^2}{2m_0}\partial_z^2 + \frac{\hbar^2 m_0}{4}\left[3\tanh^2\left(\frac{m_0}{2}z\right) - 1\right]\right\}\tilde{\eta}_\xi = \frac{\hbar^2}{2m_0}\xi\tilde{\eta}_\xi,$$

can be recognized as the Schrödingier equation for a particle in a one dimensional potential well. This is an exact soluble problem, it turns out to have both discrete and continuous eigenvalues. Together with the normalized eigenfunctions, the solutions are

$$\xi_0 = 0, \quad \tilde{\eta}_{\xi_0}(z) = \sqrt{\frac{3m_0}{8}}\mathrm{sech}^2\left(\frac{m_0}{2}z\right);$$

$$\xi_1 = \frac{3}{4}m_0^2, \quad \tilde{\eta}_{\xi_1}(z) = \sqrt{\frac{3m_0}{4}}\tanh\left(\frac{m_0}{2}z\right)\mathrm{sech}\left(\frac{m_0}{2}z\right)$$

and

$$\xi_p = m_0^2 + p^2 \text{ with } p \in \mathbb{R},$$

$$\tilde{\eta}_{\xi_p}(z) = \mathcal{N}_p e^{\mathrm{i}pz}\left[2p^2 + \frac{m_0^2}{2} - \frac{3}{2}m_0^2\tanh^2\left(\frac{m_0}{2}z\right) + 3\mathrm{i}m_0 p\tanh\left(\frac{m_0}{2}z\right)\right],$$

where the normalization constant [6]

$$\mathcal{N}_p = [2\pi(4p^4 + 5m_0^2p^2 + m_0^4)]^{-\frac{1}{2}},$$

makes sure that

$$\int \mathrm{d}z\; \tilde{\eta}_{\xi_p}(z)^*\tilde{\eta}_{\xi_{p'}}(z) = \delta(p - p').$$

As mentioned in previous sections the fluctuation operator $\mathbb{K}(\varphi_0)$ possesses one zero mode, that is

$$\eta_{\vec{0}\xi_0} = L^{\frac{1-D}{2}}\sqrt{\frac{3m_0}{8}}\mathrm{sech}^2\left(\frac{m_0}{2}z\right).$$

4.5 The propagator $\mathbb{K}'^{-1}$

With the spectrum of the fluctuation operator $\mathbb{K}$ we can construct its inverse kernel using the form

$$\mathbb{K}'^{-1}_{xx'} = \sum\!\!\!\!\!\!\int{}' \frac{1}{\lambda_{\vec{n}\xi}} \eta_{\vec{n}\xi}(x)\eta^*_{\vec{n}\xi}(x'),$$

where the prime symbol in $\sum\!\!\!\!\!\!\int{}'$ means we should throw out the translation mode here, otherwise $\mathbb{K}$ would not be invertible, then we obtain

$$\begin{aligned}
\mathbb{K}'^{-1}_{xx'} = L^{1-D}\Bigg\{ & \sum_{\vec{n}\neq\vec{0}} \frac{L^2}{4\pi^2n^2}\frac{3m_0}{8} e^{\mathrm{i}\frac{2\pi}{L}\vec{n}(\vec{x}-\vec{x}')}\mathrm{sech}^2\left(\frac{m_0}{2}z\right)\mathrm{sech}^2\left(\frac{m_0}{2}z'\right) \\
& + \sum_{\vec{n}} \frac{1}{\frac{4\pi^2n^2}{L^2}+\frac{3}{4}m_0^2}\frac{3m_0}{4} e^{\mathrm{i}\frac{2\pi}{L}\vec{n}(\vec{x}-\vec{x}')} \\
& \cdot \tanh\left(\frac{m_0}{2}z\right)\mathrm{sech}\left(\frac{m_0}{2}z\right)\tanh\left(\frac{m_0}{2}z'\right)\mathrm{sech}\left(\frac{m_0}{2}z'\right) \\
& + \int \mathrm{d}p \sum_{\vec{n}} \frac{1}{\frac{4\pi^2n^2}{L^2}+m_0^2+p^2} e^{\mathrm{i}\frac{2\pi}{L}\vec{n}(\vec{x}-\vec{x}')}\tilde{\eta}_{\xi_p}(z)\tilde{\eta}^*_{\xi_{p'}}(z')\Bigg\}. \qquad (4.15)
\end{aligned}$$

For $x = x'$, we get

$$\begin{aligned}
\mathbb{K}'^{-1}_{xx} = L^{1-D}\Bigg\{ & \sum_{\vec{n}\neq\vec{0}} \frac{L^2}{4\pi^2n^2}\frac{3m_0}{8}\mathrm{sech}^4\left(\frac{m_0}{2}z\right) \\
& + \sum_{\vec{n}} \frac{1}{\frac{4\pi^2n^2}{L^2}+\frac{3}{4}m_0^2}\frac{3m_0}{4}\tanh^2\left(\frac{m_0}{2}z\right)\mathrm{sech}^2\left(\frac{m_0}{2}z\right) \\
& + \int \mathrm{d}p \sum_{\vec{n}} \frac{1}{\frac{4\pi^2n^2}{L^2}+m_0^2+p^2}|\tilde{\eta}_{\xi_p}(z)|^2\Bigg\}, \qquad (4.16)
\end{aligned}$$

with $|\tilde{\eta}_{\xi_p}(z)|^2$ in the last term calculated as

$$\begin{aligned}
& |\tilde{\eta}_{\xi_p}(z)|^2 \\
&= \mathcal{N}_p^2\left|2p^2+\frac{m_0^2}{2}-\frac{3}{2}m_0^2\tanh^2\left(\frac{m_0}{2}z\right)+3\mathrm{i}m_0p\tanh\left(\frac{m_0}{2}z\right)\right|^2 \\
&= \mathcal{N}_p^2\left[4p^2+5m_0^2p^2+m_0^4-3(m_0^4+m_0^2p^2)\mathrm{sech}^2\left(\frac{m_0}{2}z\right)+\frac{9}{4}m_0^4\mathrm{sech}^4\left(\frac{m_0}{2}z\right)\right] \\
&= \frac{1}{2\pi}-\mathcal{N}_p^2\left[3(m_0^4+m_0^2p^2)\mathrm{sech}^2\left(\frac{m_0}{2}z\right)-\frac{9}{4}m_0^4\mathrm{sech}^4\left(\frac{m_0}{2}z\right)\right].
\end{aligned}$$

Now $\mathbb{K}'^{-1}_{xx}$ can be neatly summarized[7] as

$$\mathbb{K}'^{-1}_{xx} = C_1 \mathrm{sech}^4\left(\frac{m_0}{2}z\right) + C_2 \mathrm{sech}^2\left(\frac{m_0}{2}z\right) + C_0, \tag{4.17}$$

where

$$C_1 = L^{1-D}\left[\sum_{\vec{n}\neq\vec{0}} \frac{L^2}{4\pi^2 n^2}\frac{3m_0}{8} - \sum_{\vec{n}} \frac{1}{\frac{4\pi^2 n^2}{L^2} + \frac{3}{4}m_0^2}\frac{3m_0}{4} + \frac{9}{4}m_0^4 \int \mathrm{d}p\, \mathcal{N}_p^2 \sum_{\vec{n}} \frac{1}{\frac{4\pi^2 n^2}{L^2} + m_0^2 + p^2}\right],$$

$$C_2 = L^{1-D}\left[\sum_{\vec{n}} \frac{1}{\frac{4\pi^2 n^2}{L^2} + \frac{3}{4}m_0^2}\frac{3m_0}{4} - \int \mathrm{d}p\, 3\mathcal{N}_p^2 (m_0^4 + m_0^2 p^2) \sum_{\vec{n}} \frac{1}{\frac{4\pi^2 n^2}{L^2} + m_0^2 + p^2}\right],$$

and

$$C_0 = L^{1-D}\frac{1}{2\pi}\int \mathrm{d}p \sum_{\vec{n}} \frac{1}{\frac{4\pi^2 n^2}{L^2} + m_0^2 + p^2};$$

pay attention that C_0, C_1 and C_2 are all divergent, they need renormalization, which is not going to be included in this thesis.

5

The loop calculation

I don't want to talk about time travel because if we start talking about it then we're going to be here all day talking about it, making diagrams with straws.

— Older Joe in *Looper*

With the help of the generating functional (4.11) we can easily construct the Feynman diagrams of $\langle\eta\rangle$ using the Feynman rules, where

$$\langle\eta\rangle = \frac{1}{Z(0)} \frac{\partial Z(j)}{\partial j}\bigg|_{j=0} . \tag{5.1}$$

For the purpose of the this thesis, we only consider the Feynman diagrams up to 2-loop order.

Recall that our final goal is to calculate $\langle\varphi\rangle$, as we have shown before in (4.5), it can be further formulated as

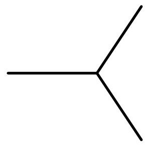
$$\begin{aligned}\langle\varphi\rangle &= \varphi_0 + \langle\eta\rangle \\ &= \sqrt{\frac{3m_0^2}{g_0}}\tanh\left(\frac{m_0}{2}z\right) + \langle\eta\rangle \\ &= \sqrt{\frac{3m_0^2}{g_0}}\left[\tanh\left(\frac{m_0}{2}z\right) + \sqrt{\frac{g_0}{3m_0^2}}\langle\eta\rangle\right] .\end{aligned} \tag{5.2}$$

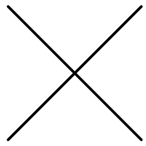
5.1 The loop expansion of $\langle\eta\rangle$

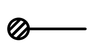
The form of (4.11) tells that we are not working with a simple φ^4-theory (in our case, η^4) any more, but an extended one with terms containing η, η^3 and

η^4, in particular we have[6]

propagator : $= \beta^{-1}\mathbb{K}'^{-1}$

3-point vertex :  $= -\beta\varphi_0 g_0 = -\beta m_0\sqrt{3g_0}\tanh\left(\frac{m_0}{2}z\right)$

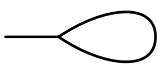
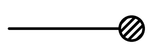
4-point vertex :  $= -\beta g_0$

internal point :  $= -\frac{1}{H[\varphi_0]}\frac{\partial^2\varphi_0}{\partial z^2} = -L^{1-D}\frac{\sqrt{3g_0}}{2m_0^2}\frac{\partial^2}{\partial z^2}\tanh\left(\frac{m_0}{2}z\right)$,

and basically the calculation of (5.1) can be sorted in loop order as

$$\begin{aligned}\langle\eta\rangle &= \frac{\alpha_1(\sqrt{g_0}) + \alpha_2(g_0\sqrt{g_0}) + \mathcal{O}(g_0^2\sqrt{g_0})}{1 + \beta_1(g_0) + \mathcal{O}(g_0^2)} \\ &= \alpha_1(\sqrt{g_0}) + (\alpha_2 - \alpha_1\beta_1)(g_0\sqrt{g_0}) + \mathcal{O}(g_0^2\sqrt{g_0}),\end{aligned} \tag{5.3}$$

where the terms are presented together with their orders in g_0 shown in the brackets, and it is straightforward to identify the diagrams contributing to α_1, β_1 and α_2. For α_1 we have

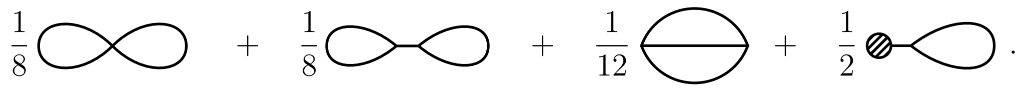
$\frac{1}{2}$  $+$  ,

with the symmetry factor directly given in front of the diagrams, and the corresponding analytical expression reads

$$-\frac{1}{2}\beta^{-1}m_0\sqrt{3g_0}\int \mathrm{d}^D x'\ \mathbb{K}'^{-1}_{xx'}\mathbb{K}'^{-1}_{x'x'}\tanh\left(\frac{m_0}{2}z'\right) \\ - \beta^{-1}L^{1-D}\frac{\sqrt{3g_0}}{2m_0^2}\int \mathrm{d}^D x'\ \mathbb{K}'^{-1}_{xx'}\frac{\partial^2}{\partial z'^2}\tanh\left(\frac{m_0}{2}z'\right).$$

It's easy to show that the order of β^{-1} corresponds to the order of g_0 in a certain way, since both of them indicate the loop order, for the sake of simplicity we just let $\beta = 1$ in the following.

The denominator in (5.3) consists of all the vacuum diagrams, and β_1 can be represented as



The product $\alpha_1\beta_1$ simply yields eight disconnected diagrams, they are all included in α_2 except the diagram in Figure 5.1.

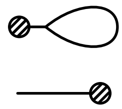
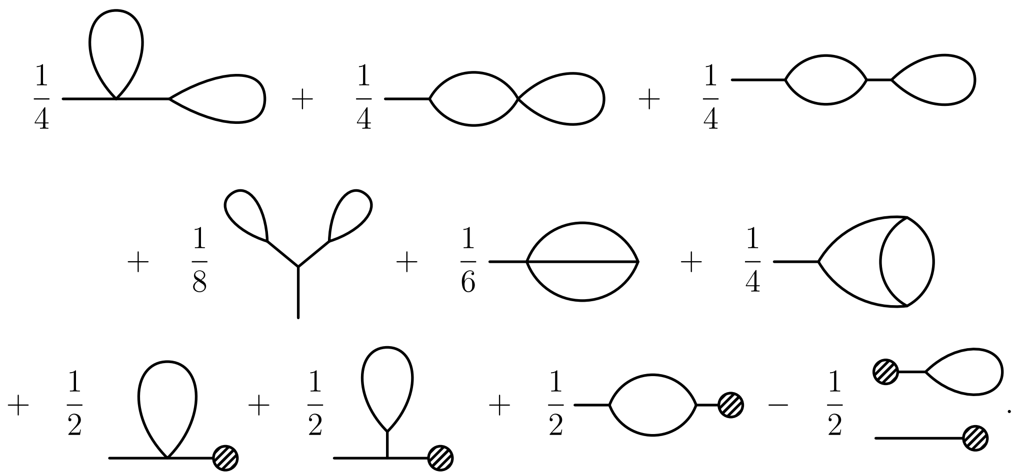


Figure 5.1

This diagram, which possesses two internal points, purely results from the division in (5.3), it cannot be acquired through perturbative calculation, since we don't expand the term

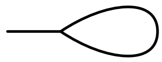
$$\frac{1}{H[\varphi_0]}\frac{\partial^2\varphi_0}{\partial z^2}\frac{\delta}{\delta j}$$

in the generating functional, thus at most one internal point can be involved in a resultant Feynman diagram. In consequence, the subtraction of $\alpha_1\beta_1$ from α_2 cancels all the disconnected diagrams in α_2, yet also attaches the extra diagram in Figure 5.1 to the order of $g_0\sqrt{g_0}$, the expression of which, that is $\alpha_2 - \alpha_1\beta_1$, as a result, can be summarized as



5.2 The Calculation of $\langle\eta\rangle$ to order $\sqrt{g_0}$

The 1-loop diagram

 $:= \alpha_1^{(1)}$

in α_1 has already been studied[7], and the solution is found as

$$\alpha_1^{(1)} = -\frac{\sqrt{3g_0}}{m_0}\left\{\left[\frac{2}{3}C_1\tanh\left(\frac{m_0}{2}z\right) + (C_0 + C_2)\frac{m_0}{2}z\right]\mathrm{sech}^2\left(\frac{m_0}{2}z\right)\right.$$
$$\left. + C_0\tanh\left(\frac{m_0}{2}z\right)\right\} \quad (5.4)$$

Now we only have to evaluate the second diagram in α_1

 $:= \alpha_1^{(2)}.$

Following the same strategy to solve the 1-loop diagram, we can create a differential equation by multiplying the diagram with the fluctuation operator $\mathbb{K}'$, and we obtain

$$\begin{aligned}
\mathbb{K}'\alpha_1^{(2)}(x) &= \int \mathrm{d}^D x''\ \mathbb{K}'_{xx''}\alpha_1^{(2)}(x'') \\
&= -L^{1-D}\frac{\sqrt{3g_0}}{2m_0^2}\int \mathrm{d}^D x'\,\mathrm{d}^D x''\ \mathbb{K}'_{xx''}\mathbb{K}'^{-1}_{x''x'}\frac{\partial^2}{\partial z'^2}\tanh\left(\frac{m_0}{2}z'\right) \\
&= -L^{1-D}\frac{\sqrt{3g_0}}{2m_0^2}\frac{\partial^2}{\partial z^2}\tanh\left(\frac{m_0}{2}z\right) \\
&= L^{1-D}\frac{\sqrt{3g_0}}{4}\tanh\left(\frac{m_0}{2}z\right)\mathrm{sech}^2\left(\frac{m_0}{2}z\right). \qquad (5.5)
\end{aligned}$$

Note that the solution of this inhomogeneous differential equation doesn't contain homogeneous part due to the definition of $\mathbb{K}'$. But in order to use the *variation of the constants method* we need to extend (5.5) to the whole fluctuation space, i.e. we replace $\mathbb{K}'$ by $\mathbb{K}$. This won't cause trouble, since if the resultant special solution contains component in the translation mode, we can simply remove it. With the ansatz

$$\alpha_1^{(2)} = \mathrm{sech}^2\left(\frac{m_0}{2}z\right) f,$$

the left-hand side of (5.5) can be rewritten as

$$2m_0 \tanh\left(\frac{m_0}{2}z\right)\operatorname{sech}^2\left(\frac{m_0}{2}z\right)f' - \operatorname{sech}^2\left(\frac{m_0}{2}z\right)f'',$$

so that (5.5) becomes

$$2m_0 \tanh\left(\frac{m_0}{2}z\right)f' - f'' = L^{1-D}\frac{\sqrt{3g_0}}{4}\tanh\left(\frac{m_0}{2}z\right),$$

and the solution for f can be easily identified as

$$f = L^{1-D}\frac{\sqrt{3g_0}}{8m_0}z,$$

hence $\alpha_1^{(2)}$ is solved with

$$\alpha_1^{(2)} = L^{1-D}\frac{\sqrt{3g_0}}{8m_0}z\operatorname{sech}^2\left(\frac{m_0}{2}z\right).$$

This solution lies obviously in $N_\perp$, since

$$\int \mathrm{d}^D x\ \alpha_1^{(2)}\eta_{\vec{0}\xi_0} = 0.$$

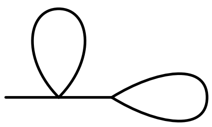
Apply the results of $\alpha_1^{(1)}$ and $\alpha_1^{(2)}$ in (5.2), with $\langle\eta\rangle$ up to order $\sqrt{g_0}$, we are able to reveal $\langle\varphi\rangle$ as

$$\begin{aligned}
\langle\varphi\rangle &= \sqrt{\frac{3m_0^2}{g_0}}\left[\tanh\left(\frac{m_0}{2}z\right) + \sqrt{\frac{g_0}{3m_0^2}}\Big(\alpha_1 + \mathcal{O}(g_0\sqrt{g_0})\Big)\right] \\
&= \sqrt{\frac{3m_0^2}{g_0}}\left[\tanh\left(\frac{m_0}{2}z\right) + \sqrt{\frac{g_0}{3m_0^2}}\left(\frac{1}{2}\alpha_1^{(1)} + \alpha_1^{(2)} + \mathcal{O}(g_0\sqrt{g_0})\right)\right] \\
&= \sqrt{\frac{3m_0^2}{g_0}}\Bigg\{\tanh\left(\frac{m_0}{2}z\right) - \frac{g_0}{2m_0^2}\Bigg[\left(\frac{2}{3}C_1\tanh\left(\frac{m_0}{2}z\right)\right. \\
&\quad \left. + \left((C_0+C_2)\frac{m_0}{2} - \frac{1}{4L^2}\right)z\right)\operatorname{sech}^2\left(\frac{m_0}{2}z\right) + C_0\tanh\left(\frac{m_0}{2}z\right)\Bigg] + \mathcal{O}(g_0^2)\Bigg\},
\end{aligned}$$

where we only considered the 3-dimensional case $D = 3$.

5.3 The Calculation of selected diagrams in order $g_0\sqrt{g_0}$

The easiest 2-loop diagram to calculate would be



$$= m_0 g_0 \sqrt{3g_0} \int \mathrm{d}^D x' \, \mathrm{d}^D x'' \; \mathbb{K}'^{-1}_{xx''} \mathbb{K}'^{-1}_{x''x''} \mathbb{K}'^{-1}_{x'x''} \mathbb{K}'^{-1}_{x'x'} \cdot \tanh\left(\frac{m_0}{2} z'\right) := \alpha_2^{(1)},$$

because we could still adopt the same method we used for the calculation in order $\sqrt{g_0}$. Multiply $\alpha_2^{(1)}$ by $\mathbb{K}'$ and we get

$$\begin{aligned}
\mathbb{K}'\alpha_2^{(1)} &= \int \mathrm{d}^D x''' \; \mathbb{K}'_{xx'''} \alpha_2^{(1)}(x''') \\
&= m_0 g_0 \sqrt{3g_0} \int \mathrm{d}^D x' \, \mathrm{d}^D x'' \, \mathrm{d}^D x''' \; \mathbb{K}'_{xx'''} \mathbb{K}'^{-1}_{x'''x''} \mathbb{K}'^{-1}_{x''x''} \mathbb{K}'^{-1}_{x'x''} \mathbb{K}'^{-1}_{x'x'} \tanh\left(\frac{m_0}{2} z'\right) \\
&= \mathbb{K}'^{-1}_{xx} m_0 g_0 \sqrt{3g_0} \int \mathrm{d}^D x' \; \mathbb{K}'^{-1}_{xx'} \mathbb{K}'^{-1}_{x'x'} \tanh\left(\frac{m_0}{2} z'\right) \\
&= -g_0 \mathbb{K}'^{-1}_{xx} \alpha_1^{(1)}, \qquad (5.6)
\end{aligned}$$

directly inserting (4.17) and (5.4) in (5.7) yields

$$\begin{aligned}
\mathbb{K}'\alpha_2^{(1)} = \frac{g_0\sqrt{3g_0}}{m_0} &\left[C_1 \mathrm{sech}^4\left(\frac{m_0}{2} z\right) + C_2 \mathrm{sech}^2\left(\frac{m_0}{2} z\right) + C_0\right] \\
&\cdot \left\{ \left[\frac{2}{3} C_1 \tanh\left(\frac{m_0}{2} z\right) + (C_0 + C_2)\frac{m_0}{2} z\right] \mathrm{sech}^2\left(\frac{m_0}{2} z\right) \right. \\
&\left. + C_0 \tanh\left(\frac{m_0}{2} z\right) \right\}. \qquad (5.7)
\end{aligned}$$

Again we replace $\mathbb{K}'$ by $\mathbb{K}$ and apply the variation of the constants method, then instead of (5.7) we write

$$
\begin{aligned}
&2m_0 \tanh\left(\frac{m_0}{2}z\right)\operatorname{sech}^2\left(\frac{m_0}{2}z\right) f' - \operatorname{sech}^2\left(\frac{m_0}{2}z\right) f'' = \\
&\qquad \frac{g_0\sqrt{3g_0}}{m_0}\Bigg[\frac{2}{3}C_1^2 \tanh\left(\frac{m_0}{2}z\right)\operatorname{sech}^6\left(\frac{m_0}{2}z\right) \\
&\qquad + \left(\frac{2}{3}C_1C_2 + C_0C_1\right)\tanh\left(\frac{m_0}{2}z\right)\operatorname{sech}^4\left(\frac{m_0}{2}z\right) \\
&\qquad + \left(\frac{2}{3}C_0C_1 + C_0C_2\right)\tanh\left(\frac{m_0}{2}z\right)\operatorname{sech}^2\left(\frac{m_0}{2}z\right) \\
&\qquad + C_1(C_0+C_2)\frac{m_0}{2}z \operatorname{sech}^6\left(\frac{m_0}{2}z\right) \\
&\qquad + C_2(C_0+C_2)\frac{m_0}{2}z \operatorname{sech}^4\left(\frac{m_0}{2}z\right) \\
&\qquad + C_0(C_0+C_2)\frac{m_0}{2}z \operatorname{sech}^2\left(\frac{m_0}{2}z\right) \\
&\qquad + C_0^2 \tanh\left(\frac{m_0}{2}z\right)\Bigg]. \qquad (5.8)
\end{aligned}
$$

The inhomogeneity on the right-hand side of (5.8) consists of seven summands s_i, $i = 1, \cdots, 7$, where

$$
\begin{aligned}
s_1 &= \frac{g_0\sqrt{3g_0}}{m_0}\frac{2}{3}C_1^2 \tanh\left(\frac{m_0}{2}z\right)\operatorname{sech}^6\left(\frac{m_0}{2}z\right) \\
s_2 &= \frac{g_0\sqrt{3g_0}}{m_0}\left(\frac{2}{3}C_1C_2 + C_0C_1\right)\tanh\left(\frac{m_0}{2}z\right)\operatorname{sech}^4\left(\frac{m_0}{2}z\right) \\
s_3 &= \frac{g_0\sqrt{3g_0}}{m_0}\left(\frac{2}{3}C_0C_1 + C_0C_2\right)\tanh\left(\frac{m_0}{2}z\right)\operatorname{sech}^2\left(\frac{m_0}{2}z\right) \\
s_4 &= \frac{g_0\sqrt{3g_0}}{m_0}C_1(C_0+C_2)\frac{m_0}{2}z \operatorname{sech}^6\left(\frac{m_0}{2}z\right) \\
s_5 &= \frac{g_0\sqrt{3g_0}}{m_0}C_2(C_0+C_2)\frac{m_0}{2}z \operatorname{sech}^4\left(\frac{m_0}{2}z\right) \\
s_6 &= \frac{g_0\sqrt{3g_0}}{m_0}C_0(C_0+C_2)\frac{m_0}{2}z \operatorname{sech}^2\left(\frac{m_0}{2}z\right) \\
s_7 &= \frac{g_0\sqrt{3g_0}}{m_0}C_0^2 \tanh\left(\frac{m_0}{2}z\right).
\end{aligned}
$$

Accordingly we can make seven ansatzes f_i for them, then we solve

$$2m_0 \tanh\left(\frac{m_0}{2}z\right)\operatorname{sech}^2\left(\frac{m_0}{2}z\right) f_i' - \operatorname{sech}^2\left(\frac{m_0}{2}z\right) f_i'' = s_i$$

separately, and the solution for (5.8) can be simply given as

$$f = \sum_i f_i.$$

The ansatzes for s_2, s_3 and s_7 are already found in the calculation of the 1-loop diagram $\alpha_1^{(1)}$, the solutions are

$$\begin{aligned}
f_2 &= \frac{2}{3}\frac{g_0\sqrt{3g_0}}{m_0^3}\left(\frac{2}{3}C_1C_2 + C_0C_1\right)\tanh\left(\frac{m_0}{2}z\right) \\
f_3 &= \frac{g_0\sqrt{3g_0}}{m_0^3}\left(\frac{2}{3}C_0C_1 + C_0C_2\right)\frac{m_0}{2}z \\
f_7 &= \frac{g_0\sqrt{3g_0}}{2m_0^3}C_0^2\left[\sinh\left(m_0 z\right) + m_0 z\right].
\end{aligned}$$

For the rest of the summands we can utilize the function *DSolve* built in *Mathematica*[25] to solve the differential equations, the results are

$$
\begin{aligned}
f_1 &= \frac{g_0\sqrt{3g_0}}{9m_0^3}C_1^2\left[2\tanh\left(\frac{m_0}{2}z\right)+\tanh\left(\frac{m_0}{2}z\right)\mathrm{sech}^2\left(\frac{m_0}{2}z\right)\right]\\
f_4 &= -\frac{g_0\sqrt{3g_0}}{105m_0^3}C_1(C_0+C_2)\Bigg\{6\sinh(m_0z)+24m_0z\cosh(m_0z)\\
&\quad+3m_0z\cosh(2m_0z)+72m_0z\ln\left(1+\mathrm{e}^{-m_0z}\right)-72\mathrm{Li}_2\left(-\mathrm{e}^{-m_0z}\right)\\
&\quad-36m_0z\ln\left[\cosh\left(\frac{m_0}{2}z\right)\right]-48\ln\left[\cosh\left(\frac{m_0}{2}z\right)\right]\sinh(m_0z)\\
&\quad-6\ln\left[\cosh\left(\frac{m_0}{2}z\right)\right]\sinh(2m_0z)-15m_0z\mathrm{sech}^2\left(\frac{m_0}{2}z\right)\\
&\quad 40\tanh\left(\frac{m_0}{2}z\right)+18m_0^2z^2\Bigg\}\\
f_5 &= -\frac{g_0\sqrt{3g_0}}{30m_0^3}C_2(C_0+C_2)\Bigg\{2\sinh(m_0z)+8m_0z\cosh(m_0z)\\
&\quad+m_0z\cosh(2m_0z)+24m_0z\ln\left(1+\mathrm{e}^{-m_0z}\right)-24\mathrm{Li}_2\left(-\mathrm{e}^{-m_0z}\right)\\
&\quad-12m_0z\ln\left[\cosh\left(\frac{m_0}{2}z\right)\right]-16\ln\left[\cosh\left(\frac{m_0}{2}z\right)\right]\sinh(m_0z)\\
&\quad-2\ln\left[\cosh\left(\frac{m_0}{2}z\right)\right]\sinh(2m_0z)+6m_0^2z^2\Bigg\}\\
f_6 &= -\frac{g_0\sqrt{3g_0}}{24m_0^3}C_0(C_0+C_2)\Bigg\{2\sinh(m_0z)+8m_0z\cosh(m_0z)\\
&\quad+m_0z\cosh(2m_0z)+12m_0z\ln\left(1+\mathrm{e}^{-m_0z}\right)-12\mathrm{Li}_2\left(-\mathrm{e}^{-m_0z}\right)\\
&\quad-12m_0z\ln\left[\cosh\left(\frac{m_0}{2}z\right)\right]-16\ln\left[\cosh\left(\frac{m_0}{2}z\right)\right]\sinh(m_0z)\\
&\quad-2\ln\left[\cosh\left(\frac{m_0}{2}z\right)\right]\sinh(2m_0z)+3m_0^2z^2-3m_0z\Bigg\}.
\end{aligned}
$$

It is still not time to take a breath yet, since, as we mentioned previously, the result may contain component in zero mode, which should have the form

$$\lambda\cdot\mathrm{sech}^2\left(\frac{m_0}{2}z\right),$$

where λ is just a constant, and in this case, we should get rid of it, that is, to find out the λ. Note that the solution in the whole fluctuation space has the form

$$f\cdot\mathrm{sech}^2\left(\frac{m_0}{2}z\right),$$

hence the final solution, which lies in $N^{\perp}$, could be written as

$$(f-\lambda)\mathrm{sech}^2\left(\frac{m_0}{2}z\right),$$

which fulfills

$$\int \mathrm{d}^D x \ (f-\lambda)\mathrm{sech}^2\left(\frac{m_0}{2}z\right)\eta_{\vec{0}\xi_0} = 0,$$

or equivalently

$$\lambda \int \mathrm{d}^D x \ \mathrm{sech}^4\left(\frac{m_0}{2}z\right) = \int \mathrm{d}^D x \ f \cdot \mathrm{sech}^4\left(\frac{m_0}{2}z\right). \tag{5.9}$$

At first glance there are indeed many terms in f_i, but as we further scrutinize each one of them with the consideration of (5.9), it is not hard to see that only the even functions in f_i contribute to the zero mode; with that in mind, the only terms need to be considered are those containing $m_0 z \ln(1+\mathrm{e}^{-m_0 z})$, $\mathrm{Li}_2(-\mathrm{e}^{-m_0 z})$ and $m_0^2 z^2$, whose coefficients seem always have the same proportion $4 : -4 : 1$ in each f_i.

We could immediately start calculating the λ, but first we need to extract the even part from $m_0 z \ln(1+\mathrm{e}^{-m_0 z})$ and $\mathrm{Li}_2(-\mathrm{e}^{-m_0 z})$, which can be separately obtained as

$$\begin{aligned}
&\frac{1}{2}\left[m_0 z \ln\left(1+\mathrm{e}^{-m_0 z}\right) - m_0 z \ln\left(1+\mathrm{e}^{m_0 z}\right)\right] \\
&= \frac{1}{2} m_0 z \ln \frac{1+\mathrm{e}^{-m_0 z}}{1+\mathrm{e}^{m_0 z}} \\
&= \frac{1}{2} m_0 z \ln \frac{\mathrm{e}^{-m_0 z}(1+\mathrm{e}^{m_0 z})}{1+\mathrm{e}^{m_0 z}} \\
&= -\frac{1}{2} m_0^2 z^2
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{2}\left[\mathrm{Li}_2\left(-\mathrm{e}^{-m_0 z}\right) + \mathrm{Li}_2\left(-\mathrm{e}^{m_0 z}\right)\right] \\
&= \frac{1}{2}\left[\mathrm{Li}_2\left(-\mathrm{e}^{-m_0 z}\right) + \mathrm{Li}_2\left(\frac{1}{-\mathrm{e}^{-m_0 z}}\right)\right] \\
&= \frac{1}{2}\left[\mathrm{Li}_2\left(-\mathrm{e}^{-m_0 z}\right) - \mathrm{Li}_2\left(-\mathrm{e}^{-m_0 z}\right) - \frac{1}{2}\ln^2\left(\mathrm{e}^{-m_0 z}\right) - \frac{\pi^2}{6}\right] \\
&= \frac{1}{2}\left[-\frac{1}{2} m_0^2 z^2 - \frac{\pi^2}{6}\right].
\end{aligned}$$

Together with the right coefficients, the relevant function in each f_i contributing to the zero mode is proportional to

$$4 \cdot \left(-\frac{1}{2} m_0^2 z^2\right) - 4 \cdot \frac{1}{2}\left[-\frac{1}{2} m_0^2 z^2 - \frac{\pi^2}{6}\right] + m_0^2 z^2 = \frac{\pi^2}{3}, \tag{5.10}$$

and the rest is to let (5.9) do the job, so we simply find that

$$\lambda_i \propto \frac{\pi^2}{3},$$

where λ_i corresponds to f_i, e.g.

$$\lambda_4 = -\frac{18 g_0 \sqrt{3 g_0}}{105 m_0^3} C_1 (C_0 + C_2) \frac{\pi^2}{3},$$

and so on. Then finally $\alpha_2^{(1)}$ can be summarized as

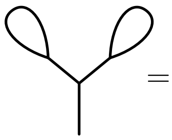
$$\alpha_2^{(1)} = \mathrm{sech}^2\left(\frac{m_0}{2} z\right) \sum_i (f_i - \lambda_i), \tag{5.11}$$

and the matching λ_i for those f_i, which don't contain even functions, are simply zero.

To be more exact, we write down the final solution

$$
\begin{aligned}
\alpha_2^{(1)} = \frac{g_0\sqrt{3g_0}}{m_0^3}\mathrm{sech}^2\left(\frac{m_0}{2}z\right)\Bigg\{ & \frac{1}{9}C_1^2\tanh\left(\frac{m_0}{2}z\right)\mathrm{sech}^2\left(\frac{m_0}{2}z\right) \\
& + \frac{1}{7}C_1(C_0+C_2)m_0 z\,\mathrm{sech}^2\left(\frac{m_0}{2}z\right) \\
& + \left(\frac{4}{63}C_1C_2 + \frac{2}{7}C_0C_1 + \frac{2}{9}C_1^2\right)\tanh\left(\frac{m_0}{2}z\right) \\
& + \left(\frac{5}{12}C_0^2 + \frac{2}{35}C_0C_1 + \frac{3}{20}C_0C_2 + \frac{2}{35}C_1C_2 + \frac{1}{15}C_2^2\right)\sinh(m_0 z) \\
& - \left(\frac{1}{3}C_0^2 + \frac{8}{35}C_0C_1 + \frac{3}{5}C_0C_2 + \frac{8}{35}C_1C_2 + \frac{4}{15}C_2^2\right)m_0 z\cosh(m_0 z) \\
& - \left(\frac{1}{24}C_0^2 + \frac{1}{35}C_0C_1 + \frac{3}{40}C_0C_2 + \frac{1}{35}C_1C_2 - \frac{1}{30}C_2^2\right)m_0 z\cosh(2m_0 z) \\
& + \left(\frac{1}{2}C_0^2 + \frac{12}{35}C_0C_1 + \frac{9}{10}C_0C_2 + \frac{12}{35}C_1C_2 + \frac{2}{5}C_2^2\right)m_0 z\ln\left[\cosh\left(\frac{m_0}{2}z\right)\right] \\
& + \left(\frac{2}{3}C_0^2 + \frac{16}{35}C_0C_1 + \frac{6}{5}C_0C_2 + \frac{16}{35}C_1C_2 + \frac{8}{15}C_2^2\right)\sinh(m_0 z)\ln\left[\cosh\left(\frac{m_0}{2}z\right)\right] \\
& + \left(\frac{1}{12}C_0^2 + \frac{2}{35}C_0C_1 + \frac{3}{20}C_0C_2 + \frac{2}{35}C_1C_2 + \frac{1}{15}C_2^2\right)\sinh(2m_0 z)\ln\left[\cosh\left(\frac{m_0}{2}z\right)\right] \\
& - \left(\frac{1}{2}C_0^2 + \frac{24}{35}C_0C_1 + \frac{13}{10}C_0C_2 + \frac{24}{35}C_1C_2 + \frac{4}{5}C_2^2\right)m_0 z\ln\left(1+\mathrm{e}^{-m_0 z}\right) \\
& + \left(\frac{1}{2}C_0^2 + \frac{24}{35}C_0C_1 + \frac{13}{10}C_0C_2 + \frac{24}{35}C_1C_2 + \frac{4}{5}C_2^2\right)\mathrm{Li}_2\left(-\mathrm{e}^{-m_0 z}\right) \\
& - \left(\frac{1}{8}C_0^2 + \frac{6}{35}C_0C_1 + \frac{13}{40}C_0C_2 + \frac{6}{35}C_1C_2 + \frac{1}{5}C_2^2\right)m_0^2 z^2 \\
& + \left(\frac{5}{8}C_0^2 + \frac{1}{3}C_0C_1 + \frac{5}{8}C_0C_2\right)m_0 z \\
& + \frac{\pi^2}{3}\left(\frac{1}{8}C_0^2 + \frac{6}{35}C_0C_1 + \frac{13}{40}C_0C_2 + \frac{6}{35}C_1C_2 + \frac{1}{5}C_2^2\right)\Bigg\}.
\end{aligned}
$$

Intuitively we would like to try this method with all the rest of the diagrams in order $g_0\sqrt{g_0}$, hoping that we could go further. The good news is that we can indeed still solve three more diagrams. The first one is



$$= -\left(m_0\sqrt{3g_0}\right)^3 \int \mathrm{d}^D x'\, \mathrm{d}^D x''\, \mathrm{d}^D x'''\, \mathbb{K}'^{-1}_{xx'''}\mathbb{K}'^{-1}_{x'''x''}\mathbb{K}'^{-1}_{x''x''}\mathbb{K}'^{-1}_{x'''x'}\mathbb{K}'^{-1}_{x'x'}$$
$$\cdot \tanh\left(\frac{m_0}{2}z'''\right)\tanh\left(\frac{m_0}{2}z''\right)\tanh\left(\frac{m_0}{2}z'\right) := \alpha_2^{(2)},$$

without repeating the arguments, we write

$$\begin{aligned}
\mathbb{K}'\alpha_2^{(2)} &= \int \mathrm{d}^D x''''\, \mathbb{K}'_{xx''''}\alpha_2^{(2)}(x'''') \\
&= -\left(m_0\sqrt{3g_0}\right)^3 \int \mathrm{d}^D x'\, \mathrm{d}^D x''\, \mathrm{d}^D x'''\, \mathrm{d}^D x''''\, \mathbb{K}'_{xx''''}\mathbb{K}'^{-1}_{x''''x'''}\mathbb{K}'^{-1}_{x'''x''}\mathbb{K}'^{-1}_{x''x''} \\
&\quad \cdot \mathbb{K}'^{-1}_{x'''x'}\mathbb{K}'^{-1}_{x'x'} \tanh\left(\frac{m_0}{2}z'''\right)\tanh\left(\frac{m_0}{2}z''\right)\tanh\left(\frac{m_0}{2}z'\right) \\
&= -\left(m_0\sqrt{3g_0}\right)^3 \tanh\left(\frac{m_0}{2}z\right) \int \mathrm{d}^D x'\, \mathrm{d}^D x''\, \mathbb{K}'^{-1}_{xx''}\mathbb{K}'^{-1}_{x''x''}\mathbb{K}'^{-1}_{xx'}\mathbb{K}'^{-1}_{x'x'} \\
&\quad \cdot \tanh\left(\frac{m_0}{2}z''\right)\tanh\left(\frac{m_0}{2}z'\right) \\
&= -m_0\sqrt{3g_0}\tanh\left(\frac{m_0}{2}z\right)\left(\alpha_1^{(1)}\right)^2 \\
&= -m_0\sqrt{3g_0}\tanh\left(\frac{m_0}{2}z\right) \\
&\quad \cdot \frac{3g_0}{m_0^2}\left\{\left[\frac{2}{3}C_1\tanh\left(\frac{m_0}{2}z\right) + (C_0+C_2)\frac{m_0}{2}z\right]\mathrm{sech}^2\left(\frac{m_0}{2}z\right)\right. \\
&\quad \left. + C_0\tanh\left(\frac{m_0}{2}z\right)\right\}^2,
\end{aligned}$$

follow the same routine, we obtain

$$\begin{aligned}
&2m_0\tanh\left(\frac{m_0}{2}z\right)\mathrm{sech}^2\left(\frac{m_0}{2}z\right)f' - \mathrm{sech}^2\left(\frac{m_0}{2}z\right)f'' \\
&\quad = -\frac{3g_0\sqrt{3g_0}}{m_0}\tanh\left(\frac{m_0}{2}z\right)\left[\frac{4}{9}C_1^2\tanh^2\left(\frac{m_0}{2}z\right)\mathrm{sech}^4\left(\frac{m_0}{2}z\right)\right. \\
&\qquad + \frac{4}{3}C_1(C_0+C_2)\frac{m_0}{2}z\tanh\left(\frac{m_0}{2}z\right)\mathrm{sech}^4\left(\frac{m_0}{2}z\right) \\
&\quad + (C_0+C_2)^2\frac{m_0^2}{4}z^2\mathrm{sech}^4\left(\frac{m_0}{2}z\right) + \frac{4}{3}C_0C_1\tanh^2\left(\frac{m_0}{2}z\right)\mathrm{sech}^2\left(\frac{m_0}{2}z\right) \\
&+ C_0(C_0+C_2)\frac{m_0}{2}z\tanh\left(\frac{m_0}{2}z\right)\mathrm{sech}^2\left(\frac{m_0}{2}z\right) + \left.C_0^2\tanh^2\left(\frac{m_0}{2}z\right)\right]. \qquad (5.12)
\end{aligned}$$

Then we try to find the ansatzes for the six summands s_i of the homogeneity in (5.12) with

$$
\begin{aligned}
s_1 &= -\frac{4g_0\sqrt{3g_0}}{3m_0} C_1^2 \tanh^3\left(\frac{m_0}{2}z\right)\operatorname{sech}^4\left(\frac{m_0}{2}z\right)\\
s_2 &= -\frac{4g_0\sqrt{3g_0}}{m_0} C_1(C_0+C_2)\frac{m_0}{2}z\tanh^2\left(\frac{m_0}{2}z\right)\operatorname{sech}^4\left(\frac{m_0}{2}z\right)\\
s_3 &= -\frac{3g_0\sqrt{3g_0}}{m_0} (C_0+C_2)^2\frac{m_0^2}{4}z^2\tanh\left(\frac{m_0}{2}z\right)\operatorname{sech}^4\left(\frac{m_0}{2}z\right)\\
s_4 &= -\frac{4g_0\sqrt{3g_0}}{m_0} C_0C_1\tanh^3\left(\frac{m_0}{2}z\right)\operatorname{sech}^2\left(\frac{m_0}{2}z\right)\\
s_5 &= -\frac{3g_0\sqrt{3g_0}}{m_0} C_0(C_0+C_2)\frac{m_0}{2}z\tanh^2\left(\frac{m_0}{2}z\right)\operatorname{sech}^2\left(\frac{m_0}{2}z\right)\\
s_6 &= -\frac{3g_0\sqrt{3g_0}}{m_0} C_0^2\tanh^3\left(\frac{m_0}{2}z\right),
\end{aligned}
$$

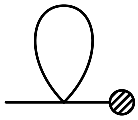
and the solutions are

$$
\begin{aligned}
f_1 &= -\frac{2g_0\sqrt{3g_0}}{9m_0^3}C_1^2\left[\tanh\left(\frac{m_0}{2}z\right)+\tanh^3\left(\frac{m_0}{2}z\right)\right]\\
f_2 &= \frac{2g_0\sqrt{3g_0}}{105m_0^3}C_1(C_0+C_2)\Bigg\{2\sinh(m_0z)+8m_0z\cosh(m_0z)\\
&\quad+m_0z\cosh(2m_0z)+24m_0z\ln\left(1+\mathrm{e}^{-m_0z}\right)-24\mathrm{Li}_2\left(-\mathrm{e}^{-m_0z}\right)\\
&\quad-12m_0z\ln\left[\cosh\left(\frac{m_0}{2}z\right)\right]-16\ln\left[\cosh\left(\frac{m_0}{2}z\right)\right]\sinh(m_0z)\\
&\qquad-2\ln\left[\cosh\left(\frac{m_0}{2}z\right)\right]\sinh(2m_0z)\\
&\qquad-80\tanh\left(\frac{m_0}{2}z\right)+6m_0^2z^2\Bigg\}\\
f_3 &= \frac{g_0\sqrt{3g_0}}{30m_0^3}(C_0+C_2)^2\Bigg\{2\sinh(m_0z)+8m_0z\cosh(m_0z)\\
&\quad+m_0z\cosh(2m_0z)+84m_0z\ln\left(1+\mathrm{e}^{-m_0z}\right)-84\mathrm{Li}_2\left(-\mathrm{e}^{-m_0z}\right)\\
&\quad-16m_0z\ln\left[\cosh\left(\frac{m_0}{2}z\right)\right]-16\ln\left[\cosh\left(\frac{m_0}{2}z\right)\right]\sinh(m_0z)\\
&\qquad-2\ln\left[\cosh\left(\frac{m_0}{2}z\right)\right]\sinh(2m_0z)\\
&\qquad-15m_0^2z^2\tanh\left(\frac{m_0}{2}z\right)+21m_0^2z^2\Bigg\}\\
f_4 &= -\frac{2g_0\sqrt{3g_0}}{3m_0^3}C_0C_1\left[3m_0z-4\tanh\left(\frac{m_0}{2}z\right)\right]\\
f_5 &= \frac{g_0\sqrt{3g_0}}{40m_0^3}C_0(C_0+C_2)\Bigg\{2\sinh(m_0z)+8m_0z\cosh(m_0z)\\
&\quad+m_0z\cosh(2m_0z)-36m_0z\ln\left(1+\mathrm{e}^{-m_0z}\right)+36\mathrm{Li}_2\left(-\mathrm{e}^{-m_0z}\right)\\
&\quad-12m_0z\ln\left[\cosh\left(\frac{m_0}{2}z\right)\right]-16\ln\left[\cosh\left(\frac{m_0}{2}z\right)\right]\sinh(m_0z)\\
&\qquad-2\ln\left[\cosh\left(\frac{m_0}{2}z\right)\right]\sinh(2m_0z)\\
&\qquad-9m_0^2z^2-15m_0z\Bigg\}\\
f_6 &= -\frac{3g_0\sqrt{3g_0}}{2m_0^3}C_0^2\sinh\left(\frac{m_0}{2}z\right).
\end{aligned}
$$

After removing the component in zero mode using (5.11), we get the final solution

$$
\begin{aligned}
\alpha_2^{(2)} = \frac{g_0\sqrt{3g_0}}{m_0^3}\operatorname{sech}^2\left(\frac{m_0}{2}z\right)\Bigg\{&\left(\frac{8}{7}C_0C_1-\frac{32}{21}C_1C_2-\frac{2}{9}C_1^2\right)\tanh\left(\frac{m_0}{2}z\right)\\
&-\frac{2}{9}C_1^2\tanh^3\left(\frac{m_0}{2}z\right)-\frac{3}{2}C_0^2\sinh\left(\frac{m_0}{2}z\right)\\
&+\left(\frac{7}{60}C_0^2+\frac{4}{105}C_0C_1+\frac{11}{60}C_0C_2+\frac{4}{105}C_1C_2+\frac{1}{15}C_2^2\right)\sinh(m_0z)\\
&+\left(\frac{7}{15}C_0^2+\frac{16}{105}C_0C_1+\frac{11}{15}C_0C_2+\frac{16}{105}C_1C_2+\frac{1}{15}C_2^2\right)m_0z\cosh(m_0z)\\
&+\left(\frac{7}{120}C_0^2+\frac{2}{105}C_0C_1+\frac{11}{120}C_0C_2+\frac{2}{105}C_1C_2+\frac{1}{30}C_2^2\right)m_0z\cosh(2m_0z)\\
&-\left(\frac{5}{6}C_0^2+\frac{8}{35}C_0C_1+\frac{41}{30}C_0C_2+\frac{8}{35}C_1C_2+\frac{8}{15}C_2^2\right)m_0z\ln\left[\cosh\left(\frac{m_0}{2}z\right)\right]\\
&-\left(\frac{14}{15}C_0^2+\frac{32}{105}C_0C_1+\frac{22}{15}C_0C_2+\frac{32}{105}C_1C_2+\frac{8}{15}C_2^2\right)\sinh(m_0z)\ln\left[\cosh\left(\frac{m_0}{2}z\right)\right]\\
&-\left(\frac{7}{60}C_0^2+\frac{4}{105}C_0C_1+\frac{11}{60}C_0C_2+\frac{4}{105}C_1C_2+\frac{1}{15}C_2^2\right)\sinh(2m_0z)\ln\left[\cosh\left(\frac{m_0}{2}z\right)\right]\\
&+\left(\frac{19}{10}C_0^2+\frac{16}{35}C_0C_1+\frac{47}{10}C_0C_2+\frac{16}{35}C_1C_2+\frac{14}{5}C_2^2\right)m_0z\ln\left(1+\mathrm{e}^{-m_0z}\right)\\
&-\left(\frac{19}{10}C_0^2+\frac{16}{35}C_0C_1+\frac{47}{10}C_0C_2+\frac{16}{35}C_1C_2+\frac{14}{5}C_2^2\right)\mathrm{Li}_2\left(-\mathrm{e}^{-m_0z}\right)\\
&+\left(\frac{19}{40}C_0^2+\frac{4}{35}C_0C_1+\frac{47}{40}C_0C_2+\frac{4}{35}C_1C_2+\frac{7}{10}C_2^2\right)m_0^2z^2\\
&-\left(\frac{3}{8}C_0^2+2C_0C_1+\frac{3}{8}C_0C_2\right)m_0z\\
&-\frac{\pi^2}{3}\left(\frac{19}{40}C_0^2+\frac{4}{35}C_0C_1+\frac{47}{40}C_0C_2+\frac{4}{35}C_1C_2+\frac{7}{10}C_2^2\right)\Bigg\}.
\end{aligned}
$$

The diagram



$$
= L^{1-D}\frac{g_0\sqrt{3g_0}}{2m_0^2}\int \mathrm{d}^Dx'\,\mathrm{d}^Dx''\;\mathbb{K}'^{-1}_{xx''}\mathbb{K}'^{-1}_{x''x''}\mathbb{K}'^{-1}_{x''x'}
\cdot\frac{\partial^2}{\partial z^2}\tanh\left(\frac{m_0}{2}z'\right) := \alpha_2^{(3)}
$$

can also be solved, start with

$$\begin{aligned}
\mathbb{K}'\alpha_2^{(3)} &= \int \mathrm{d}^D x'''\ \mathbb{K}'_{xx'''}\alpha_2^{(3)}(x''') \\
&= L^{1-D}\frac{g_0\sqrt{3g_0}}{2m_0^2}\int \mathrm{d}^D x'\,\mathrm{d}^D x''\,\mathrm{d}^D x'''\ \mathbb{K}'_{xx'''}\mathbb{K}'^{-1}_{x'''x''}\mathbb{K}'^{-1}_{x''x''}\mathbb{K}'^{-1}_{x''x'} \\
&\quad \cdot \frac{\partial^2}{\partial z^2}\tanh\left(\frac{m_0}{2}z'\right) \\
&= L^{1-D}\frac{g_0\sqrt{3g_0}}{2m_0^2}\mathbb{K}'^{-1}_{xx}\int \mathrm{d}^D x'\ \mathbb{K}'^{-1}_{xx'}\frac{\partial^2}{\partial z^2}\tanh\left(\frac{m_0}{2}z'\right) \\
&= g_0\mathbb{K}'^{-1}_{xx}\alpha_1^{(2)} \\
&= L^{1-D}\frac{g_0\sqrt{3g_0}}{8m_0}\left[C_1\mathrm{sech}^4\left(\frac{m_0}{2}z\right)+C_2\mathrm{sech}^2\left(\frac{m_0}{2}z\right)+C_0\right]z\,\mathrm{sech}^2\left(\frac{m_0}{2}z\right)
\end{aligned}$$

we obtain

$$\begin{aligned}
&2m_0\tanh\left(\frac{m_0}{2}z\right)\mathrm{sech}^2\left(\frac{m_0}{2}z\right)f' - \mathrm{sech}^2\left(\frac{m_0}{2}z\right)f'' \\
&\qquad = L^{1-D}\frac{g_0\sqrt{3g_0}}{8m_0}\left[C_1\mathrm{sech}^4\left(\frac{m_0}{2}z\right)+C_2\mathrm{sech}^2\left(\frac{m_0}{2}z\right)+C_0\right] \\
&\qquad\qquad \cdot z\,\mathrm{sech}^2\left(\frac{m_0}{2}z\right), \quad (5.13)
\end{aligned}$$

and for the summands s_i of the homogeneity in (5.14) with

$$\begin{aligned}
s_1 &= L^{1-D}\frac{g_0\sqrt{3g_0}}{8m_0}C_1 z\,\mathrm{sech}^6\left(\frac{m_0}{2}z\right) \\
s_2 &= L^{1-D}\frac{g_0\sqrt{3g_0}}{8m_0}C_2 z\,\mathrm{sech}^4\left(\frac{m_0}{2}z\right) \\
s_3 &= L^{1-D}\frac{g_0\sqrt{3g_0}}{8m_0}C_0 z\,\mathrm{sech}^2\left(\frac{m_0}{2}z\right),
\end{aligned}$$

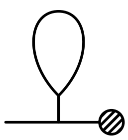
we separately found the solutions

$$\begin{aligned}
f_1 = -L^{1-D}\frac{g_0\sqrt{3g_0}}{420m_0^4}C_1\Big\{&6\sinh(m_0z) + 24m_0z\cosh(m_0z) \\
&+ 3m_0z\cosh(2m_0z) + 72m_0z\ln\left(1+\mathrm{e}^{-m_0z}\right) - 72\mathrm{Li}_2\left(-\mathrm{e}^{-m_0z}\right) \\
&- 36m_0z\ln\left[\cosh\left(\frac{m_0}{2}z\right)\right] - 48\ln\left[\cosh\left(\frac{m_0}{2}z\right)\right]\sinh(m_0z) \\
&- 6\ln\left[\cosh\left(\frac{m_0}{2}z\right)\right]\sinh(2m_0z) - 15m_0z\mathrm{sech}^2\left(\frac{m_0}{2}z\right) \\
&+ 40\tanh\left(\frac{m_0}{2}z\right) + 18m_0^2z^2\Big\} \\
f_2 = -L^{1-D}\frac{g_0\sqrt{3g_0}}{120m_0^4}C_2\Big\{&2\sinh(m_0z) + 8m_0z\cosh(m_0z) \\
&+ m_0z\cosh(2m_0z) + 24m_0z\ln\left(1+\mathrm{e}^{-m_0z}\right) - 24\mathrm{Li}_2\left(-\mathrm{e}^{-m_0z}\right) \\
&- 12m_0z\ln\left[\cosh\left(\frac{m_0}{2}z\right)\right] - 16\ln\left[\cosh\left(\frac{m_0}{2}z\right)\right]\sinh(m_0z) \\
&- 2\ln\left[\cosh\left(\frac{m_0}{2}z\right)\right]\sinh(2m_0z) + 6m_0^2z^2\Big\} \\
f_3 = -L^{1-D}\frac{g_0\sqrt{3g_0}}{96m_0^4}C_0\Big\{&2\sinh(m_0z) + 8m_0z\cosh(m_0z) \\
&+ m_0z\cosh(2m_0z) + 12m_0z\ln\left(1+\mathrm{e}^{-m_0z}\right) - 12\mathrm{Li}_2\left(-\mathrm{e}^{-m_0z}\right) \\
&- 12m_0z\ln\left[\cosh\left(\frac{m_0}{2}z\right)\right] - 16\ln\left[\cosh\left(\frac{m_0}{2}z\right)\right]\sinh(m_0z) \\
&- 2\ln\left[\cosh\left(\frac{m_0}{2}z\right)\right]\sinh(2m_0z) + 3m_0^2z^2 - 3m_0z\Big\}.
\end{aligned}$$

Again we need to remove the component in zero mode, same as before, to obtain the final result

$$\begin{aligned}\alpha_2^{(3)} = L^{1-D}\frac{g_0\sqrt{3g_0}}{m_0^4}\mathrm{sech}^2\left(\frac{m_0}{2}z\right)\Bigg\{&\frac{3}{84}C_1 m_0 z\,\mathrm{sech}^2\left(\frac{m_0}{2}z\right)-\frac{2}{21}C_1\tanh\left(\frac{m_0}{2}z\right)\\
&-\left(\frac{1}{48}C_0+\frac{1}{70}C_1+\frac{1}{60}C_2\right)\sinh(m_0 z)\\
&-\left(\frac{1}{12}C_0+\frac{2}{35}C_1+\frac{1}{15}C_2\right)m_0 z\cosh(m_0 z)\\
&-\left(\frac{1}{96}C_0+\frac{1}{140}C_1+\frac{1}{120}C_2\right)m_0 z\cosh(2m_0 z)\\
&+\left(\frac{1}{8}C_0+\frac{3}{35}C_1+\frac{1}{10}C_2\right)m_0 z\ln\left[\cosh\left(\frac{m_0}{2}z\right)\right]\\
&+\left(\frac{1}{6}C_0+\frac{4}{35}C_1+\frac{2}{15}C_2\right)\sinh(m_0 z)\ln\left[\cosh\left(\frac{m_0}{2}z\right)\right]\\
&+\left(\frac{1}{48}C_0+\frac{1}{70}C_1+\frac{1}{60}C_2\right)\sinh(2m_0 z)\ln\left[\cosh\left(\frac{m_0}{2}z\right)\right]\\
&-\left(\frac{1}{8}C_0+\frac{6}{35}C_1+\frac{1}{5}C_2\right)m_0 z\ln\left(1+\mathrm{e}^{-m_0 z}\right)\\
&+\left(\frac{1}{8}C_0+\frac{6}{35}C_1+\frac{1}{5}C_2\right)\mathrm{Li}_2\left(-\mathrm{e}^{-m_0 z}\right)\\
&-\left(\frac{1}{32}C_0+\frac{3}{70}C_1+\frac{1}{20}C_2\right)m_0^2 z^2+\frac{1}{32}C_0 m_0 z\\
&+\frac{\pi^2}{3}\left(\frac{1}{32}C_0+\frac{3}{70}C_1+\frac{1}{20}C_2\right)\Bigg\}.\end{aligned}$$

Another diagram



$$\begin{aligned}&= -L^{1-D}\frac{g_0\sqrt{3g_0}}{2m_0^2}\left(m_0\sqrt{3g_0}\right)^2\\
&\cdot\int \mathrm{d}^D x'\,\mathrm{d}^D x''\,\mathrm{d}^D x'''\;\mathbb{K}'^{-1}_{xx'''}\mathbb{K}'^{-1}_{x'''x''}\mathbb{K}'^{-1}_{x''x''}\mathbb{K}'^{-1}_{x'''x'}\\
&\cdot\tanh\left(\frac{m_0}{2}z'''\right)\tanh\left(\frac{m_0}{2}z''\right)\frac{\partial^2}{\partial z'^2}\tanh\left(\frac{m_0}{2}z'\right) := \alpha_2^{(4)},\end{aligned}$$

is solved as well, actually its structure is quite similar to $\alpha_2^{(2)}$, it might be boring, but we have to repeat the same routine over and over again, that is

$$
\begin{aligned}
\mathbb{K}'\alpha_2^{(4)} &= \int \mathrm{d}^D x'''' \; \mathbb{K}'_{xx''''}\alpha_2^{(4)}(x'''') \\
&= -L^{1-D}\frac{g_0\sqrt{3g_0}}{2m_0^2}\left(m_0\sqrt{3g_0}\right)^2 \int \mathrm{d}^D x' \, \mathrm{d}^D x'' \, \mathrm{d}^D x''' \; \mathrm{d}^D x'''' \; \mathbb{K}'_{xx''''}\mathbb{K}'^{-1}_{x''''x'''} \\
&\quad \cdot \mathbb{K}'^{-1}_{x'''x''}\mathbb{K}'^{-1}_{x''x''}\mathbb{K}'^{-1}_{x'''x'} \tanh\left(\frac{m_0}{2}z'''\right)\tanh\left(\frac{m_0}{2}z''\right)\frac{\partial^2}{\partial z'^2}\tanh\left(\frac{m_0}{2}z'\right) \\
&= -L^{1-D}\frac{g_0\sqrt{3g_0}}{2m_0^2}\left(m_0\sqrt{3g_0}\right)^2 \tanh\left(\frac{m_0}{2}z\right)\int \mathrm{d}^D x' \, \mathrm{d}^D x'' \; \mathbb{K}'^{-1}_{xx''}\mathbb{K}'^{-1}_{x''x''}\mathbb{K}'^{-1}_{xx'} \\
&\quad \cdot \tanh\left(\frac{m_0}{2}z''\right)\frac{\partial^2}{\partial z'^2}\tanh\left(\frac{m_0}{2}z'\right) \\
&= -m_0\sqrt{3g_0}\tanh\left(\frac{m_0}{2}z\right)\alpha_1^{(1)}\alpha_1^{(2)} \\
&= m_0\sqrt{3g_0}\tanh\left(\frac{m_0}{2}z\right) \\
&\quad \cdot L^{1-D}\frac{3g_0}{8m_0^2}z\,\mathrm{sech}^2\left(\frac{m_0}{2}z\right)\left\{\left[\frac{2}{3}C_1\tanh\left(\frac{m_0}{2}z\right) + (C_0+C_2)\frac{m_0}{2}z\right]\mathrm{sech}^2\left(\frac{m_0}{2}z\right)\right. \\
&\quad \left. + C_0\tanh\left(\frac{m_0}{2}z\right)\right\},
\end{aligned}
$$

then we have

$$
\begin{aligned}
2m_0\tanh&\left(\frac{m_0}{2}z\right)\mathrm{sech}^2\left(\frac{m_0}{2}z\right)f' - \mathrm{sech}^2\left(\frac{m_0}{2}z\right)f'' \\
&= L^{1-D}\frac{3g_0\sqrt{3g_0}}{8m_0}z\,\tanh\left(\frac{m_0}{2}z\right)\mathrm{sech}^2\left(\frac{m_0}{2}z\right) \\
&\quad \cdot\left\{\left[\frac{2}{3}C_1\tanh\left(\frac{m_0}{2}z\right) + (C_0+C_2)\frac{m_0}{2}z\right]\mathrm{sech}^2\left(\frac{m_0}{2}z\right)\right. \\
&\qquad\qquad \left. + C_0\tanh\left(\frac{m_0}{2}z\right)\right\}, \qquad (5.14)
\end{aligned}
$$

again we write down the three summands s_i of the homogeneity in (5.14) as

$$
\begin{aligned}
s_1 &= L^{1-D}\frac{g_0\sqrt{3g_0}}{4m_0}C_1 z\tanh^2\left(\frac{m_0}{2}z\right)\mathrm{sech}^4\left(\frac{m_0}{2}z\right) \\
s_2 &= L^{1-D}\frac{3g_0\sqrt{3g_0}}{16}(C_0+C_2)z^2\tanh\left(\frac{m_0}{2}z\right)\mathrm{sech}^4\left(\frac{m_0}{2}z\right) \\
s_3 &= L^{1-D}\frac{3g_0\sqrt{3g_0}}{8m_0}C_0 z\tanh^2\left(\frac{m_0}{2}z\right)\mathrm{sech}^2\left(\frac{m_0}{2}z\right),
\end{aligned}
$$

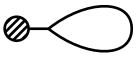
then we found the solutions

$$
\begin{aligned}
f_1 = -L^{1-D}\frac{g_0\sqrt{3g_0}}{420m_0^4}C_1\Big\{&2\sinh(m_0z)+8m_0z\cosh(m_0z)\\
&+m_0z\cosh(2m_0z)+24m_0z\ln\left(1+\mathrm{e}^{-m_0z}\right)-24\mathrm{Li}_2\left(-\mathrm{e}^{-m_0z}\right)\\
&-12m_0z\ln\left[\cosh\left(\frac{m_0}{2}z\right)\right]-16\ln\left[\cosh\left(\frac{m_0}{2}z\right)\right]\sinh(m_0z)\\
&-2\ln\left[\cosh\left(\frac{m_0}{2}z\right)\right]\sinh(2m_0z)+30m_0z\mathrm{sech}^2\left(\frac{m_0}{2}z\right)\\
&-80\tanh\left(\frac{m_0}{2}z\right)+6m_0^2z^2\Big\}\\
f_2 = -L^{1-D}\frac{g_0\sqrt{3g_0}}{120m_0^4}(C_0+C_2)\Big\{&2\sinh(m_0z)+8m_0z\cosh(m_0z)\\
&+m_0z\cosh(2m_0z)+84m_0z\ln\left(1+\mathrm{e}^{-m_0z}\right)-84\mathrm{Li}_2\left(-\mathrm{e}^{-m_0z}\right)\\
&-12m_0z\ln\left[\cosh\left(\frac{m_0}{2}z\right)\right]-16\ln\left[\cosh\left(\frac{m_0}{2}z\right)\right]\sinh(m_0z)\\
&-2\ln\left[\cosh\left(\frac{m_0}{2}z\right)\right]\sinh(2m_0z)+21m_0^2z^2-15m_0^2z^2\tanh\left(\frac{m_0}{2}z\right)\Big\}\\
f_3 = -L^{1-D}\frac{g_0\sqrt{3g_0}}{160m_0^4}C_0\Big\{&2\sinh(m_0z)+8m_0z\cosh(m_0z)\\
&+m_0z\cosh(2m_0z)-36m_0z\ln\left(1+\mathrm{e}^{-m_0z}\right)+36\mathrm{Li}_2\left(-\mathrm{e}^{-m_0z}\right)\\
&-12m_0z\ln\left[\cosh\left(\frac{m_0}{2}z\right)\right]-16\ln\left[\cosh\left(\frac{m_0}{2}z\right)\right]\sinh(m_0z)\\
&-2\ln\left[\cosh\left(\frac{m_0}{2}z\right)\right]\sinh(2m_0z)-9m_0^2z^2-15m_0z\Big\},
\end{aligned}
$$

and after taking out the component in zero mode, we receive the final expression

$$\alpha_2^{(3)} = -L^{1-D}\frac{g_0\sqrt{3g_0}}{m_0^4}\text{sech}^2\left(\frac{m_0}{2}z\right)\left\{\frac{1}{14}C_1 m_0 z\text{sech}^2\left(\frac{m_0}{2}z\right)\right.$$
$$-\frac{1}{8}(C_0+C_2)m_0^2z^2\tanh\left(\frac{m_0}{2}z\right)$$
$$+\left(\frac{7}{240}C_0+\frac{1}{210}C_1+\frac{1}{60}C_2\right)\sinh(m_0z)$$
$$+\left(\frac{7}{60}C_0+\frac{2}{105}C_1+\frac{1}{15}C_2\right)m_0z\cosh(m_0z)$$
$$+\left(\frac{7}{480}C_0+\frac{1}{420}C_1+\frac{1}{120}C_2\right)m_0z\cosh(2m_0z)$$
$$-\left(\frac{7}{40}C_0+\frac{1}{35}C_1+\frac{1}{10}C_2\right)m_0z\ln\left[\cosh\left(\frac{m_0}{2}z\right)\right]$$
$$-\left(\frac{7}{30}C_0+\frac{4}{105}C_1+\frac{2}{15}C_2\right)\sinh(m_0z)\ln\left[\cosh\left(\frac{m_0}{2}z\right)\right]$$
$$-\left(\frac{7}{240}C_0+\frac{1}{210}C_1+\frac{1}{60}C_2\right)\sinh(2m_0z)\ln\left[\cosh\left(\frac{m_0}{2}z\right)\right]$$
$$+\left(\frac{19}{40}C_0+\frac{4}{70}C_1+\frac{7}{10}C_2\right)m_0z\ln\left(1+\mathrm{e}^{-m_0z}\right)$$
$$-\left(\frac{19}{40}C_0+\frac{4}{70}C_1+\frac{7}{10}C_2\right)\mathrm{Li}_2\left(-\mathrm{e}^{-m_0z}\right)$$
$$+\left(\frac{19}{160}C_0+\frac{1}{70}C_1+\frac{7}{40}C_2\right)m_0^2z^2-\frac{3}{32}C_0m_0z$$
$$\left.-\frac{\pi^2}{3}\left(\frac{19}{160}C_0+\frac{1}{70}C_1+\frac{7}{40}C_2\right)\right\}.$$

So far we have worked out four diagrams in order $g_0\sqrt{g_0}$, yet trivially, we can still solve the disconnected diagram in Figure (5.1), we name it $\alpha_2^{(5)}$, which consists of two parts, note that we have already calculated one of them, that is $\alpha_1^{(2)}$, and the other part is already done within the framework of a dimensional regulation by Hoppe[6], i.e.



$$= -\frac{g_0}{m_0}\left[0.795774716(2)\cdot 10^{-2}\ln(m_0L)+0.578689881(6)\cdot 10^{-1}\right]$$
$$+\mathcal{O}\left(m_0Le^{-m_0L}\right)+\mathcal{O}(\varepsilon),$$

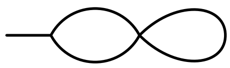
where $\varepsilon = 3 - D$, hence

$$\alpha_2^{(5)} = -L^{1-D}\frac{g_0\sqrt{3g_0}}{8m_0^2} z\,\mathrm{sech}^2\left(\frac{m_0}{2}z\right) \cdot \Big\{\left[0.795774716(2)\cdot 10^{-2}\ln(m_0L) + 0.578689881(6)\cdot 10^{-1}\right] + \mathcal{O}\left(m_0Le^{-m_0L}\right) + \mathcal{O}(\varepsilon)\Big\}.$$

The rest unsolved diagrams in 2-loop order require different approach, which might exceed the scope of a diploma thesis.

5.4 A glimpse of the unsolved diagram

You might be curious about how to riddle the rest of the 2-loop diagrams, personally I would prefer to look for a shortcut to ease the problems ahead, but wether it's feasible really depends on the luck of the draw. However, as a backup plan, we could always try to solve the integral more directly, and I can illustrate this point by knocking on the diagram



$$= m_0g_0\sqrt{3g_0}\int \mathrm{d}^Dx'\,\mathrm{d}^Dx''\;\mathbb{K}'^{-1}_{xx''}\left(\mathbb{K}'^{-1}_{x'x''}\right)^2\mathbb{K}'^{-1}_{x'x'} \cdot \tanh\left(\frac{m_0}{2}z''\right) := \alpha_2^{(6)}$$

a bit.

Remember the spectrum representations of the propagator $\mathbb{K}'^{-1}_{xx'}$ and $\mathbb{K}'^{-1}_{xx}$ are already given in (4.15) and (4.16) respectively; we could of course apply them in $\alpha_2^{(6)}$ straightforwardly, or we could first convert the integral to a differential equation just like before

$$\begin{aligned}\mathbb{K}'\alpha_2^{(6)} &= \int \mathrm{d}^Dx'''\;\mathbb{K}'_{xx'''}\alpha_2^{(6)}(x''')\\ &= m_0g_0\sqrt{3g_0}\int \mathrm{d}^Dx'\,\mathrm{d}^Dx''\,\mathrm{d}^Dx'''\;\mathbb{K}'_{xx'''}\mathbb{K}'^{-1}_{x'''x''}\left(\mathbb{K}'^{-1}_{x'x''}\right)^2\mathbb{K}'^{-1}_{x'x'}\tanh\left(\frac{m_0}{2}z''\right)\\ &= m_0g_0\sqrt{3g_0}\tanh\left(\frac{m_0}{2}z\right)\int \mathrm{d}^Dx'\;\left(\mathbb{K}'^{-1}_{xx'}\right)^2\mathbb{K}'^{-1}_{x'x'},\end{aligned}$$

then, as you can see, it all comes down to the evaluation of the integral

$$\int \mathrm{d}^Dx'\;\left(\mathbb{K}'^{-1}_{xx'}\right)^2\mathbb{K}'^{-1}_{x'x'}\;, \tag{5.15}$$

but doing which is not as easy as it looks like. After inserting (4.15) and (4.16) in (5.15) we obtain the integral

$$\int \mathrm{d}^D x' \; L^{3(1-D)} \Bigg\{ \sum_{\vec{n}\neq\vec{0}} \frac{L^2}{4\pi^2 n^2} \frac{3m_0}{8} e^{\mathrm{i}\frac{2\pi}{L}\vec{n}(\vec{x}-\vec{x}')} \mathrm{sech}^2\left(\frac{m_0}{2}z\right) \mathrm{sech}^2\left(\frac{m_0}{2}z'\right)$$

$$+ \sum_{\vec{n}} \frac{1}{\frac{4\pi^2 n^2}{L^2} + \frac{3}{4}m_0^2} \frac{3m_0}{4} e^{\mathrm{i}\frac{2\pi}{L}\vec{n}(\vec{x}-\vec{x}')}$$

$$\cdot \tanh\left(\frac{m_0}{2}z\right) \mathrm{sech}\left(\frac{m_0}{2}z\right) \tanh\left(\frac{m_0}{2}z'\right) \mathrm{sech}\left(\frac{m_0}{2}z'\right)$$

$$+ \int \mathrm{d}p \; \sum_{\vec{n}} \frac{1}{\frac{4\pi^2 n^2}{L^2} + m_0^2 + p^2} e^{\mathrm{i}\frac{2\pi}{L}\vec{n}(\vec{x}-\vec{x}')} \tilde{\eta}_{\xi_p}(z) \tilde{\eta}^*_{\xi_{p'}}(z') \Bigg\}^2$$

$$\cdot \Bigg\{ \sum_{\vec{n}\neq\vec{0}} \frac{L^2}{4\pi^2 n^2} \frac{3m_0}{8} \mathrm{sech}^4\left(\frac{m_0}{2}z'\right)$$

$$+ \sum_{\vec{n}} \frac{1}{\frac{4\pi^2 n^2}{L^2} + \frac{3}{4}m_0^2} \frac{3m_0}{4} \tanh^2\left(\frac{m_0}{2}z'\right) \mathrm{sech}^2\left(\frac{m_0}{2}z'\right)$$

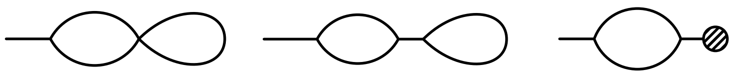
$$+ \int \mathrm{d}p \; \sum_{\vec{n}} \frac{1}{\frac{4\pi^2 n^2}{L^2} + m_0^2 + p^2} |\tilde{\eta}_{\xi_p}(z')|^2 \Bigg\},$$

which can be further divided into 18 integrals. To carry out the full calculation, approximations and numerical methods might be needed; the relevant techniques could be found in Hoppe's doctoral thesis [6].

Closing Words

I was sort of relieved as Mr. Münster allowed me to close my work like this, which means I must have done something right. It took me around two years to finally grind out this thesis, that is definitely much longer than what is really needed, and then I realized that no great achievement is possible without persistent work; ennui might be inevitable at some point, no matter how overwhelming the initial interest seems to be, therefore certain willingness of enduring boredom becomes essential.

Anyway, this thesis can be regarded as the follow-up study of Köpf's thesis[7]. We followed a statistical field theory approach to study the interfacial roughening. The perturbative calculation based on the kink solution defines the interfacial profile, which is the key to investigate the interface width; attention ought to be paid while establishing the Feynman diagrams, which can all be traced back to the carefully prepared generating functional. The correction I added in the one-loop approximation is negligible for large interface size L; I only managed to solve some of the Feynman diagrams in 2-loop order, basically I adopted the same technique that Köpf had used in his thesis, that is, I turned the integrals into inhomogeneous differential equations, which can be worked out via the variation of the constants method, the solutions to the corresponding Feynman diagrams then easily follow. The rest of the Feynman diagrams are more stubborn to deal with, in general, solving them must be a tour de force if no tricks are applied in advance, yet in my opinion the following diagrams



should have the same difficulty due to the similarity in their topologies. On a more complex level lie these two diagrams,

especially the latter one, which is probably the most difficult diagram in the 2-loop category.

References

[1] M. Le Bellac, *Quantum and Statistical Field Theory* (Oxford University Press, 1991).

[2] A. Zee, *Quantum Field Theory in a Nutshell*, second edition (Princeton University Press, 2010).

[3] C. Kittel, *Introduction to Solid State Physics*, eighth edition (John Wiley & Sons, 2005).

[4] W. Nolting, *Grundkurs Theoretische Physik*, **6**, sixth edition (Springer, 2007).

[5] E.S. Abers, B.W. Lee, Generating functionals for Green's functions and proper vertices. *Physics Reports* **9**, 91-99 (1973).

[6] P. Hoppe, *Analytische Berechnung der Oberflächenspannung im dreidimensionalen ϕ^4-Modell in 2-Loop-Ordnung*, PhD thesis, University Münster (1997).

[7] M. Köpf, *Rauhigkeit von Grenzflächen in der Ising-Universalitätsklasse*, Diplomarbeit, University Münster (2008).

[8] F. König, *Lösung der Feldgleichung einer durch Randbedingungen fixierten kritischen Grenzfläche*, Diplomarbeit, University Münster (2012).

[9] E. Ising, Beitrag zur Theorie des Ferromagnetismus. *Z. Phys.* **31**, 253-258 (1925).

[10] V.L. Ginzburg and L.D. Landau, On the theory of superconductivity. *Zh. Eksp. Teor. Fiz.* **20**, 1064-1082 (1950).

[11] L.D. Landau, On the theory of phase transitions. *Zh. Eksp. Teor. Fiz.* **7**, 19-32 (1937).

[12] J.D. van der Waals, The thermodynamic theory of capillarity under the hypothesis of a continuous variation of density (translated by J.S. Rowlinson). *J. Stat. Phys.* **20**, 197-200 (1979)

[13] F. Buff, R. Lovett, F. Stillinger, Interfacial density profile for fluids in the critical region. *Phys. Rev. Lett.* **15**, 621-623 (1965).

[14] M. Köpf, G. Münster, Interfacial roughening in field theory. *J. Stat. Phys.* **132**, 417-430 (2008).

[15] E. Bürkner and D. Stauffer, Monte Carlo study of surface roughening in the three-dimensional Ising model. *Z. Phys. B* **53**, 241-243 (1983).

[16] K. Mon, D. Landau and D. Stauffer, Interface roughening in the three-dimensional Ising model. *Phys. Rev. B* **42**, 545-547 (1990).

[17] M. Hasenbusch and K. Pinn, Surface tension, surface stiffness, and surface width of the 3-dimensional Ising model on a cubic lattice. *Physica A* **192**, 342-374 (1992).

[18] M. Müller and G. Münster, Profile and width of rough interfaces. *J. Stat. Phys.* **118**, 669-686 (2005).

[19] P. Ehrenfest, Phasenumwandlungen im üblichen und erweiterten Sinn, classifiziert nach den entsprechenden Singularitäten des thermodynamischen Potentiales. *Proc. Kon. Akad. v. Wetensch., Amsterdam* **36**, 153-157 (1933).

[20] R. Peierls, Ising's model of ferromagnetism. *Math. Proc. Cambridge. Philos. Soc.* **32**, 477-481 (1936).

[21] L. Onsager, Crystal statistics. I. A two-Dimensional model with an order-Disorder transition. *Phys. Rev.* **65**, 117-149 (1944).

[22] P. Weiss, L'hypothèse du champ moléculaire et la propriété ferromagnétique. *J. de Phys. Rad.* **6**, 661-690 (1907) (The hypothesis of the molecular field and the property of ferromagnetism)

[23] J.W. Cahn and J.E. Hilliard, Free energy of a nonuniform system. I. Interfacial free energy. *J. chem. Phys.* **28**, 258 (1958).

[24] R. Rajaraman, Some non-perturbative semi-classical methods in quantum field theory (a pedagogical review). *Physics Reports* **21**, 227-313 (1975).

[25] *Mathematica 8.0* (Wolfram Research, Inc., 2010).

Plagiatserklärung des Studierenden

Hiermit versichere ich, dass ich die vorliegende Arbeit mit dem Title *Perturbative Calculation around the Kink Interface in 2-loop Order* selbständig verfasst habe, und dass ich keine anderen Quellen und Hilfsmittel als die angegebenen benutzt habe und dass die Stellen der Arbeit, die anderen Werken - auch elektronischen Medien - dem Wortlaut oder Sinn nach entnommen wurden, auf jeden Fall unter Angabe der Quelle als Entlehnung kenntlich gemacht worden sind.

Münster, July 4, 2013

Ich erkläre mich mit einem Abgleich der Arbeit mit anderen Texten zwecks Auffindung von Übereinstimmungen sowie mit einer zu diesem Zweck vorzunehmenden Speicherung der Arbeit in eine Datenbank einverstanden.

Münster, July 4, 2013