Nächst-höhere Strukturfunktionen in MHD-Turbulenzen

Next-order structure function in MHD turbulence

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Contents

Appendix:

Chapter 1

Introduction

The subject of magneto-hydrodynamics is the study of the interaction between an electrically conducting fluid with a magnetic field. The fluid motion induces electric currents which lead to a modification of the magnetic field; at the same time these modifications end up in a stretch-mechanism of the lines of force and thus react on the fluid motion in terms of the Lorentz force. MHD turbulence is of great importance for the theoretical treatment of all kinds of cosmic problems: the explanation of the existence of magnetic fields in spiral arms of certain galaxies [Cha53], the occurrence of sunspots as a consequence of the Sun's magnetic field [Alf50], or simply the presence of the Earth's magnetic field.

Especially the particular behavior of the Sun's magnetic field can have tremendous influences on our everyday lifes. The so-called solar wind, a stream of charged particles ejected from the upper atmosphere of the Sun, can cause power grid outages or the loss of GPS signals, crucial for the GPS aircraft landing on foggy days. Thereby, massive bursts of solar wind, so-called coronal mass ejections that are preceded by a sudden brightening of the Sun's surface, can occur. Such a brightening and a following large mass ejection along the lines of force of the sun's magnetic field is depicted in fig. 1.1.
Another important phenomenon that occurs on a more regular basis, is the sunspot-cycle already mentioned above. The observation of sun spots has a history of about 400 years: Every 11 years, an increasing sun activity that manifests itself in the occurrence of sunspots, like the one depicted in fig. 1.2, can be observed.

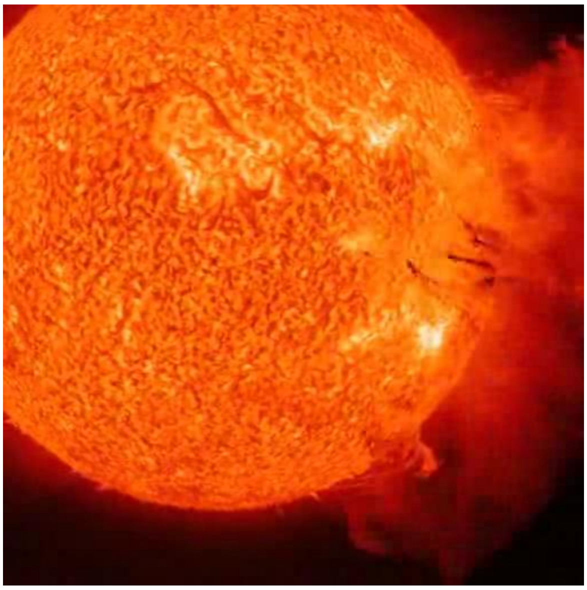
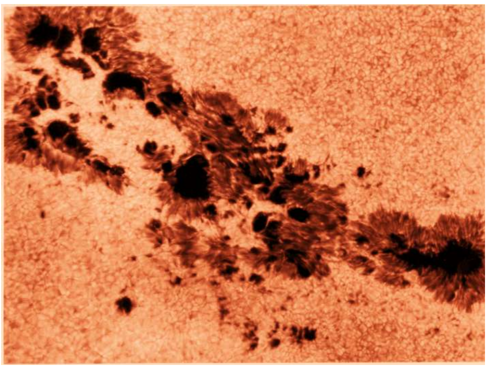


Figure 1.1: A medium-sized (M-2) solar flare provokes a spectacular coronal mass ejection observed by the Solar Dynamics Observatory (SDO) [Nas11] on June 7, 2011. The cloud of particles was thrown out along the lines of force and fell back down covering an area of more than a quarter of the solar surface.

These spots are the center of strong magnetic fields, which was first observed by Zeeman splitting and a first theory based on the MHD equations was given by [Alf50]. Thereby, the temperature in these spots is inferior to the temperature outside and the missing thermal pressure is balanced by an additional magnetic pressure.



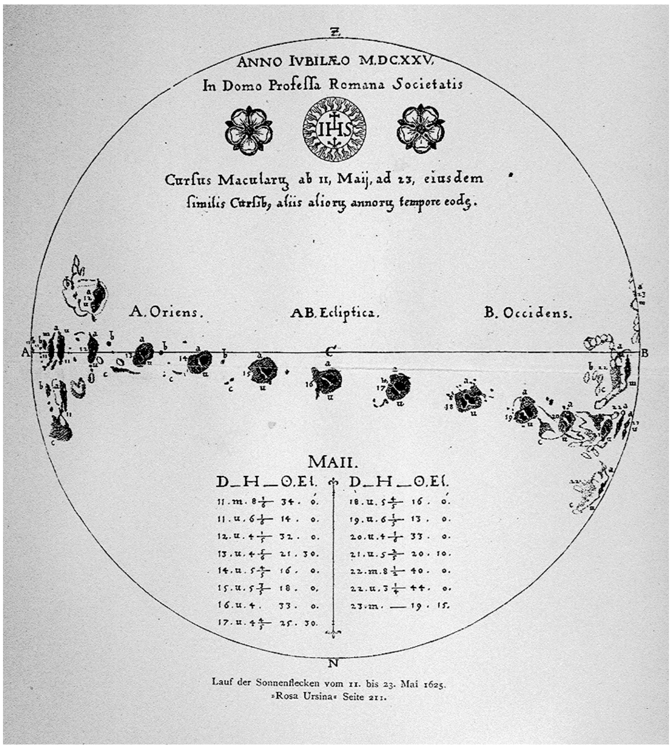


Figure 1.2: Above: Sunspots that occur in groups at the surface of the sun. The size of a spot is comparable to the size of the Earth [Nas98].
Right: A sketch of the course of the sunspots made by the German astronomer Christoph Scheiner from the 11th to the 23rd Mai 1685 [She70]. Scheiner described the inclination of the axis of rotation of the sunspots with respect to the plane of the ecliptic, which he accurately determined as $7°30'$, the modern value is $7°15'$.

A major problem for the theory of MHD turbulence is its relation to hydrodynamic turbulence: The fluid motion is governed by the Navier-Stokes equation under the additional influence of the Lorentz force. Although the fundamental equations of fluid mechanics are known since the 18th century, their implications for statistical physics and mathematical physics are far from being fully understood. Exact solutions exist only for some special cases and common perturbation theories, which work well in quantum mechanics or field theory, can not reproduce the basic character of the nonlinearity in the equation.
The characteristics of a turbulent flow can be summarized as follows

- chaotic spatio-temporal behavior,
- the involvement of a multitude of length scales,
- the equations are very sensitive with respect to small perturbations; fluid particles that were very close initially can rapidly disperse,
- increasing transport of momentum and energy that exceeds simple diffusion processes.

The aim of this work is to examine the structural and scaling properties in MHD turbulence in the same manner, as it has been performed for the Navier-Stokes equation in hydrodynamics. In this context one has to focus on the year 1941, where Kolmogorov's similarity hypothesis and the concept of energy cascades introduced by Richardson led the way to a whole new interpretation of the theory of turbulence (for references see [Arg07]). Thereby, moments of spatial velocity differences play a central role and the theory predicts the scaling behavior of these so-called structure functions. A similar treatment of the MHD equations was first performed by Batchelor [Bat50] and continued in a more rigorous way by Chandrasekhar [Cha50] who made use of the invariant theory of isotropic turbulence introduced by Kármán and Howarth [Kar38], Chandrasekhar [Cha51], Robertson [Rob40] and Batchelor [Bat60]. The contributions of Chandrasekhar to magneto-hydrodynamics and references to his papers are collected in a review by E.N. Parker [Par96].

The thesis is organized as follows: First-of-all, the MHD equations and their fluid- and electrodynamical implications are discussed in chapter 2. The statistical description of MHD turbulence follows in chapter 3. A hierarchy of structure function equations is presented in chapter 4 in analogy to equations that were recently derived for hydrodynamic turbulence [Hil01a], [Yak02]. The obtained relations are finally compared to direct numerical simulations of the 2D MHD equations, as discussed in chapter 5.

Chapter 2

The governing equations of MHD turbulence

The MHD equations combine the ordinary electromagnetic and hydrodynamic equations, in order to take into account the interaction between the magnetic field and the fluid motion. In a first approximation, the accumulation of charges and thus Maxwell's displacement current are neglected. This is justified in considering the equation of continuity of electric charge in which the term representing the rate of change of charge density is proportional to u^2/c^2. Maxwell's equations thus read

$$\nabla \times \mathbf{H} = 4\pi \mathbf{j}, \tag{2.1}$$

$$\nabla \times \mathbf{E} = -\mu \frac{\partial}{\partial t}\mathbf{H}, \tag{2.2}$$

$$\nabla \cdot \mathbf{j} = 0, \tag{2.3}$$

$$\nabla \cdot \mathbf{H} = 0. \tag{2.4}$$

Thereby, $\mathbf{H}$ is the magnetic field, $\mathbf{E}$ the electric field and $\mathbf{j}$ is the current density. If the material moves with the velocity $\mathbf{u}$, we have to calculate the electric and magnetic fields $\mathbf{E}'$ and $\mathbf{H}'$ by means of the Lorentz transformations. Since even in astronomical plasmas, the velocities are much smaller than the velocity of light, it is appropriate to use the non-relativistic approximation , and the transformations read

$$\mathbf{E}' = \mathbf{E} + \mu \mathbf{u} \times \mathbf{H}, \tag{2.5}$$

$$\mathbf{H}' = \mathbf{H}. \tag{2.6}$$

The magnetic field is therefore independent of the frame of reference, whereas the definition of the electric field is meaningless without making reference to its corresponding coordinate system. The current density thus reads

$$\mathbf{j} = \sigma(\mathbf{E} + \mu \mathbf{u} \times \mathbf{H}), \tag{2.7}$$

where σ is the electric conductivity.
The electromagnetic consequences of the equations can be seen by inserting (2.7) in (2.2). We obtain the induction equation for the magnetic field $\mathbf{H}$

$$\begin{aligned} \frac{\partial}{\partial t}\mathbf{H} &= \nabla \times \left(\mathbf{u} \times \mathbf{H} - \frac{1}{\mu\sigma}\mathbf{j} \right) \qquad (2.8) \\ &= -\mathbf{u} \cdot \nabla \mathbf{H} + \mathbf{H} \cdot \nabla \mathbf{u} + \lambda \nabla^2 \mathbf{H}, \end{aligned}$$

where we have introduced the magnetic diffusivity $\lambda = (4\pi\mu\sigma)^{-1}$.
Turning next to the fluid dynamical implications, the ordinary Navier-Stokes equation has to be modified to take into account the body force $\mu\mathbf{j} \times \mathbf{H}$ which is acting on the fluid motion per unit mass. The evolution equation for the velocity thus reads

$$\frac{\partial}{\partial t}\mathbf{u} + \mathbf{u} \cdot \nabla\mathbf{u} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\mathbf{u} + \frac{\mu}{\rho}\mathbf{j} \times \mathbf{H}. \tag{2.9}$$

where ν is the kinematic viscosity.
The supplementary term can be interpreted in terms of Maxwell's stresses

$$\mu\mathbf{j} \times \mathbf{H} = -\nabla\left(\frac{\mu H^2}{8\pi}\right) + \nabla \cdot \left(\frac{\mu\mathbf{HH}}{4\pi}\right). \tag{2.10}$$

The body force thus consists of a hydrostatic lateral pressure $\frac{\mu H^2}{8\pi}$ together with a tension $\frac{\mu\mathbf{HH}}{4\pi}$ along the lines of force.
Inserting equation (2.10) into (2.11) yields

$$\frac{\partial}{\partial t}\mathbf{u} + \mathbf{u} \cdot \nabla\mathbf{u} - \frac{\mu}{4\pi\rho}\mathbf{H} \cdot \nabla\mathbf{H} = -\frac{1}{\rho}\nabla\left(p + \frac{\mu H^2}{8\pi}\right) + \nu\nabla^2\mathbf{u}. \tag{2.11}$$

In the following we introduce the quantity

$$\mathbf{h} = \left(\frac{\mu}{4\pi\rho}\right)^{\frac{1}{2}}\mathbf{H}, \tag{2.12}$$

which has the dimension of a velocity.
The MHD equations can now be written as

$$\frac{\partial}{\partial t}\mathbf{u} + \mathbf{u} \cdot \nabla\mathbf{u} - \mathbf{h} \cdot \nabla\mathbf{h} = -\frac{1}{\rho}\nabla\left(p + \frac{1}{2}\rho|\mathbf{h}|^2\right) + \nu\nabla^2\mathbf{u}, \tag{2.13}$$

$$\frac{\partial}{\partial t}\mathbf{h} + \mathbf{u} \cdot \nabla\mathbf{h} - \mathbf{h} \cdot \nabla\mathbf{u} = \lambda\nabla^2\mathbf{h}, \tag{2.14}$$

Equation (2.13) and (2.14) in addition to the incompressibility conditions[1] for the velocity and magnetic field

$$\nabla \cdot \mathbf{u} = 0, \qquad \nabla \cdot \mathbf{h} = 0, \tag{2.15}$$

are the basic equations of our problem.

2.1 Fluid dynamical properties

In the following, the focus lies on the fluid dynamical implications of the MHD equations.

[1]The incompressibility of the magnetic field is directly given by equation (2.4). The incompressibility of the velocity field can be justified as follows: In general, pressure changes in a fluid imply changes in the density. These changes however are negligible if the occurring velocities are small compared to the velocity of propagation of sound [Bat60]. This is the case in laboratory plasmas which are usually confined by a strong magnetic field. Also the motion in the core of the Earth can be considered as incompressible. However, in the interstellar medium, which is rather cold so that density changes can become important, we have to deal with compressible velocity fields [Bis03].

2.1.1 The magnetic and fluid Reynolds numbers

Transitions from a laminar state of the fluid into a turbulent state can be characterized by the so-called Reynolds number. Therefore we introduce the following dimensionless quantities

$$\tilde{\mathbf{u}} = \frac{\mathbf{u}}{U}, \quad \tilde{\mathbf{h}} = \frac{\mathbf{h}}{U}, \quad \tilde{t} = \frac{t}{T}, \quad \tilde{\mathbf{x}} = \frac{\mathbf{x}}{L}, \quad \tilde{p} = \frac{p}{\rho}\frac{L}{U^2}, \quad U = \frac{L}{T}, \tag{2.16}$$

where L, T and $U = L/T$ are characteristic length, time and velocity scales of the flow and the density ρ is assumed to be constant. Omitting the tilde signs, we get the MHD equations in the form

$$\frac{\partial}{\partial t}\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} - \mathbf{h}\cdot\nabla\mathbf{h} = -\frac{1}{\rho}\nabla\left(p + \frac{1}{2}|\mathbf{h}|^2\right) + \frac{1}{\mathrm{Re}}\nabla^2\mathbf{u}, \tag{2.17}$$

$$\frac{\partial}{\partial t}\mathbf{h} + \mathbf{u}\cdot\nabla\mathbf{h} - \mathbf{h}\cdot\nabla\mathbf{u} = \frac{1}{\mathrm{Rm}}\nabla^2\mathbf{h}, \tag{2.18}$$

which solely depend on the magnetic and fluid Reynolds number

$$\mathrm{Rm} = \frac{UL}{\lambda} \qquad \text{and} \qquad \mathrm{Re} = \frac{UL}{\nu}. \tag{2.19}$$

At high Reynolds numbers, the influence of the nonlinearities in (2.17) and (2.18) dominate the dissipative terms and we expect turbulent behavior. Furthermore there exists a geometrical similarity between flows with different scales but equal Reynolds number, so that experimental data can adequately be compared to real flows, for instance behind a wind turbine. In MHD turbulence the ratio between fluid and magnetic Reynolds number is given by the so-called magnetic Prandtl number

$$\mathrm{Pm} = \frac{\mathrm{Rm}}{\mathrm{Re}} = \frac{\nu}{\lambda}. \tag{2.20}$$

Typical Reynolds and magnetic Prandtl numbers are listed in table 2.1.

	Rm	Re	Pm
Liquid sodium experiments	10^2	10^8	10^{-6}
Liquid Earth's core	10^3	10^9	10^{-6}
Jupiter's core	10^6	10^{12}	10^{-6}
Sun's convective zone	10^{10}	10^{13}	10^{-3}
Numerical simulations	10^3	10^3	1
Accretion Disks	10^{12}	10^9	10^3
Galaxies	10^{13}	10^4	10^9
Air passing a car		10^7	
Water passing a ship		$10^7 - 10^{10}$	

Table 2.1: Typical fluid and magnetic Reynolds numbers and their corresponding magnetic Prandtl number in astrophysics [Plu12].

2.1.2 The role of pressure in MHD turbulence

As Elsässer [Els46-50] has pointed out, the MHD equations can be symmetrized by the introduction of the variables

$$\mathbf{z}^+ = \mathbf{u} + \mathbf{h} \qquad \text{and} \qquad \mathbf{z}^- = \mathbf{u} - \mathbf{h}. \tag{2.21}$$

The ideal MHD equations ($\nu = \lambda = 0$) in terms of the Elsässer fields thus read

$$\frac{\partial}{\partial t}\mathbf{z}^{+} + \mathbf{z}^{-} \cdot \nabla \mathbf{z}^{+} = -\frac{1}{\rho}\nabla\left(p + \frac{1}{2}\rho|\mathbf{h}|^2\right), \tag{2.22}$$

$$\frac{\partial}{\partial t}\mathbf{z}^{-} + \mathbf{z}^{+} \cdot \nabla \mathbf{z}^{-} = -\frac{1}{\rho}\nabla\left(p + \frac{1}{2}\rho|\mathbf{h}|^2\right) . \tag{2.23}$$

These equations treat $\mathbf{z}^{\pm}$ and hence $\mathbf{u}$ and $\mathbf{h}$ with complete symmetry, with one exception: On the right-hand side of equation (2.22) and (2.23) the term $\frac{1}{2}\rho|\mathbf{h}|^2$ appears directly, but the pressure term p appears instead of $\frac{1}{2}\rho|\mathbf{u}|^2$. This underlines the important role of the pressure term for the spatio-temporal behavior of magnetohydrodynamic flows.
In taking the divergence of equation (2.13), and by making use of the incompressibility conditions for the two fields, we can derive a Poisson equation for the pressure term at a given point $\mathbf{x}$

$$\nabla^2\left(p(\mathbf{x},t) + \frac{1}{2}\rho|\mathbf{h}(\mathbf{x},t)|^2\right) = -\rho\nabla \cdot (\mathbf{u}(\mathbf{x},t)\cdot\nabla\mathbf{u}(\mathbf{x},t) - \mathbf{h}(\mathbf{x},t)\cdot\nabla\mathbf{h}(\mathbf{x},t)), \tag{2.24}$$

which can be solved by making use of a Green function, introduced in the appendix A.1.
For an infinite three-dimensional region we get

$$p(\mathbf{x},t) = -\frac{1}{2}\rho|\mathbf{h}(\mathbf{x},t)|^2 + \rho\int \mathrm{d}\mathbf{x}'\frac{\nabla_{\mathbf{x}'}\cdot(\mathbf{u}(\mathbf{x}',t)\cdot\nabla_{\mathbf{x}'}\mathbf{u}(\mathbf{x}',t) - \mathbf{h}(\mathbf{x}',t)\cdot\nabla_{\mathbf{x}'}\mathbf{h}(\mathbf{x}',t))}{4\pi|\mathbf{x}-\mathbf{x}'|}. \tag{2.25}$$

The nonlocal character of the pressure term is next to the nonlinearities in the equations of motion, the main reason for the complexity of the fluid motion. Due to the incompressibility condition, the velocity of a turbulent fluid at a point $\mathbf{x}$ is not only influenced by the fluid motion in its surrounding, but instantaneously by the velocity and magnetic field over the whole space. In addition to hydrodynamic turbulence, the pressure $p(\mathbf{x},t)$ has a local contribution that arises from the magnetic pressure $\frac{1}{2}\rho|\mathbf{h}(\mathbf{x},t)|^2$.

2.1.3 The vorticity equation of MHD turbulence

In order to determine the occurrence of coherent structures of a turbulent flow, it is sometimes helpful to introduce the vorticity, which is defined as the curl of the velocity

$$\boldsymbol{\omega}(\mathbf{x},t) = \nabla \times \mathbf{u}(\mathbf{x},t) . \tag{2.26}$$

A strong vorticity can be found in regions, where there is an accumulation of several vortices. Whereas the nonlocality of the pressure avoids a local accumulation of the velocity, the vorticity in two dimensional fluid dynamics can organize itself into bigger clusters.
In taking the curl of (2.13) and setting $\sqrt{4\pi\mu/\rho}$ to unity, we arrive at

$$\begin{aligned}\frac{\partial}{\partial t}\boldsymbol{\omega}(\mathbf{x},t) + \mathbf{u}(\mathbf{x},t)\cdot\nabla\boldsymbol{\omega}(\mathbf{x},t) &= \boldsymbol{\omega}(\mathbf{x},t)\cdot\nabla\mathbf{u}(\mathbf{x},t)\\ &\quad+\mathbf{h}(\mathbf{x},t)\cdot\nabla\mathbf{j}(\mathbf{x},t) - \mathbf{j}(\mathbf{x},t)\cdot\nabla\mathbf{h}(\mathbf{x},t) + \nu\nabla^2\boldsymbol{\omega}(\mathbf{x},t).\end{aligned} \tag{2.27}$$

The second term on the left-hand side is a convective term, where the vorticity is transported by the velocity field. The first term on the right-hand side is the so-called vortex stretching term, which organizes the vorticity in thin vortex tubes and sheets. This term vanishes in two dimensional fluid dynamics, and the vorticity can organize itself into bigger clusters.

The other terms in (2.27) are pure magnetic influences and can also result in a stretching mechanism, which can be seen in 2D MHD. Instead one observes the formation and decay of so-called current sheets, which will be discussed in chapter 5 on 2D MHD turbulence. It is important to mention that although the pressure term dropped out of the vorticity equation, the nonlocality of the equations is still conserved due to the velocity, which is related to the vorticity over the whole space by the Biot-Savart law, as described in the appendix A.1.
In contrary to MHD flows, the vorticity equation in fluid mechanics exhibits several exact solutions. For instance, for inviscid fluid flows in two dimensions, the partial differential equation can be reduced to a Hamiltonian system of ordinary differential equations for the temporal evolution of so-called point vortices, provided that the vorticity initially consists in a superposition of delta-distributions [Are83-07]. Another important vortex solution for the case of viscous three dimensional flows, is the Lamb-Oseen vortex, a singular azimuthal-symmetric vortex that reduces the vorticity equation to a simple heat equation. The vortex solution therefore decays in time due to dissipation. Contrary to this particular solution, the so-called Burgers vortex consists in a stationary solution in the presence of dissipation, since it is exposed to an azimuthal strain field. For further discussion of vortex solutions and vortex methods the reader is referred to [Arg07] and [Saf92].

2.1.4 Kelvin's circulation theorem in fluid mechanics

An important consequence which follows from the vorticity equation of ideal fluid dynamics

$$\frac{\partial}{\partial t}\boldsymbol{\omega}(\mathbf{x},t) + \mathbf{u}(\mathbf{x},t)\cdot\nabla\boldsymbol{\omega}(\mathbf{x},t) = \boldsymbol{\omega}(\mathbf{x},t)\cdot\nabla\mathbf{u}(\mathbf{x},t), \tag{2.28}$$

is the conservation of the line integral of the velocity $\mathbf{u}$ along a closed curve C, which is known as the circulation Γ

$$\Gamma = \oint_C \mathrm{d}\mathbf{r}\cdot\mathbf{u} = \int_A \mathrm{d}\mathbf{a}\cdot\boldsymbol{\omega}. \tag{2.29}$$

In the last step the line integral was written as a surface integral of the vorticity, according to Stokes's theorem. A is thereby a surface bounded by the closed curve C. Kelvin's circulation theorem now states that any circulation Γ about a closed curve that moves within an incompressible, inviscid and barotropic[1] fluid, is a conserved quantity. If the fluid is subjected to dissipation, the circulation along the trajectories of particles moving with the fluid is no longer conserved.

2.2 Electromagnetic effects

In the following, we consider the implications of the MHD equations that originate from the induction equation in its original form (2.8). If the material is at rest, we get

$$\frac{\partial}{\partial t}\mathbf{H}(\mathbf{x},t) = \lambda\nabla^2\mathbf{H}(\mathbf{x},t), \tag{2.30}$$

which has the form of a diffusion equation. This leads to a decay of the magnetic field and dimensional arguments indicate a time of decay of the order $\tau = L^2\lambda^{-1} = 4\pi\sigma L^2$, where L is a length scale comparable with the region in which current flows. In laboratory conductors, for instance in a copper sphere of a radius of 1m, the time of decay is less than 10s, whereas for cosmic conductors these decay times can be very long. Elsässer calculated the free decay time

[1] A fluid is called barotropic, if the pressure is independent of the temperature and only depends on the density. In addition, the fluid should only be subjected to conservative volume forces

of the earth's magnetic field, assuming that its core consists of molten iron [Cow76]. He obtains a time of decay of 15000 years, and these times increase further for larger stellar objects.
The opposite case, where the material is exposed to a strong turbulent motion leads to

$$\frac{\partial}{\partial t}\mathbf{H}(\mathbf{x},t) = \nabla \times (\mathbf{u}(\mathbf{x},t) \times \mathbf{H}(\mathbf{x},t)), \tag{2.31}$$

which has the same form as the vorticity equation of inviscid fluid dynamics. The interpretation that the vortex lines move with the fluid, can therefore be adapted for the magnetic field. A motion of material relative to the lines of force generates an induced electric force. In the case of high-conductivity, the induced electric force must vanish, and with it the motion of the material relative to the lines of force. Only a transverse motion affects the magnetic field, and carries its lines of force with it. According to Alfvén [Alf50] one might also say that the lines of force are frozen into the material and carried about by its motion. The implications for frozen-in fields are discussed in considering a tube with a normal cross section $\mathrm{d}a$. Since the lines of force are frozen into the material the strength $H\mathrm{d}a$ has to remain constant. If $\mathrm{d}l$ is the distance between two neighboring segments, the mass $\rho\mathrm{d}a\mathrm{d}l$ of this segment remains constant as the tube is carried about by the motion and we arrive at

$$H \sim \rho \mathrm{d}l. \tag{2.32}$$

Therefore, the stretching of the lines of force increases also H.
Turning again to the mechanical effects of the MHD equations, we have seen that the body force $\mu\mathbf{j}(\mathbf{x},t) \times \mathbf{H}(\mathbf{x},t)$ of electromagnetic origin is perpendicular to the magnetic field. It thus has no influence on the fluid motion in the direction of the magnetic field. This can also be seen in dividing $\mathbf{u}(\mathbf{x},t)$ into a part longitudinal to the magnetic field $\mathbf{u}_l(\mathbf{x},t)$ and a part $\mathbf{u}_t(\mathbf{x},t)$ transverse to it

$$\mathbf{u}(\mathbf{x},t) = \mathbf{u}_l(\mathbf{x},t) + \mathbf{u}_t(\mathbf{x},t), \tag{2.33}$$

where

$$\mathbf{u}_l = \frac{\mathbf{H}}{H} \cdot \left(\frac{\mathbf{H}}{H} \cdot \mathbf{u}\right), \tag{2.34}$$

$$\mathbf{u}_t = -\frac{\mathbf{H}}{H} \times \left(\frac{\mathbf{H}}{H} \times \mathbf{u}\right). \tag{2.35}$$

Therefore, the longitudinal equation reads

$$\frac{\partial}{\partial t}\mathbf{u}_l(\mathbf{x},t) + \mathbf{u}(\mathbf{x},t)\cdot\nabla\mathbf{u}_l(\mathbf{x},t) = -\frac{1}{\rho}[\nabla p(\mathbf{x},t)]_l + \nu\nabla^2\mathbf{u}_l(\mathbf{x},t), \tag{2.36}$$

and the fluid motion in the field parallel direction is governed by the Navier-Stokes equation of fluid dynamics, with a pressure gradient $[\nabla p(\mathbf{x},t)]_l$ in the direction of the magnetic field.
For the discussion of the transverse motion it is convenient to focus on two cases:

i.) The case where the resistance is important:

In replacing $\mathbf{j}(\mathbf{x},t)$ according to (2.7), we get

$$\mu\left[\mathbf{j}(\mathbf{x},t) \times \mathbf{H}(\mathbf{x},t)\right]_t = \mu\sigma\left(\left[\mathbf{E}(\mathbf{x},t) \times \mathbf{H}(\mathbf{x},t)\right]_t - \mu H^2(\mathbf{x},t)\mathbf{u}_t(\mathbf{x},t)\right), \tag{2.37}$$

for the transverse part of the Lorentz force. Inserting this in (2.11) yields

$$\frac{\partial}{\partial t}\mathbf{u}_t(\mathbf{x},t) + \mathbf{u}(\mathbf{x},t)\cdot\nabla\mathbf{u}_t(\mathbf{x},t) = \mathbf{F}_t(\mathbf{x},t) + \mu\sigma\left(\left[\mathbf{E}(\mathbf{x},t) \times \mathbf{H}(\mathbf{x},t)\right]_t - \mu H^2(\mathbf{x},t)\mathbf{u}_t(\mathbf{x},t)\right), \tag{2.38}$$

where $\mathbf{F}(\mathbf{x},t)$ takes into account all non-electromagnetic forces. If the two terms on the right-hand side of (2.38) are rather small, the transverse motion is damped due to the induction drag represented by the last term, which can be regarded as a strong magnetic 'viscosity'. This quantity is characterized by the so-called Hartmann number

$$\mathrm{M} = \mu \bar{H} L \sqrt{\sigma} \rho \nu, \tag{2.39}$$

where L is a characteristic length scale and $\bar{H}$ is a mean magnetic field profile.

ii.) The case where the resistance is unimportant:

In the case, where the lines of force are frozen into the material, the current $\mathbf{j}(\mathbf{x},t)$ is expressed by

$$\mathbf{j}(\mathbf{x},t) = \frac{1}{4\pi} \nabla \times \mathbf{H}(\mathbf{x},t), \tag{2.40}$$

instead of equation (2.7), and again by the use of Maxwell's stresses, we obtain

$$\frac{\partial}{\partial t}\mathbf{u}_t(\mathbf{x},t) + \mathbf{u}(\mathbf{x},t)\cdot\nabla\mathbf{u}_t(\mathbf{x},t) - \frac{\mu}{4\pi\rho}\mathbf{H}(\mathbf{x},t)\cdot\nabla\mathbf{H}(\mathbf{x},t) = -\frac{1}{\rho}\nabla_t\left(p(\mathbf{x},t) + \frac{\mu H^2(\mathbf{x},t)}{8\pi}\right) + \nu\nabla^2\mathbf{u}_t(\mathbf{x},t).$$

The mechanical effects can now be separated. First-of-all, the lateral pressure $\frac{H^2(\mathbf{x},t)}{8\pi}$ inhibits the compression of a bundle of lines of force, whereas the tension $\frac{\mathbf{H}(\mathbf{x},t)\mathbf{H}(\mathbf{x},t)}{4\pi}$ along the lines of force shrinks themselves. Both effects pose certain restraints on the material motions, since if the lines of force are displaced from equilibrium the force $\mu\mathbf{j}(\mathbf{x},t) \times \mathbf{H}(\mathbf{x},t)$ has a restoring effect and is always normal to $\mathbf{H}(\mathbf{x},t)$. Therefore the presence of a magnetic field imparts to the fluid a certain rigidity. On the other hand it also imparts to it certain degrees of freedom, like the property of elasticity, where disturbances are transmitted by new modes of wave propagation, which are discussed in the following section.

2.3 Alfvén waves

In contrast to hydrodynamic turbulence, in MHD turbulence we can find solutions that represent the propagation of waves. Suppose that a uniform magnetic field $\mathbf{H}_0$ is present in an infinite homogeneous medium of an incompressible fluid and consider a velocity and a magnetic field $\mathbf{u}(\mathbf{x},t)$ and $\mathbf{h}(\mathbf{x},t)$ that are produced by small disturbances. The linearized MHD equations in their original form (2.8) and (2.11) thus read

$$\begin{aligned}
\frac{\partial}{\partial t}\mathbf{u}(\mathbf{x},t) - \mathbf{H}_0\cdot\nabla\mathbf{h}(\mathbf{x},t) &= -\frac{1}{\rho}\nabla\left(p(\mathbf{x},t) + \frac{\mu\mathbf{H}_0\cdot\mathbf{h}(\mathbf{x},t)}{4\pi}\right) + \nu\nabla^2\mathbf{u}(\mathbf{x},t), \\
\frac{\partial}{\partial t}\mathbf{h}(\mathbf{x},t) - \mathbf{H}_0\cdot\nabla\mathbf{u}(\mathbf{x},t) &= \lambda\nabla^2\mathbf{h}(\mathbf{x},t).
\end{aligned} \tag{2.41}$$

Taking the divergence of the first equation yields the Laplace equation

$$\nabla^2\left(p(\mathbf{x},t) + \frac{\mu\mathbf{H}_0\cdot\mathbf{h}(\mathbf{x},t)}{4\pi}\right) = 0. \tag{2.42}$$

Since the disturbance $\mathbf{h}(\mathbf{x},t)$ vanishes outside the region that is unperturbed, and the equilibrium condition yields $\nabla p(\mathbf{x},t) = 0$, the solution of the Laplace equation has to be constant everywhere in space and we get

$$\nabla\left(p(\mathbf{x},t) + \frac{\mu\mathbf{H}_0\cdot\mathbf{h}(\mathbf{x},t)}{4\pi}\right) = 0. \tag{2.43}$$

Inserting this in (2.41) yields

$$\frac{\partial}{\partial t}\mathbf{u}(\mathbf{x},t) - \mathbf{H}_0 \cdot \nabla \mathbf{h}(\mathbf{x},t) = \nu \nabla^2 \mathbf{u}(\mathbf{x},t). \tag{2.44}$$

Making the plane wave-ansatz

$$\begin{aligned} \mathbf{u}(\mathbf{x},t) &= \tilde{\mathbf{u}} e^{i\mathbf{k}\cdot\mathbf{x} - i\omega(k)t}, \\ \text{and} \quad \mathbf{h}(\mathbf{x},t) &= \tilde{\mathbf{h}} e^{i\mathbf{k}\cdot\mathbf{x} - i\omega(k)t}, \end{aligned} \tag{2.45}$$

for the magnetic and velocity field fluctuations yields

$$(\omega(k) - i\nu k^2)\tilde{\mathbf{u}} = \frac{\mu}{4\pi\rho}(\mathbf{k}\cdot\mathbf{H}_0)\tilde{\mathbf{h}}, \tag{2.46}$$

$$(\omega(k) - i\lambda k^2)\tilde{\mathbf{h}} = (\mathbf{k}\cdot\mathbf{H}_0)\tilde{\mathbf{u}}, \tag{2.47}$$

and

$$\mathbf{k}\cdot\tilde{\mathbf{u}} = 0, \qquad \mathbf{k}\cdot\tilde{\mathbf{h}} = 0. \tag{2.48}$$

Combining (2.46) and (2.47) gives a dispersion relation for the magnetohydrodynamic waves, namely

$$(\omega(k) - i\nu k^2)(\omega(k) - i\lambda k^2) = V_A^2 k^2, \tag{2.49}$$

where

$$V_A = \left(\frac{\mu}{4\pi}\right)^{\frac{1}{2}} H_0 \cos\theta, \tag{2.50}$$

is the so called Alfvén velocity and θ is the angle between $\mathbf{H}_0$ and the direction of propagation. As one readily can see from the equations (2.48), the Alfvén waves are transverse waves. This is indicated in fig 2.1, where the Alfvén waves propagate in the direction of a magnetic field $\mathbf{H}_0$.

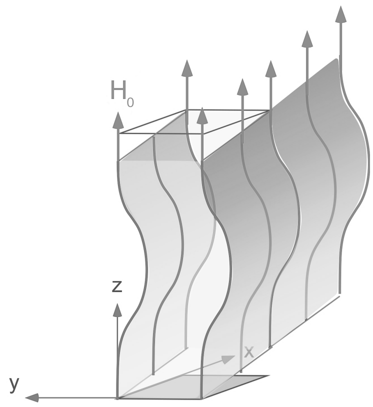


Figure 2.1: Alfvén waves moving in the direction of a magnetic field $\mathbf{H}_0$, which points in z-direction.

In the following we distinguish two cases:

i.) The case when there is no viscosity and resistivity $\nu = \lambda = 0$:

The waves are undamped and the dispersion relation has two solutions

$$\frac{\omega(k)}{k} = \pm V_A, \tag{2.51}$$

and in this case

$$\mathbf{u}(\mathbf{x},t) = \frac{\mu k}{4\pi\rho\omega(k)} H_0 \cos\theta \; \mathbf{h}(\mathbf{x},t), \tag{2.52}$$

which implies an equipartition of the mean energy of the wave over the kinetic and magnetic energy.

$$\frac{1}{2}\rho|\mathbf{u}(\mathbf{x},t)|^2 = \frac{\mu}{8\pi}|\mathbf{h}(\mathbf{x},t)|^2. \tag{2.53}$$

The group velocity of the waves is

$$\mathbf{v}_G = \nabla_{\mathbf{k}}\omega(k) = \pm\left(\frac{\mu}{4\pi\rho}\right)^{\frac{1}{2}}\mathbf{H}_0, \qquad (2.54)$$

which corresponds to wave-packets traveling parallel and anti-parallel to the lines of force, under the assumption that $\theta = 0$. This effect stands in analogy with the theory of stretched strings, where tensions lead to the possibility of transverse waves along the lines of force.

ii.) The case when there is finite viscosity and resistivity $\nu \neq 0$ and $\lambda \neq 0$.

The general solution of (2.49) reads

$$\omega(k) = \pm k\sqrt{V_A^2 - \frac{1}{4}(\nu-\lambda)^2k^2} + \frac{1}{2}i(\nu+\lambda)k^2, \qquad (2.55)$$

which has the effect of a damping of the waves, where ν and λ occur additively and a phase shift of the waves, where ν and λ occur differentially. In the limit of small ν and λ, we obtain the following approximation

$$\omega(k) = \pm V_A k\left(1 - \frac{(\nu-\lambda)^2}{8V_A^2}k^2\right) + \frac{1}{2}i(\nu+\lambda)k^2. \qquad (2.56)$$

Thus waves with different wave fronts do not all move with the same velocity along the lines of force. The time required for the diffusion of the Alfvén waves is in the order of the time of decay of magnetic fields mentioned in 2.2. This is caused by the fact that in general the electric resistance is much more important then the viscosity. In mercury under laboratory conditions one obtains for example magnetic Prandtl numbers Pm=ν/λ of $1,6\cdot 10^{-7}$, which is also comparable with Pm= 10^{-3} in the solar material.
The discussion of Alfvén waves can be extended to a nonuniform field $\mathbf{H}_0 = \mathbf{H}_0(\mathbf{x})$. The consequence of such a field is, that different parts of the waves move along separate lines of force with a local velocity $\mathbf{v}_G(\mathbf{x}) = \sqrt{\frac{\mu}{4\pi\rho}}\mathbf{H}_0(\mathbf{x})$. This will be of importance for the discussion of the hierarchy of structure function equations in chapter 4, in which a large-scale magnetic field is present.

2.4 Exact solutions of the MHD equations

Contrary to the vortex solutions, mentioned in 5.4, the MHD equations do exhibit far less exact solutions. Therefore we have to restrict ourselves to steady-state solutions. In the following we consider three special solutions of this kind.

2.4.1 The equipartition solution

A simple steady state solution of (2.8) and (2.11) is given by

$$\mathbf{u}(\mathbf{x}) = \pm\left(\frac{\mu}{4\pi\rho}\right)^{\frac{1}{2}}\mathbf{H}(\mathbf{x}), \qquad (2.57)$$

and

$$\nabla\left(p(\mathbf{x}) + \frac{\mu}{8\pi}|\mathbf{H}(\mathbf{x})|^2\right) = 0. \qquad (2.58)$$

This means that the fluid velocity is parallel or anti-parallel to the direction of the magnetic field at every point $\mathbf{x}$ in space. Furthermore (2.57) implies that there is equipartition between kinetic and magnetic energy

$$\frac{1}{2}\rho|\mathbf{u}(\mathbf{x})|^2 = \frac{\mu}{8\pi}|\mathbf{H}(\mathbf{x})|^2. \tag{2.59}$$

Although this solution seems to be arbitrary, it is worth noticing that only the condition of hydrostatic equilibrium (5.12) imposes a restriction on the spatial dependence of the field and the motion. The condition in equation (2.57) is a priori not forbidden.

2.4.2 The vanishing of the Lorentz force

If the Lorentz force in (2.11) vanishes, there is no back-reaction of the magnetic field on the fluid motion: Under these circumstances the conditions of hydrostatic equilibrium is satisfied if

$$[\nabla \times \mathbf{H}(\mathbf{x})] \times \mathbf{H}(\mathbf{x}) = 0, \qquad \mathbf{u}(\mathbf{x}) = 0, \qquad \text{and} \;\; \nabla p(\mathbf{x}) = 0. \tag{2.60}$$

This implies that the magnetic field and the current density are aligned at every point in space, which can be satisfied if

$$\nabla \times \mathbf{H}(\mathbf{x}) = \alpha \mathbf{H}(\mathbf{x}). \tag{2.61}$$

These so-called force-free fields, when the stresses vanish inside a given region, do not automatically imply that the stresses vanish outside this region, since they impose more complicated boundary conditions.

2.4.3 The Taylor-Proudman theorem for MHD turbulence

In the presence of a uniform magnetic field $\mathbf{H}_0$, we can ask the question how perturbations behave when there are no fluid motions at the beginning. We denote $\mathbf{h}(\mathbf{x}, t)$ as the perturbation in the magnetic field and $\mathbf{u}(\mathbf{x}, t)$ as the velocity perturbation. If a steady state is reached (2.8) and (2.11) read

$$\begin{aligned} \mu \mathbf{H}_0 \cdot \nabla \mathbf{h}(\mathbf{x}) &= \nabla \left(p(\mathbf{x}) + \mu \frac{\mathbf{H}_0 \cdot \mathbf{h}(\mathbf{x})}{4\pi\rho} \right), & (2.62) \\ \mathbf{H}_0 \cdot \nabla \mathbf{u}(\mathbf{x}) &= 0. & (2.63) \end{aligned}$$

An interesting feature of this system can be seen from (2.63). The fluid motions cannot vary in the direction of the uniform magnetic field $\mathbf{H}_0$ and all steady slow motions become necessarily two-dimensional. An analogy can be found in the Taylor-Proudman theorem for rotating fluids [Cha81]. It states that the velocity field of a fluid that is rotated with a high angular frequency Ω is uniform along any line parallel to the axis of rotation.

2.5 Conserved quantities

In addition to the basic conservation laws of ordinary inviscid fluid dynamics, the ideal MHD equations exhibit further conservation laws involving the magnetic field. We consider the MHD equations in the form of (2.17) and (2.18).

2.5.1 Fluid invariants

The total energy E_{tot} in a fluid element of the volume V and constant density $\rho = 1$ is composed of the kinetic energy $E_{kin} = \int_V \mathrm{d}\mathbf{x} \frac{\mathbf{u}^2}{2}$ and the magnetic energy $E_{mag} \int_V \mathrm{d}\mathbf{x} \frac{\mathbf{b}^2}{2}$, according to

$$E_{tot} = E_{kin} + E_{mag} = \int_V \mathrm{d}\mathbf{x} \left(\frac{\mathbf{u}^2}{2} + \frac{\mathbf{h}^2}{2} \right). \tag{2.64}$$

The density of the total energy $e_{tot} = \frac{\mathbf{u}^2}{2} + \frac{\mathbf{h}^2}{2}$ has to satisfy the following balance equation

$$\frac{\partial}{\partial t} e_{tot}(\mathbf{x}, t) + \nabla \cdot \mathbf{J}^{tot}(\mathbf{x}, t) = q(\mathbf{x}, t), \tag{2.65}$$

where

$$\int_V \mathrm{d}\mathbf{x} \; q(\mathbf{x}, t), \tag{2.66}$$

is the total energy per time that is generated or destroyed in the volume V and

$$\int_V \mathrm{d}\mathbf{x} \; \nabla \cdot \mathbf{J}^{tot}(\mathbf{x}, t) = \int_S \mathrm{d}\mathbf{a} \cdot \mathbf{J}^{tot}(\mathbf{x}, t), \tag{2.67}$$

is the transported energy throughout the surface S of the fixed volume V.
The current density $\mathbf{J}^{tot}(\mathbf{x}, t)$ and the source term follow from the MHD equations and are derived in the appendix A.2.

$$\begin{aligned} \mathbf{J}^{tot}(\mathbf{x}, t) &= \mathbf{u}(\mathbf{x}, t) \left[\frac{\mathbf{u}^2(\mathbf{x}, t)}{2} + \mathbf{h}^2(\mathbf{x}, t) + p(\mathbf{x}, t) \right] - \mathbf{h}(\mathbf{x}, t)\mathbf{u}(\mathbf{x}, t) \cdot \mathbf{h}(\mathbf{x}, t) \\ &\quad - \frac{\nu}{2} \nabla \mathbf{u}^2(\mathbf{x}, t) - \nu \mathbf{u}(\mathbf{x}, t) \cdot \nabla \mathbf{u}(\mathbf{x}, t) \\ &\quad - \frac{\lambda}{2} \nabla \mathbf{h}^2(\mathbf{x}, t) - \lambda \mathbf{h}(\mathbf{x}, t) \cdot \nabla \mathbf{h}(\mathbf{x}, t), \end{aligned} \tag{2.68}$$

$$q(\mathbf{x}, t) = \mathbf{f}(\mathbf{x}, t) \cdot \mathbf{u}(\mathbf{x}, t) + \mathbf{g}(\mathbf{x}, t) \cdot \mathbf{h}(\mathbf{x}, t) - \varepsilon^{\nu}(\mathbf{x}, t) - \varepsilon^{\lambda}(\mathbf{x}, t). \tag{2.69}$$

The two first terms in the brackets of (2.68) denote the kinetic and magnetic energy which are transported throughout the surface. It is important to notice, that, due to the pressure contribution, there is always the double of the magnetic energy which is transported. The second term $\mathbf{u}(\mathbf{x}, t)p(\mathbf{x}, t)$ is the work from the fluid against the pressure. The next term is the so-called cross helicity $\mathbf{u}(\mathbf{x}, t) \cdot \mathbf{h}(\mathbf{x}, t)$ which is transported anti-parallel along the lines of force.
The source term takes into account the power density which stems from an external forcing mechanism, $\mathbf{f}(\mathbf{x}, t)$ and $\mathbf{g}(\mathbf{x}, t)$, and contains the local energy dissipation rates $\varepsilon^{\nu}(\mathbf{x}, t)$ and $\varepsilon^{\lambda}(\mathbf{x}, t)$, which are defined as

$$\varepsilon^{\nu}(\mathbf{x}, t) = \frac{\nu}{2} \sum_{i,j} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)^2, \tag{2.70}$$

$$\varepsilon^{\lambda}(\mathbf{x}, t) = \frac{\lambda}{2} \sum_{i,j} \left(\frac{\partial h_i}{\partial x_j} + \frac{\partial h_j}{\partial x_i} \right)^2. \tag{2.71}$$

The local energy dissipation rates play an important role in the statistical theory of isotropic turbulence discussed in chapter 3. They are linked with the gradient of the velocity and the magnetic field so that an important amount of energy is dissipated in regions of high velocity and magnetic field gradients, like for example in the region of currents sheets, discussed in the

section 5.4 about the vorticity equation in MHD.
In the case of ideal MHD turbulence ($\nu = \lambda = 0$), the source term $q(\mathbf{x}, t)$ on the right-hand side of (2.65) vanishes and E_{tot} is a conserved quantity, provided that the current density $\mathbf{J}^{tot}(\mathbf{x}, t)$ vanishes at the domain boundary S.

Another important conserved quantity in ideal MHD is the so-called cross helicity

$$E_{cross} = \int_V \mathrm{d}\mathbf{x}\ \mathbf{u} \cdot \mathbf{h}. \tag{2.72}$$

The corresponding balance equation for the cross helicity density $e_{cross} = \mathbf{u} \cdot \mathbf{h}$

$$\frac{\partial}{\partial t} e_{cross}(\mathbf{x}, t) + \nabla \cdot \mathbf{J}^{cross}(\mathbf{x}, t) = q(\mathbf{x}, t), \tag{2.73}$$

with

$$\begin{aligned} \mathbf{J}^{cross}(\mathbf{x}, t) &= [\mathbf{u}(\mathbf{x}, t) + \mathbf{h}(\mathbf{x}, t)] \frac{\mathbf{u}(\mathbf{x}, t) \cdot \mathbf{h}(\mathbf{x}, t)}{2} \\ &\quad - \mathbf{h}(\mathbf{x}, t) \left[\frac{\mathbf{u}^2(\mathbf{x}, t)}{2} - p(\mathbf{x}, t) \right] - \nu A^\dagger \mathbf{h}(\mathbf{x}, t) - \lambda B^\dagger \mathbf{u}(\mathbf{x}, t) \qquad (2.74) \\ q(\mathbf{x}, t) &= \mathbf{f}(\mathbf{x}, t) \cdot \mathbf{h}(\mathbf{x}, t) + \mathbf{g}(\mathbf{x}, t) \cdot \mathbf{u}(\mathbf{x}, t) + (\nu + \lambda) \mathbf{j}(\mathbf{x}, t) \cdot \boldsymbol{\omega}(\mathbf{x}, t) \qquad (2.75) \end{aligned}$$

For the viscous terms we have introduced the velocity and magnetic gradient tensors

$$A_{ij} = \frac{\partial u_i}{\partial x_j}, \tag{2.76}$$

$$B_{ij} = \frac{\partial h_i}{\partial x_j}. \tag{2.77}$$

From (2.74) we conclude that there is a transport process of cross helicity along the vector sum of the magnetic and velocity field $\mathbf{u}(\mathbf{x}, t) + \mathbf{h}(\mathbf{x}, t)$. Furthermore, a transport of kinetic energy occurs against the lines of force next to a work by the magnetic field lines during a translation against the pressure.

2.5.2 Magnetic invariants

The induction equation (2.8) for $\mathbf{H}$ itself provides the conservation of two purely magnetic invariants in the case of vanishing resistivity: the magnetic flux ϕ and the magnetic helicity H_{mag}. The first one stands in analogy with the circulation Γ in hydrodynamics, since there is a similarity between the induction equation and the vorticity equation in hydrodynamic turbulence, as mentioned in 2.1.4. In the case where the lines of force are frozen into the material, the surface integral over $\mathbf{H}$, defined as the magnetic flux

$$\phi = \int_S \mathrm{d}\mathbf{a} \cdot \mathbf{H}, \tag{2.78}$$

is conserved.
The other invariant, the magnetic helicity

$$H_{mag} = \int_V \mathrm{d}\mathbf{x}\ \mathbf{A} \cdot \mathbf{H}, \tag{2.79}$$

is a measure for the twist and linkage of the lines of force, where $\mathbf{A}$ is the vector potential, so that

$$\mathbf{H} = \nabla \times \mathbf{A}. \tag{2.80}$$

Again, H_{mag} is similar to the kinetic helicity

$$H_{kin} = \int_V \mathrm{d}\mathbf{x}\ \mathbf{u} \cdot \boldsymbol{\omega}, \tag{2.81}$$

which is a conserved quantity in ideal hydrodynamics. However the kinetic helicity and the circulation are no longer conserved quantities in ideal MHD turbulence.

Chapter 3

Statistical description of MHD turbulence

In the preceding chapter, we have discussed the basic properties of the MHD equations. In this chapter, the focus lies on a statistical description of MHD turbulence which treats the velocity and the magnetic field as random functions. Since turbulent fluid motion involves a high number of active non-locally-coupled degrees of freedom, it is intuitively clear that a combination between a deterministic and a statistical treatment of the basic equations would mean further progress in the long-standing problem of turbulence. In the first section however, the need for a statistical description is discussed in detail. Most of the viewpoints presented in this chapter are taken from statistical fluid mechanics, mainly related with the name of A. N. Kolmogorov, but also with people like C. F. Weiszäcker, W. Heisenberg and G. I. Taylor.

3.1 The need for a statistical description

The Navier-Stokes equation, and also the MHD equations, are deterministic, nonlinear partial differential equations. If we know the velocity field $\mathbf{u}(\mathbf{x}, t)$ at a given instant t at every point in space $\mathbf{x}$, we should able to determine the field at a later time t'. The deterministic character of those kind of differential equations is best resumed in a note by Laplace, who talks about a fictive omniscient daemon:

"Une intelligence qui pour un instant donné, connaîtrait toutes les forces dont la nature est animée et la situation respective des êtres qui la composent, si d'ailleurs elle était assez vaste pour soumettre ces données à l'analyse, embrasserait dans la même formule, les mouvements des plus grands corps de l'univers et ceux du plus léger atome: rien ne serait incertain pour elle, et l'avenir comme le passé serait présent à ses yeux."

However, we know from dynamic systems theory that some differential equations are very sensitive with respect to their initial conditions. The Navier-Stokes equation shows this feature, too: small deviations in the initial conditions can have a great influence on the dynamics for later times. This is demonstrated in a simple example by Frisch ([Fri95] section 3.2, p.31). Nevertheless, Laplace's argumentation is still valid, there exists simply not such a creature, which possesses the whole information of the velocity field at a given instant t over the whole space. Similar problems arise in quantum theory where we are not able to determine the momentum and the location of a small particle, due to Heisenberg's uncertainty principle. We can only make

[1]This note is taken from [Arg07] p.16. An adequate translation would be: *A creature that knows at a given instant all forces acting on (fluid) elements, whose positions in an object of nature are known additionally, possesses the whole information about movements from the largest objects in the universe to the lightest atoms: nothing would be uncertain for it and the future as well as the past would be present to it.*

statements which have their foundations in probability theory. It seems therefore necessary, to develop a statistical description of turbulence. To this end, we suppose that the velocity in a turbulent flow takes random values of $\mathbf{u}(\mathbf{x}, t)$ and are mainly interested in the mean value of this random function. This immediately brings up the question of what we mean by a random function: Since we know that it is impossible to determine the exact value of $\mathbf{u}(\mathbf{x}, t)$, we rather assume that the random values of $\mathbf{u}(\mathbf{x}, t)$ are distributed according to certain probability laws. Then the method of taking averages consists in finding an appropriate statistical ensemble. This takes into account the totality of all possible realizations of a turbulent flow, which only differ in their initial conditions $\mathbf{u_0}(\mathbf{x})$ at all points $\mathbf{x}$ at a time t. The ensemble is then characterized by a statistical distribution for the initial field $\mathbf{u}_0(\mathbf{x})$ and we can derive a probability density function $p(\mathbf{u}, \mathbf{x}, t)$ for the velocity field $\mathbf{u}$ at position $\mathbf{x}$ and time t.
In general there will be a statistical connection between several velocities $\mathbf{u}_i$ at a point $\mathbf{x}_i$, and we have to consider the N-point probability function $p(\mathbf{u}_1, ..., \mathbf{u}_n, \mathbf{x}_1, ..., \mathbf{x}_n, t)$ and then perform the continuum limit. In order to take an average of some function $F(\mathbf{u}_1, ..., \mathbf{u}_n, t)$ an integration over all possible realizations $\mathbf{u}_i$ with a weight of the joint-probability function has to be performed according to

$$\langle F \rangle_{ensemble} = \int \mathrm{d}\mathbf{x}_1 ... \mathrm{d}\mathbf{x}_n F(\mathbf{u}_1, ..., \mathbf{u}_n, t) p(\mathbf{u}_1, ..., \mathbf{u}_n, \mathbf{x}_1, ..., \mathbf{x}_n, t) \tag{3.1}$$

This rather mathematical method of taking averages is not useful for applications, since the totality of all realizations is never accessible in experiments or in direct numerical simulations. Therefore the ergodicity hypothesis is used, which states that the ensemble average is replaced by temporal averages

$$\langle F \rangle_{temporal} = \langle F(t) \rangle = \lim_{T \to \infty} \frac{1}{T} \int_0^T \mathrm{d}t' F(t + t'). \tag{3.2}$$

The probability functions provide an adequate mean to characterize a turbulent flow with respect to the following aspects

- **Homogeneity:**
  The multi-point distributions are functions of the relative distance $\mathbf{r}_{ij} = \mathbf{x}_i - \mathbf{x}_j$ only. This can be interpreted as the invariance of the probability densities under translations $\mathbf{X}$

  $$p(\mathbf{u}_1, ..., \mathbf{u}_n, \mathbf{x}_1 + \mathbf{X}, ..., \mathbf{x}_n + \mathbf{X}, t) = p(\mathbf{u}_1, ..., \mathbf{u}_n, \mathbf{x}_1, ..., \mathbf{x}_n, t). \tag{3.3}$$

  Take for example the two-point velocity field correlation function

  $$C_{ij}^{\mathbf{uu}}(\mathbf{x}, \mathbf{x}', t) = \langle u_i(\mathbf{x}) u_j(\mathbf{x}') \rangle, \tag{3.4}$$

  under the assumption of homogeneity, it solely depends on the relative distance $\mathbf{r} = \mathbf{x} - \mathbf{x}'$ between the velocity field $u_i(\mathbf{x})$ at point $\mathbf{x}$ and the velocity field $u_i(\mathbf{x}')$ at point $\mathbf{x}'$, or

  $$C_{ij}^{\mathbf{uu}}(\mathbf{x}, \mathbf{x}', t) = C_{ij}^{\mathbf{uu}}(\mathbf{r}, t). \tag{3.5}$$

- **Stationarity:**
  The probability density function is not dependent on the time t and is thus invariant under time translations τ

  $$p(\mathbf{u}_1, ..., \mathbf{u}_n, \mathbf{x}_1, ..., \mathbf{x}_n, t + \tau) = p(\mathbf{u}_1, ..., \mathbf{u}_n, \mathbf{x}_1, ..., \mathbf{x}_n, t). \tag{3.6}$$

- **Isotropy:**
The probability density function is invariant under rotations $C \in \mathrm{SO}(3)$ with respect to an arbitrary axis of the coordinate system. This invariance can be broken by the presence of buoyancy and rotational forces, which suppress vertical and horizontal motion.

$$p(C\mathbf{u}_1, ..., C\mathbf{u}_n, C\mathbf{x}, ..., C\mathbf{x}_n, t) = p(\mathbf{u}_1, ..., \mathbf{u}_n, \mathbf{x}_1, ..., \mathbf{x}_n, t). \tag{3.7}$$

The assumption of isotropy and homogeneity has also a great influence on the tensorial form for the two-point velocity correlation function $C^{\mathbf{uu}}_{ij}(\mathbf{r}, t)$. For instance, it follows from the calculus of isotropic tensors, discussed in the appendix B.1, that its tensorial form is given according to

$$C^{\mathbf{uu}}_{ij}(\mathbf{r}, t) = (C^{\mathbf{uu}}_{rr}(r, t) - C^{\mathbf{uu}}_{tt}(r, t))\,\frac{r_i r_j}{r^2} + C^{\mathbf{uu}}_{tt}(r, t)\delta_{ij}, \tag{3.8}$$

where $C^{\mathbf{uu}}_{rr}(r, t)$ and $C^{\mathbf{uu}}_{tt}(r, t)$ are the longitudinal and transverse correlation functions, with respect to the relative distance $\mathbf{r}$.

- **Mirror Symmmetry:**
This symmetry is closely related to the condition of isotropy. In the invariant theory of turbulence and especially in the invariant theory of MHD turbulence, we have to deal several times with statistical quantities that are not invariant under the full rotation group. This is demonstrated at the example of the vorticity-velocity correlation function $B_{ij}(\mathbf{x}, \mathbf{x}', t) = \langle u_i(\mathbf{x}, t)\omega_j(\mathbf{x}', t)\rangle$. For the following a basic knowledge about isotropic tensor fields from the appendix B.1 is needed, in order to understand the different symmetry behavior of the tensor $B_{ij}(\mathbf{r}, t)$ compared to the tensor $C^{\mathbf{uu}}_{ij}(\mathbf{r}, t)$: This difference arises due to $\boldsymbol{\omega}$ being an axial vector, which is unchanged under a reflexion, contrary to the true polar vector $\mathbf{u}$, which changes signs, as indicated in fig. 3.1. Tensors which involve an odd

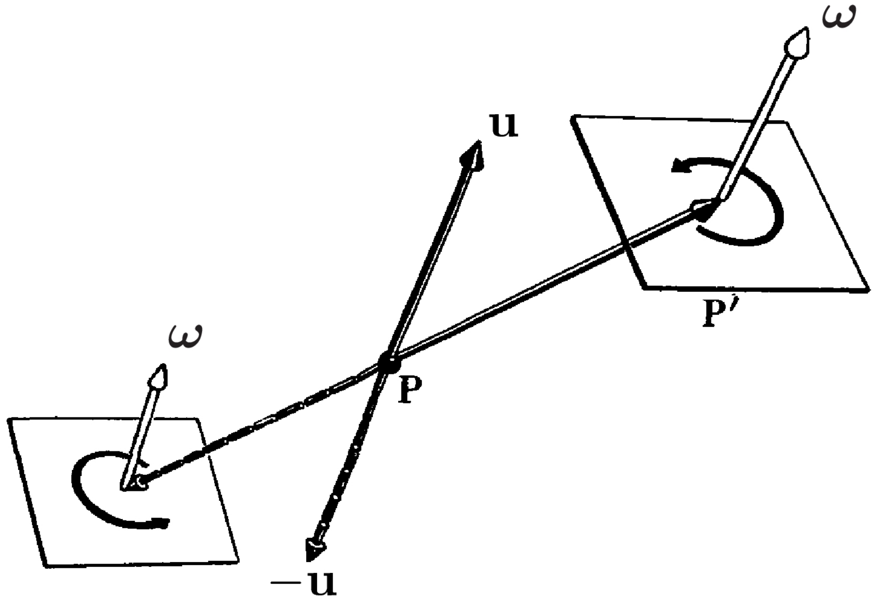


Figure 3.1: Symmetry behavior of $\mathbf{u}$ and $\boldsymbol{\omega}$ under reflections [Rob40].

number of polar vector quantities, like[1]

$$B_{ij}(\mathbf{r}, t) = \langle u_i(\mathbf{x}, t)\omega_j(\mathbf{x}', t)\rangle = B(r, t)\varepsilon_{ijk}\frac{r_k}{r}, \tag{3.9}$$

[1]The summation over equal indices is implied in the following.

thus lack the mirror symmetry, whereas tensors like $C^{\mathbf{uu}}_{ij}(\mathbf{r},t)$ that involve an even number of polar vectorial quantities can be written in the usual form according to equation (3.8). Since the magnetic field is an axial vector, similar to the vorticity, the invariant theory of MHD turbulence has to deal with correlations, like

$$C^{\mathbf{uh}}_{ij}(\mathbf{r},t) = \langle u_i(\mathbf{x},t)h_j(\mathbf{x}',t)\rangle = C^{\mathbf{uh}}(r,t)\varepsilon_{ijk}\frac{r_k}{r}. \tag{3.10}$$

that lack the mirror symmetry. The extension of the invariant theory to MHD turbulence was first introduced by Chandrasekhar [Cha51] and is discussed in section 3.4.4

3.2 Moment equations and the closure problem of turbulence

Important quantities in the statistical treatment of turbulence are correlation functions, like for example the two-point velocity correlation function $C^{\mathbf{uu}}_{ij}(\mathbf{x},\mathbf{x}',t) = \langle u_i(\mathbf{x},t)u_j(\mathbf{x}',t)\rangle$. An evolution equation for this moment of second order can be derived from the Navier-Stokes equation by making use of the incompressibility condition for the velocity field. This can also be done for moments of order n. The infinite chain of evolution equations for the n-point correlation functions in hydrodynamic turbulence is called the Friedmann-Keller hierarchy. However, these equations are not closed, and one has to deal with what is called the *closure problem in turbulence*: Due to the nonlinearity in the Navier-Stokes equation the evolution equation of the correlation function of order n is dependent on the divergence of the correlation function of order $n+1$, highlighting the fact that there is physics in the equations of order $n+1$ that is not contained up to n-th order. However, it is important to mention that the closure problem originates not only from the convective part, but mainly arises due to the nonlocality of pressure contributions in the moment equations.
In a later section 3.4.4 we will derive similar equations for three correlation functions of second order that occur in MHD turbulence, along the lines of Chandrasekhar [Cha51].
In the next chapter, the MHD equations are treated by a statistical description that was introduced for hydrodynamic turbulence by Osborne Reynolds in 1883. Its focus lies on the evolution equation for the mean field $\bar{\mathbf{u}}(\mathbf{x},t)$.

3.3 Reynolds averaged MHD equations

For most practical purposes, for instance in engineering, it is not necessary to know every detail of the fluctuating fields. In fact, one is rather interested in the behavior of the mean fields of the flow. This is also the case in dynamo theory, described in section 3.7, which is primarily concerned with the growth or maintenance of a mean magnetic field.
To this end, we assume that the magnetic and velocity field can be divided into mean and fluctuating parts, according to

$$\mathbf{u}(\mathbf{x},t) = \bar{\mathbf{u}}(\mathbf{x},t) + \mathbf{u}'(x,t), \tag{3.11}$$
$$\mathbf{h}(\mathbf{x},t) = \bar{\mathbf{h}}(\mathbf{x},t) + \mathbf{h}'(x,t), \tag{3.12}$$
$$p(\mathbf{x},t) = \bar{p}(\mathbf{x},t) + p'(\mathbf{x},t), \tag{3.13}$$

where the temporal mean values of the fluctuating parts vanish

$$\langle \mathbf{u}'(\mathbf{x},t)\rangle = \lim_{T\to\infty}\frac{1}{T}\int_t^{t+T} \mathrm{d}t'\ \mathbf{u}'(\mathbf{x},t') = 0, \tag{3.14}$$
$$\langle \mathbf{h}'(\mathbf{x},t)\rangle = \lim_{T\to\infty}\frac{1}{T}\int_t^{t+T} \mathrm{d}t'\ \mathbf{h}'(\mathbf{x},t') = 0. \tag{3.15}$$

Furthermore, terms like $\langle \mathbf{u}'(\mathbf{x},t) \cdot \nabla \mathbf{u}(\mathbf{x},t) \rangle$ vanish, since the mean streaming profile is assumed remain constant over the interval T. In addition, the mean and fluctuating fields are taken to be incompressible and ρ is set to $\rho = 1$. The Reynolds equations for MHD turbulence read

$$\begin{aligned} &\frac{\partial}{\partial t}\bar{\mathbf{u}}(\mathbf{x},t) + \bar{\mathbf{u}}(\mathbf{x},t) \cdot \nabla \bar{\mathbf{u}}(\mathbf{x},t) - \bar{\mathbf{h}}(\mathbf{x},t) \cdot \nabla \bar{\mathbf{h}}(\mathbf{x},t) + \nabla \cdot \langle \mathbf{u}'(\mathbf{x},t)\mathbf{u}'(\mathbf{x},t) - \mathbf{h}'(\mathbf{x},t)\mathbf{h}'(\mathbf{x},t) \rangle \\ = \quad & -\nabla \bar{p}(\mathbf{x},t) - \nabla \left(\frac{\bar{\mathbf{h}}(\mathbf{x},t)^2}{2} + \frac{\langle \mathbf{h}'(\mathbf{x},t)^2 \rangle}{2} \right) + \nu \nabla^2 \bar{\mathbf{u}}(\mathbf{x},t), \end{aligned} \tag{3.16}$$

$$\frac{\partial}{\partial t}\bar{\mathbf{h}}(\mathbf{x},t) - \nabla \times (\bar{\mathbf{u}}(\mathbf{x},t) \times \bar{\mathbf{h}}(\mathbf{x},t)) - \nabla \times \langle \mathbf{u}'(\mathbf{x},t) \times \mathbf{h}'(\mathbf{x},t) \rangle = \lambda \nabla^2 \bar{\mathbf{h}}(\mathbf{x},t). \tag{3.17}$$

In introducing the derivative $\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + \bar{u}_n \frac{\partial}{\partial x_n}$, these equations can be written in component form according to

$$\frac{\mathrm{D}}{\mathrm{D}t}\bar{u}_i = \frac{\partial}{\partial x_j}\left[\nu \left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i} \right) - \left(\bar{p} + \frac{\bar{h}^2}{2}\delta_{ij} \right) - \langle u_i' u_j' - h_i' h_j' \rangle - \frac{\langle h'^2 \rangle}{2}\delta_{ij} \right], \tag{3.18}$$

$$\frac{\mathrm{D}}{\mathrm{D}t}\bar{h}_i = \frac{\partial}{\partial x_i}\left[\lambda \left(\frac{\partial \bar{h}_i}{\partial x_j} + \frac{\partial \bar{h}_j}{\partial x_i} \right) + \bar{u}_i \bar{h}_j + \varepsilon_{ijk}\varepsilon_{kmn} \langle u_m' h_n' \rangle \right], \tag{3.19}$$

which corresponds to the MHD equations of the mean fields with supplementary nonlinearities. These terms represent the turbulent pulsations and have a non-negligible influence on the mean fields. The tensor which enters in equation (3.18)

$$R_{ij} = \langle u_i' u_j' - h_i' h_j' \rangle - \frac{\langle h'^2 \rangle}{2}\delta_{ij}, \tag{3.20}$$

is composed of the Reynolds and Maxwell stresses as well as an isotropic contribution from the lateral pressure $\frac{\langle h'^2 \rangle}{2}$.
The effect of the turbulent pulsations on the mean magnetic field is contained in the tensor

$$S_{ij} = \varepsilon_{ijk}\varepsilon_{kmn} \langle u_m' h_n' \rangle = \varepsilon_{ijk}\epsilon_k, \tag{3.21}$$

where $\boldsymbol{\epsilon} = \langle \mathbf{u}' \times \mathbf{h}' \rangle$ is the electromotive force, which is (apart from resistive effects) the induced electric field. It can be considered as an extra mean electric force which arises from the interaction of the turbulent motion with the magnetic field and plays an important role in the theory of dynamos.
The knowledge of these tensors in terms of functions of the mean fields would signify a great step forward in the calculations of mean streaming profiles, as for example the flow behind a wedge, or the explanation of the growth of a mean magnetic field due to small-scale turbulent motion, crucial for dynamo theories. To the present day there exists no such knowledge, and one has to model the turbulent stresses by an effective damping term that characterizes the energy flux from the mean field to the small-scale turbulent fine structure.
In ordinary turbulence the Reynolds stresses are written in a form analogous to the viscous stress tensor used in the derivation of the Navier-Stokes equation

$$R_{ij} = \langle u_i' u_j' \rangle = -\nu_t \left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i} \right) + \frac{1}{3}\langle u_k' u_k' \rangle \delta_{ij}. \tag{3.22}$$

The so-called eddy viscosity ν_t is thereby much larger than the kinematic viscosity ν and in the simplest model one assumes that it depends only on the the turbulent kinetic energy $K = \frac{1}{2}\langle u_i' u_i' \rangle$ and the local energy dissipation rate ε,

$$\nu_t = \frac{CK^2}{\langle \varepsilon \rangle}, \tag{3.23}$$

where C is a free numerical parameter.
Another interesting model goes back to Prandtl and is based on the mixing length r_m,

$$\nu_t = |\mathbf{u}'| r_m, \tag{3.24}$$

where r_m is typical mixing length. The mixing length can be seen as a turbulent 'mean free path' and an additional transport equation for the turbulent kinetic energy has to be solved. For the case of MHD turbulence an ansatz for the turbulent tensors was first given by [Yos90] as

$$R_{ij} = \frac{1}{3}\langle u'_k u'_k - h'_k h'_k \rangle \delta_{ij} - \nu_t^{kin}\left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}\right) + \nu_t^{mag}\left(\frac{\partial \bar{h}_i}{\partial x_j} + \frac{\partial \bar{h}_j}{\partial x_i}\right), \tag{3.25}$$

$$S_{ij} = -\alpha_t \epsilon_{ijk} \bar{h}_k + \beta_t^{mag}\left(\frac{\partial \bar{h}_j}{\partial x_i} - \frac{\partial \bar{h}_i}{\partial x_j}\right) - \beta_t^{kin}\left(\frac{\partial \bar{u}_j}{\partial x_i} - \frac{\partial \bar{u}_i}{\partial x_j}\right). \tag{3.26}$$

The last equation can be rewritten in terms of the electromotive force as

$$\boldsymbol{\epsilon} = \alpha_t \bar{\mathbf{h}} - \beta_t^{mag} \bar{\mathbf{j}} + \beta_t^{kin} \boldsymbol{\omega}, \tag{3.27}$$

with the current density $\mathbf{j} = \nabla \times \mathbf{h}$ and the vorticity $\boldsymbol{\omega} = \nabla \times \mathbf{u}$. The interpretations of the transport coefficients are the following: ν_t^{kin} is an eddy viscosity similar to that in (3.22), and β_t^{kin} is a turbulent resistivity. The term ν_t^{mag} represents the influence of the magnetic fluctuations $\mathbf{h}'(\mathbf{x}, t)$ on the mean field $\bar{\mathbf{u}}(\mathbf{x}, t)$ and similarly β_t^{kin} represents the influence of the turbulent velocity field $\mathbf{u}'(\mathbf{x}, t)$ on the mean magnetic field $\bar{\mathbf{h}}(\mathbf{x}, t)$. Furthermore the coefficients ν_t^{kin} and β_t^{mag} are connected with the turbulent energy $E = \frac{1}{2}\langle u'_k u'_k + h'_k h'_k$, whereas ν_t^{mag} and β_t^{kin} are connected with the turbulent cross helicity $H^C = \langle \mathbf{v}' \cdot \mathbf{h}' \rangle$. We will come back to the issue of mean magnetic fields in the section about mean field electrodynamics 3.7.2, where the α_t-term in equation (3.26) will play an important role for the maintenance of a mean magnetic field.

3.4 The concept of turbulent cascades

The observation that the turbulent motion manifests itself by the presence of many vortices of different sizes, which decay after a certain time, led the way to the concept of energy cascades. These observations were already made by Leonarda da Vinci, who tried to qualify the complex streaming profiles in sketches like that in fig. 3.2.

Figure 3.2: Leonardo da Vinci's sketch of water passing a row. A separation of vortices with different length scales and intensities can clearly be observed.

Richardson interpreted the emergence and the decay of vortices in three dimension as an energy transfer between vortices of different length scales. In the following, it will be seen that the nonlinearity is responsible for this energy flux from large to small scales, a characteristic feature of a direct cascade. It is important to stress that during this process no energy is lost. Only when a smallest scale is reached, the kinetic energy is dissipated by molecular viscosity. The concept of energy cascades was then followed by Kolmogorov. He made quantitative statements about the rate of energy transfer and characteristic scales in turbulence: In his first hypothesis of local isotropy he assumed that in the limit of high Reynolds numbers, the system forgets about large-scale anisotropies during the cascading process. These large scale anisotropies can be imposed by a forcing mechanism or by certain boundary conditions.
A direct consequence of his hypothesis is that every turbulent flow with equal Reynolds number should show the same statistical properties at scales small compared to the so called integral length scale L, where energy is injected into the system. Homogeneity and isotropy is then reached at smaller scales, where the energy transfer to smaller scales takes place. In his first similarity hypothesis Kolmogorov postulates that the processes in this range are uniquely determined by the averaged rate of energy dissipation $\langle\varepsilon\rangle$ and the kinematic viscosity. From dimensional arguments we can therefore define characteristic length, velocity and time scales,

at which the hypothesis is fulfilled. It follows from the Navier-Stokes equation that ν and $\langle\epsilon\rangle$ have the following units

$$[\nu] = \frac{\mathrm{m}^2}{\mathrm{s}} \qquad [\ \langle\varepsilon\rangle\ ] = \frac{\mathrm{m}^2}{\mathrm{s}^3}. \tag{3.28}$$

If we define a length scale η that depends only on ν and ϵ, its dimensional form is given as

$$[\eta] = [\ \eta(\nu, \langle\varepsilon\rangle)\ ] = \left(\frac{\mathrm{m}^2}{\mathrm{s}}\right)^a \left(\frac{\mathrm{m}^2}{\mathrm{s}^3}\right)^b. \tag{3.29}$$

Since the dimension of η has to be m, the exponents in (3.29) have to be chosen as $a = \frac{3}{4}$ and $b = -\frac{1}{4}$. The so-called dissipation length η is therefore given as

$$\eta = \left(\frac{\nu^3}{\langle\varepsilon\rangle}\right)^{\frac{1}{4}}. \tag{3.30}$$

Similar considerations can be made for the time and velocity scales. The so-called Kolmogorov length scales thus read

$$\eta = \left(\frac{\nu^3}{\langle\varepsilon\rangle}\right)^{\frac{1}{4}}, \qquad u_\eta = (\langle\varepsilon\rangle\nu)^{\frac{1}{4}} \qquad \text{and} \quad \tau_\eta = \left(\frac{\nu}{\langle\varepsilon\rangle}\right)^{\frac{1}{2}}. \tag{3.31}$$

The corresponding Reynolds number of these scales is

$$\mathrm{Re}_\eta = \frac{u_\eta \eta}{\nu} = 1. \tag{3.32}$$

They can thus be considered as a characteristic scale of a laminar fluid motion.
In the following we want to introduce further characteristic length scales that play an important role in the statistical treatment of turbulence.

3.4.1 The integral length scale

As it was mentioned in the preceding section, Kolmogorov's first similarity is only fulfilled on scales that are small compared to the integral length scale L, which defines the typical length scales of the biggest vortex structures and also the scale at which energy is injected into the system. A formal definition of the integral length scale makes use of the longitudinal velocity correlation function

$$C^{\mathbf{uu}}_{r\,r}(r) = \langle u_r(\mathbf{x}+\mathbf{r})u_r(\mathbf{x})\rangle, \tag{3.33}$$

and is given as

$$L = \int_0^\infty \mathrm{d}r \frac{C^{\mathbf{uu}}_{r\,r}(r)}{C^{\mathbf{uu}}_{r\,r}(0)}. \tag{3.34}$$

The integral scale depends on the kinetic energy E and the energy dissipation rate $\langle\epsilon\rangle$, $L = L(E, \epsilon)$. By dimensional analysis, we find

$$L = \frac{E^{\frac{3}{2}}}{\langle\varepsilon\rangle} = \frac{(\frac{1}{2}u_{rms})^{\frac{3}{2}}}{\langle\varepsilon\rangle}, \tag{3.35}$$

where $u_{rms} = \sqrt{\langle\mathbf{u}^2\rangle}$ is the root mean square velocity.

3.4.2 Taylor microscale

It is sometimes helpful to use another scale then the integral length scale, for instance for the definition of Reynolds numbers in experiments that have no comparable boundary conditions. This so-called Taylor microscale λ is defined by the radius of curvature of the longitudinal velocity correlation function. A Taylor-Reynolds number can therefore be defined as

$$Re_\lambda = \frac{u_{rms}\lambda}{\nu}, \tag{3.36}$$

where $\sigma = \sqrt{\langle \mathbf{u}'^2 \rangle}$ denotes the standard deviation of the velocity fluctuations. The advantage of the Taylor-Reynolds number is its independence of the integral length scale L. Therefore, experiments that have different boundary conditions or a different energy injection mechanism can be compared more reasonably.
The integral length scale and the Taylor microscale limit the so-called inertial range, at which Kolmogorov's theory is fulfilled. At scales that are smaller then the Taylor microscale, dissipative effects play an important role. This range is therefore called the dissipation range and λ is the appropriate length scale.
Since $\lambda = \lambda(E, \nu, \langle\varepsilon\rangle)$, this gives by dimensional analysis

$$\lambda = \sqrt{\frac{E\nu}{\langle\varepsilon\rangle}}. \tag{3.37}$$

3.4.3 Taylor hypothesis

Structure functions based on spatial velocity increments $\mathbf{v}(\mathbf{x}, \mathbf{x}', t) = \mathbf{u}(\mathbf{x}, t) - \mathbf{u}(\mathbf{x}', t)$ play an important role in the statistical treatment of turbulence. In experimental measurements, these increments are often not accessible and velocity increments are measured rather in the temporal domain. The obtained temporal increments Δt can be transformed into spatial increments $\mathbf{r} = \mathbf{x} - \mathbf{x}'$ by the use of Taylor's hypothesis:
Let us consider a wind tunnel where the velocity $\mathbf{u}'(\mathbf{x}, t)$ is measured in the frame of reference of the mean flow, which points in the direction of $\mathbf{x}$. In the frame of reference of the laboratory we would therefore measure the velocity

$$\mathbf{u}(\mathbf{x}, t) = \mathbf{u}'(\mathbf{x} - \mathbf{U}t, t) + \mathbf{U}, \tag{3.38}$$

where $\mathbf{U}$ denotes the mean flow $\mathbf{U} = \langle \mathbf{u}(\mathbf{x}, t) \rangle$. Under the assumption, that the mean flow is large compared to the fluctuations

$$\frac{|\mathbf{u}'|^2}{|\mathbf{U}|^2} \ll 1, \tag{3.39}$$

most of the time-dependence in $\mathbf{u}(\mathbf{x}, t)$ arises from $\mathbf{x} - \mathbf{U}t$ in $\mathbf{u}'(\mathbf{x}, t)$. Therefore, the temporal variation of $\mathbf{u}(\mathbf{x}, t)$ can be interpreted as a spatial variation of $\mathbf{u}'(\mathbf{x}, t)$ and one gets

$$\mathbf{r} = \mathbf{U}\Delta t. \tag{3.40}$$

3.4.4 The Kármán-Howarth equation of MHD turbulence

In analogy to the equation of motion of the two-point correlation function $C_{ij}^{\mathbf{uu}}(\mathbf{x}, \mathbf{x}', t) = \langle u_i(\mathbf{x}, t) u_j(\mathbf{x}', t) \rangle$ resulting from the Navier-Stokes equation of fluid dynamics mentioned in section 3.2, we can derive three equations for the three correlation functions occurring in MHD turbulence, namely

$$\begin{aligned} C^{\mathbf{uu}}_{i\,j}(\mathbf{x},\mathbf{x}',t) &= \langle u_i(\mathbf{x},t)u_j(\mathbf{x}',t)\rangle, \\ C^{\mathbf{hh}}_{i\,j}(\mathbf{x},\mathbf{x}',t) &= \langle h_i(\mathbf{x},t)h_j(\mathbf{x}',t)\rangle, \\ \text{and} \quad C^{\mathbf{uh}}_{i\,j}(\mathbf{x},\mathbf{x}',t) &= \langle u_i(\mathbf{x},t)h_j(\mathbf{x}',t)\rangle. \end{aligned} \tag{3.41}$$

To this end we consider the MHD equations in component form

$$\frac{\partial}{\partial t}u_i + \frac{\partial}{\partial x_k}(u_i u_k - h_i h_k) = -\frac{1}{\rho}\frac{\partial}{\partial x_i}P + \nu\nabla^2_{\mathbf{x}}u_i, \tag{3.42}$$

$$\frac{\partial}{\partial t}h_i + \frac{\partial}{\partial x_k}(h_i u_k - u_i h_k) = \lambda\nabla^2_{\mathbf{x}}h_i, \tag{3.43}$$

where $P = p + \frac{1}{2}\rho h^2$ is the total pressure exerted on the fluid. Furthermore we have introduced the Laplace operator $\nabla^2_{\mathbf{x}} = \frac{\partial}{\partial x_n}\frac{\partial}{\partial x_n}$ acting in $\mathbf{x}$-space and sum over equal indices.
By a procedure developed by von Kármán and Howarth [Kar38], we multiply (3.42) by $u'_j = u_j(\mathbf{x}',t)$ and take the ensemble average

$$\langle u'_j\frac{\partial}{\partial t}u_i\rangle + \frac{\partial}{\partial x_k}(\langle u_i u_k u'_j\rangle - \langle h_i h_k u'_j\rangle) = -\frac{1}{\rho}\frac{\partial}{\partial x_i}\langle P u'_j\rangle + \nu\nabla^2_{\mathbf{x}}\langle u_i u'_j\rangle, \tag{3.44}$$

where it is assumed that the ensemble average commutes with the partial derivatives.
In interchanging the indices and the primed and unprimed quantities, we arrive at a second equation

$$\langle u_i\frac{\partial}{\partial t}u'_j\rangle + \frac{\partial}{\partial x'_k}(\langle u'_j u'_k u_i\rangle - \langle h'_j h'_k u_i\rangle) = -\frac{1}{\rho}\frac{\partial}{\partial x'_j}\langle P' u_i\rangle + \nu\nabla^2_{\mathbf{x}'}\langle u_i u'_j\rangle. \tag{3.45}$$

Before we add the two equations to one another in order to get the evolution equation for $C^{\mathbf{uu}}_{i\,j}(\mathbf{x},\mathbf{x}',t) = \langle u_i(\mathbf{x},t)u_j(\mathbf{x}',t)\rangle$, we can simplify some terms on the basis of symmetry arguments:

- Since we are assuming homogeneity, the correlation functions are only dependent on the relative distance $\mathbf{r} = \mathbf{x}' - \mathbf{x}$, so that

$$\begin{aligned} C^{\mathbf{uu}}_{i\,j}(\mathbf{r},t) &= \langle u_i(\mathbf{x},t)u_j(\mathbf{x}',t)\rangle = C^{\mathbf{uu}}_{i\,j}(-\mathbf{r},t), \\ C^{\mathbf{uuu}}_{k\,i,j}(\mathbf{r},t) &= \langle u_k(\mathbf{x},t)u_i(\mathbf{x},t)u_j(\mathbf{x}',t),\rangle \\ C^{\mathbf{uuu}}_{k\,j,i}(-\mathbf{r},t) &= \langle u_k(\mathbf{x},t)u_j(\mathbf{x},t)u_i(\mathbf{x}',t).\rangle \end{aligned} \tag{3.46}$$

  Therefore, we can rewrite the viscous terms as[1]

$$(\nabla^2_{\mathbf{x}} + \nabla^2_{\mathbf{x}'})\langle u_i(\mathbf{x},t)u_j(\mathbf{x}',t)\rangle = 2\nabla^2_{\mathbf{r}}\langle u_i(\mathbf{x},t)u_j(\mathbf{x}',t)\rangle \tag{3.47}$$

  The same relations hold for the tensors

$$\begin{aligned} C^{\mathbf{hh}}_{i\,j}(\mathbf{r},t) &= \langle h_i(\mathbf{x},t)h_j(\mathbf{x}',t)\rangle, \\ C^{\mathbf{hhu}}_{k\,i,j}(\mathbf{r},t) &= \langle h_k(\mathbf{x},t)h_i(\mathbf{x},t)u_j(\mathbf{x}',t)\rangle, \end{aligned} \tag{3.48}$$

  since they are isotropic and mirror symmetric.
  Consequently, the triple correlation tensors fulfill the following relation

$$\begin{aligned} \langle u_i(\mathbf{x},t)u_j(\mathbf{x},t)u_k(\mathbf{x}',t)\rangle &= -\langle u_i(\mathbf{x}',t)u_j(\mathbf{x}',t)u_k(\mathbf{x},t)\rangle, \\ \langle h_i(\mathbf{x},t)h_j(\mathbf{x},t)u_k(\mathbf{x}',t)\rangle &= -\langle h_i(\mathbf{x}',t)h_j(\mathbf{x}',t)u_k(\mathbf{x},t)\rangle. \end{aligned} \tag{3.49}$$

[1] This relationship is derived in the appendix B.2.3 in equation (B.79) and is only valid for homogeneous correlation functions.

- Turning to the total pressure, we conclude that the pressure correlation $\frac{1}{\rho}\langle P(\mathbf{x},t)u_j(\mathbf{x}',t)\rangle$, is a vectorial quantity that depends on the relative distance $\mathbf{r}$. On the basis of isotropy it can thus be written as
$$\frac{1}{\rho}\langle P(\mathbf{x},t)u_j(\mathbf{x}',t)\rangle = C^P(r,t)\frac{r_j}{r}. \tag{3.50}$$
The incompressibility condition for the velocity field implies that
$$\frac{\partial}{\partial x'_j}\frac{1}{\rho}\langle P(\mathbf{x},t)u_j(\mathbf{x}',t)\rangle = -\frac{\partial}{\partial r_j}C^P(r,t)\frac{r_j}{r} = 0, \tag{3.51}$$
which immediately yields $C^P(r,t) = 0$, so that the pressure correlations vanish.

Respecting these considerations, we arrive at a balance equation for the two-point velocity correlation function, which reads
$$\frac{\partial}{\partial t}\langle u_i u'_j\rangle - 2\frac{\partial}{\partial r_k}(\langle u_i u_k u'_j\rangle - \langle h_i h_k u'_j\rangle) = 2\nu\nabla^2_{\mathbf{r}}\langle u_i u'_j\rangle, \tag{3.52}$$
or in shorter form
$$\frac{\partial}{\partial t}C^{\mathbf{uu}}_{i\,j}(\mathbf{r},t) - 2\frac{\partial}{\partial r_k}(C^{\mathbf{uuu}}_{k\,i,j}(\mathbf{r},t) - C^{\mathbf{hhu}}_{k\,i,j}(\mathbf{r},t)) = 2\nu\nabla^2_{\mathbf{r}}C^{\mathbf{uu}}_{i\,j}(\mathbf{r},t). \tag{3.53}$$
In the same manner, we get an evolution equation for the two-point magnetic field correlation from (3.43)
$$\frac{\partial}{\partial t}\langle h_i h'_j\rangle - 2\frac{\partial}{\partial r_k}\langle(h_i u_k - h_k u_i)h'_j\rangle = 2\lambda\nabla^2_{\mathbf{r}}\langle h_i h'_j\rangle, \tag{3.54}$$
which can be rewritten as
$$\frac{\partial}{\partial t}C^{\mathbf{hh}}_{i\,j}(\mathbf{r},t) - 2\frac{\partial}{\partial r_k}C^{\mathbf{uhh}}_{k\,i,j}(\mathbf{r},t) = 2\lambda\nabla^2_{\mathbf{r}}C^{\mathbf{hh}}_{i\,j}(\mathbf{r},t). \tag{3.55}$$
Turning to equation (3.53), the defining scalars of each tensor can be rewritten in terms of its longitudinal correlation function only. This is possible due to the incompressibility condition and it is derived in the appendix B.1.2. One obtains
$$\begin{aligned}
C^{\mathbf{uu}}_{i\,j}(\mathbf{r},t) &= \left(C^{\mathbf{uu}}_{r\,r}(r,t) - \frac{1}{2r}\frac{\partial}{\partial r}(r^2 C^{\mathbf{uu}}_{r\,r}(r,t))\right)\frac{r_i r_j}{r^2} + \frac{1}{2r}\frac{\partial}{\partial r}(r^2 C^{\mathbf{uu}}_{r\,r}(r,t))\delta_{ij},\\
C^{\mathbf{uuu}}_{k\,i,j}(\mathbf{r},t) &= -\frac{r^2}{2}\frac{\partial}{\partial r}\left(\frac{C^{\mathbf{uuu}}_{r\,r\,r}(r,t)}{r}\right)\frac{r_i r_j r_k}{r^3}\\
&\quad + \frac{1}{4r}\frac{\partial}{\partial r}\left(r^2 C^{\mathbf{uuu}}_{r\,r\,r}(r,t)\right)\left(\frac{r_i}{r}\delta_{kj} + \frac{r_k}{r}\delta_{ij}\right) - \frac{C^{\mathbf{uuu}}_{r\,r\,r}(r,t)}{2}\frac{r_j}{r}\delta_{ik},\\
C^{\mathbf{hhu}}_{k\,i,j}(\mathbf{r},t) &= -\frac{r^2}{2}\frac{\partial}{\partial r}\left(\frac{C^{\mathbf{hhu}}_{r\,r\,r}(r,t)}{r}\right)\frac{r_i r_j r_k}{r^3}\\
&\quad + \frac{1}{4r}\frac{\partial}{\partial r}\left(r^2 C^{\mathbf{hhu}}_{r\,r\,r}(r,t)\right)\left(\frac{r_i}{r}\delta_{kj} + \frac{r_k}{r}\delta_{ij}\right) - \frac{C^{\mathbf{hhu}}_{r\,r\,r}(r,t)}{2}\frac{r_j}{r}\delta_{ik}.
\end{aligned} \tag{3.56}$$
The scalars can be interpreted as indicated in fig. 3.3, where the longitudinal direction lies in x_1 direction: The correlation between u_1^2 (or u_2^2) at $M = (0,0,0)$ and u'_1 at $M = (r,0,0)$ is determined from equation (3.56) as
$$\langle u_1^2 u'_1\rangle = -4C^{\mathbf{uuu}}_{r\,r\,r}(r,t) \quad \text{and} \quad \langle u_2^2 u'_1\rangle = 2rC^{\mathbf{uuu}}_{r\,r\,r}(r,t) = -4C^{\mathbf{uuu}}_{t\,t\,r}(r,t). \tag{3.57}$$

A similar consideration can be made for the correlations between h_1^2 (or h_2^2) at $M = (0,0,0)$ and u_1' at $M = (r,0,0)$. We get

$$\langle h_1^2 u_1' \rangle = -4C^{\mathbf{hhu}}_{r\,r\,r}(r,t) \quad \text{and} \quad \langle h_2^2 u_1' \rangle = 2rC^{\mathbf{hhu}}_{r\,r\,r}(r,t) = -4C^{\mathbf{hhu}}_{t\,t\,r}(r,t). \tag{3.58}$$

In fig. 3.3 the mixed correlation function $C^{\mathbf{hhu}}_{t\,t\,r}(r,t) = -\frac{1}{2}C^{\mathbf{hhu}}_{t\,t\,r}(r,t)$ is depicted, since it is more illustrative.

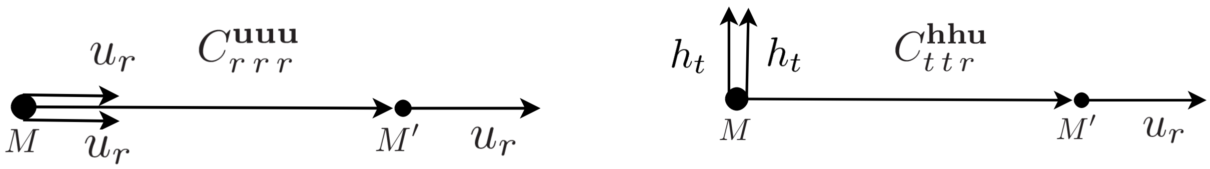


Figure 3.3: The third order correlation functions involved in the Kármán-Howarth equation of MHD turbulence.

It follows from a Taylor expansion of $C^{\mathbf{uuu}}_{r\,r\,r}(r,t)$ that the longitudinal correlation function scales as r^3 near the origin[1]. This is however not the case for $C^{\mathbf{hhu}}_{r\,r\,r}(r,t)$ which can be expanded as

$$C^{\mathbf{hhu}}_{r\,r\,r}(r,t) = C_0 r, \tag{3.59}$$

where

$$C_0 = -\frac{1}{4}\langle h_1^2 \frac{\partial u_1}{\partial x_1} \rangle = \frac{1}{2}\langle h_2^2 \frac{\partial u_1}{\partial x_1} \rangle \tag{3.60}$$

can be seen as a contribution to the magnetic energy from the stretching of the magnetic field lines by the velocity field.

Let us sum over equal indices i and j in equation (3.53). We define

$$Q_{kin}(r,t) = \sum_{i=j} C^{\mathbf{uu}}_{i\,j}(\mathbf{r},t), \tag{3.61}$$

$$J^{kin}_k(r,t) = \sum_{i=j} [C^{\mathbf{hhu}}_{k\,i,j}(\mathbf{r},t) - C^{\mathbf{uuu}}_{k\,i,j}(\mathbf{r},t)], \tag{3.62}$$

and arrive at a balance equation for the quantity $Q_{kin}(r,t)$ with a corresponding current $\mathbf{J}^{kin}(\mathbf{r},t)$

$$\frac{\partial}{\partial t} Q_{kin}(r,t) + 2\frac{\partial}{\partial r_k} J^{kin}_k(\mathbf{r},t) = 2\nu \nabla^2_{\mathbf{r}} Q_{kin}(r,t). \tag{3.63}$$

$Q_{kin}(r,t)$ and its corresponding current $\mathbf{J}^{kin}(\mathbf{r},t)$ can be expressed in terms of $C^{\mathbf{uu}}_{r\,r}(r,t)$, $C^{\mathbf{uuu}}_{r\,r\,r}(r,t)$ and $C^{\mathbf{hhu}}_{t\,t\,r}(r,t)$ as it is shown in the appendix B.1.2. In exchanging the derivatives after t and r, we arrive at

$$\begin{aligned}
\frac{\partial}{\partial t} Q_{kin}(r,t) &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^3 \frac{\partial}{\partial t} C^{\mathbf{uu}}_{r\,r}(r,t), \right), \\
2\frac{\partial}{\partial r_k} J^{kin}_k(\mathbf{r},t) &= -\frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\left[r^4 (C^{\mathbf{uuu}}_{r\,r\,r}(r,t) + 2C^{\mathbf{hhu}}_{t\,t\,r}(r,t)) \right] \right), \\
\nabla^2_{\mathbf{r}} Q_{kin}(r,t) &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r} Q_{kin}(r,t) \right).
\end{aligned} \tag{3.64}$$

[1]See for instance in [Mon71] Vol. II p. 65.

Inserting these relations in equation (3.56) yields the von Kármán-Howarth equation of MHD turbulence

$$\frac{\partial}{\partial t}C^{\mathbf{uu}}_{r\,r}(r,t) = \frac{1}{r^4}\frac{\partial}{\partial r}\left[r^4\left(C^{\mathbf{uuu}}_{r\,r\,r}(r,t) + 2C^{\mathbf{hhu}}_{t\,t\,r}(r,t) + 2\nu\frac{\partial}{\partial r}C^{\mathbf{uu}}_{r\,r}(r,t)\right)\right]. \tag{3.65}$$

This equation contains the longitudinal correlations $C^{\mathbf{uu}}_{r\,r}(r,t)$ and $C^{\mathbf{uuu}}_{r\,r\,r}(r,t)$ as well as the mixed correlation $C^{\mathbf{hhu}}_{t\,t\,r}(r,t)$. It indicates therefore how the kinetic turbulent energy is transferred: In addition to the energy transfer from large to small scales in fluid dynamics we have another process, ending up in stretch mechanism of the lines of force caused by the velocity field.

Let us now turn to the evolution equation of the two-point magnetic field correlation function (3.55). We have to deal with the antisymmetric tensor

$$C^{\mathbf{uhh}}_{k\,i,j}(\mathbf{r},t) = \langle (h_i u_k - h_k u_i)h'_j\rangle = C^{\mathbf{uhh}}(r,t)\left(\frac{r_i}{r}\delta_{kj} - \frac{r_k}{r}\delta_{ij}\right) \tag{3.66}$$

Its significance can best be seen in considering the configuration depicted in fig. 3.4.

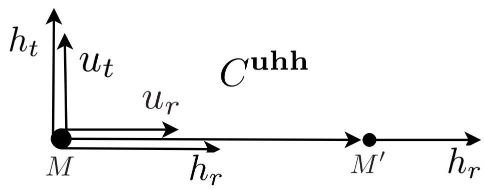


Figure 3.4: Antisymmetric correlation tensor occurring in the evolution equation of the two-point magnetic field correlation function.

Therefore, one obtains

$$\langle (h_1 u_2 - h_2 u_1)h'_2\rangle = C^{\mathbf{uhh}}(r,t)\frac{r_1}{r}. \tag{3.67}$$

From a Taylor expansion we find that

$$C^{\mathbf{uhh}}(r,t) = C^{\mathbf{uhh}}_0(r,t)r, \tag{3.68}$$

where $C^{\mathbf{uhh}}_0(r,t) = \langle (h_1 u_2 - h_2 u_1)\frac{\partial h_2}{\partial x_1}\rangle$. It can be shown under the assumption of isotropy [Cha51], that as $r \to 0$

$$\langle (h_1 u_2 - h_2 u_1)\frac{\partial h_2}{\partial x_1}\rangle = -\frac{5}{4}\langle h_1^2\frac{\partial u_1}{\partial x_1}\rangle, \tag{3.69}$$

or

$$C^{\mathbf{uhh}}(r,t) = 5C^{\mathbf{hhu}}_{r\,r\,r}(r,t). \tag{3.70}$$

This is an interesting result, since at small r we find a connection between the evolution equation of the longitudinal two-point velocity correlation function and the evolution equation of the longitudinal two-point magnetic function. Generally speaking, we expect an exchange between kinetic energy and magnetic energy at small values of r (thus large values of k).
In order to get a second balance equation, we take the sum over i,j of equation (3.66) and introduce the following quantities

$$Q_{mag}(r,t) = \sum_{i=j} C^{\mathbf{uu}}_{i\,j}(r,t), \tag{3.71}$$

$$J^{mag}_k(\mathbf{r},t) = -\sum_{i=j} C^{\mathbf{uhh}}_{k\,i,j}(\mathbf{r},t)\frac{r_k}{r}. \tag{3.72}$$

The balance equation for the magnetic correlation can be derived as

$$\frac{\partial}{\partial t}Q_{mag}(r,t) + 2\frac{\partial}{\partial r_k}J_k^{mag}(\mathbf{r},t) = \lambda\nabla_{\mathbf{r}}^2 Q_{mag}(r,t). \tag{3.73}$$

Analogous to (3.64) we insert the following relations and get

$$\frac{\partial}{\partial t}Q_{mag}(r,t) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^3\frac{\partial}{\partial t}C_{r\,r}^{\mathbf{hh}}(r,t)\right), \tag{3.74}$$

$$2\frac{\partial}{\partial r_k}J_k^{mag}(\mathbf{r},t) = -2\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 C^{\mathbf{uhh}}(r,t)\right), \tag{3.75}$$

$$\nabla_{\mathbf{r}}^2 Q_{mag}(r,t) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}Q_{mag}(r,t)\right). \tag{3.76}$$

After the calculation of the divergence we get an evolution equation for the longitudinal two-point magnetic field correlation function

$$\frac{\partial}{\partial t}C_{r\,r}^{\mathbf{hh}}(r,t) = \frac{4}{r}C^{\mathbf{uhh}}(r,t) + \frac{2\lambda}{r^4}\frac{\partial}{\partial r}\left(r^4\frac{\partial}{\partial r}C_{r\,r}^{\mathbf{hh}}(r,t)\right). \tag{3.77}$$

The third evolution equation is that for the correlation tensor of the cross helicity

$$\frac{\partial}{\partial t}\langle u_i h_j'\rangle - \frac{\partial}{\partial r_k}(\langle u_i u_k h_j'\rangle - \langle h_i h_k h_j'\rangle) + \frac{\partial}{\partial r_k}\langle u_i(h_j' u_k' - u_j' h_k')\rangle = -\frac{1}{\rho}\frac{\partial}{\partial x_i}\langle P h_j'\rangle + (\lambda+\nu)\nabla_{\mathbf{r}}^2\langle u_i h_j'\rangle \tag{3.78}$$

Since this equation involves only skew tensors, the mirror symmetry is broken. The pressure and the magnetic pressure correlations vanish on the basis of isotropy. The equation can be rewritten as

$$\frac{\partial}{\partial t}C_{i\,j}^{\mathbf{uh}}(\mathbf{r},t) - \frac{\partial}{\partial r_k}(C_{k\,i,j}^{\mathbf{uuh}}(\mathbf{r},t) - C_{k\,i,j}^{\mathbf{hhh}}(\mathbf{r},t) + C_{j\,k,i}^{\mathbf{uhu}}(\mathbf{r},t)) = 2(\nu+\lambda)\nabla_{\mathbf{r}}^2 C_{i\,j}^{\mathbf{hh}}(\mathbf{r},t). \tag{3.79}$$

The tensor of the cross helicity correlation has the form

$$C_{i\,j}^{\mathbf{uh}}(\mathbf{r},t) = C^{\mathbf{uh}}(r,t)\epsilon_{ijl}\frac{r_l}{r}. \tag{3.80}$$

The first two triple tensors on the left-hand side are skew, symmetric in the indices i and j and solenoidal in k, whereas the third one is skew, antisymmetric in j and k and solenoidal in i, so that its defining scalar is restricted by an incompressibility condition that is evaluated in the appendix B.1.2. One obtains

$$C_{k\,i,j}^{\mathbf{uuh}}(\mathbf{r},t) = \langle u_i u_k h_j'\rangle = C^{\mathbf{uuh}}(r,t)\left(\frac{r_k}{r}\epsilon_{ijl}\frac{r_l}{r} + \frac{r_i}{r}\epsilon_{kjl}\frac{r_l}{r}\right). \tag{3.81}$$

$$C_{k\,i,j}^{\mathbf{hhh}}(\mathbf{r},t) = \langle h_i h_k h_j'\rangle = C^{\mathbf{hhh}}(r,t)\left(\frac{r_k}{r}\epsilon_{ijl}\frac{r_l}{r} + \frac{r_i}{r}\epsilon_{kjl}\frac{r_l}{r}\right). \tag{3.82}$$

$$C_{j\,k,i}^{\mathbf{uhu}}(\mathbf{r},t) = \langle u_i(h_j' u_k' - u_j' h_k')\rangle = 2C^{\mathbf{uhu}}(r,t)\epsilon_{ijk} + r\left(\frac{\partial}{\partial r}C^{\mathbf{uhu}}(r,t)\right)\left(\frac{r_j}{r}\epsilon_{kil}\frac{r_l}{r} - \frac{r_k}{r}\epsilon_{jil}\frac{r_l}{r}\right). \tag{3.83}$$

By inserting these tensors into (3.79), we get an evolution equation for the defining scalar of the cross helicity correlation function

$$\frac{\partial}{\partial t}C^{\mathbf{uh}}(r,t) = \frac{1}{r^3}\frac{\partial}{\partial r}\left[r^3(C^{\mathbf{uuh}}(r,t) - C^{\mathbf{hhh}}(r,t)) - r^4\frac{\partial}{\partial r}\left(C^{\mathbf{uhu}}(r,t) - 2(\nu+\lambda)\frac{C^{\mathbf{uh}}(r,t)}{r}\right)\right]. \tag{3.84}$$

3.5 Kolmogorov's theory of locally isotropic homogeneous turbulence

As we will see in chapter 4, it is possible to derive an exact relation for the third order longitudinal structure function $D_{rrr}(\mathbf{r}) = \langle |\mathbf{u}_r(\mathbf{x}+\mathbf{r}) - \mathbf{u}_r(\mathbf{x})|^3\rangle = \langle v_r(\mathbf{x}, \mathbf{x}+\mathbf{r})^3\rangle$ in hydrodynamic turbulence directly from the Navier-Stokes equation. Thereby, $\mathbf{u}_r(\mathbf{x})$ is the part of the velocity $\mathbf{u}(\mathbf{x})$ that points in the direction of the relative distance $\mathbf{r}$ between the velocities $\mathbf{u}(\mathbf{x})$ and $\mathbf{u}(\mathbf{x}+\mathbf{r})$. The corresponding tensor calculus for the structure functions is discussed in the appendix under B.2. The important relation was derived by Kolmogorov in 1941 and is known as *Kolmogorov's 4/5 law*:

In the limit of infinite Reynolds number, the third order longitudinal structure function of homogeneous isotropic turbulence evaluated for increments r small compared to the integral scale L is given in terms of the mean energy dissipation per unit mass $\langle\varepsilon\rangle$ by

$$D_{rrr}(r) = -\frac{4}{5}\langle\varepsilon\rangle r. \tag{3.85}$$

In the following we want to discuss the significance of this scaling behavior of the third order structure function, for structure functions of order n. First-of-all, equation (3.85) signifies that in turbulence one has to deal with a non-Gaussian statistic, since all odd order moments should vanish for these kind of statistic. The extrapolation of this exact result is of great importance especially for the second order longitudinal structure function $D_{rr}(r)$, since it can be seen as a measure of the kinetic energy of an eddy of size r.

Let us consider the probability distribution $p(v_r, r)$ of the longitudinal increment v_r. According to Kolmogorov's first similarity hypothesis for $r \ll L$ this can only be a function

$$p(v_r, r) = F(v_r, r, \langle\epsilon\rangle, \nu). \tag{3.86}$$

By introducing the Kolmogorov length and time scales, according to (3.31), we arrive at dimensionless unities

$$\tilde{v}_r = \frac{v_r}{u_\eta} = \frac{v_r}{(\nu\langle\varepsilon\rangle)^{1/4}} \qquad \text{and} \quad \tilde{r} = \frac{r}{\eta} = \frac{r}{(\nu^3/\langle\epsilon\rangle)^{1/4}}. \tag{3.87}$$

and we can write a probability distribution $P(\tilde{v}_r, \tilde{r})$ for these unities, which is related to the old one by

$$p(v_r, r) = P(\tilde{v}_r, \tilde{r})\frac{\mathrm{d}\tilde{v}_r}{\mathrm{d}v_r} = \frac{1}{u_\eta}P\left(\frac{v_r}{u_\eta}, \frac{r}{\eta}\right). \tag{3.88}$$

In the inertial range, $p(v_r, r)$ should be independent from ν, so that

$$\frac{\partial}{\partial\nu}p(v_r, r) = \frac{\partial}{\partial\nu}\left[\frac{1}{u_\eta}P\left(\frac{v_r}{u_\eta}, \frac{r}{\eta}\right)\right] = 0. \tag{3.89}$$

By making use of the chain rule we arrive at a differential equation for $P(\tilde{v}_r, \tilde{r})$, namely

$$\frac{\partial}{\partial\tilde{v}_r}\left[\tilde{v}_r P(\tilde{v}_r, \tilde{r})\right] + 3\tilde{r}\frac{\partial}{\partial\tilde{r}}P(\tilde{v}_r, \tilde{r}) = 0, \tag{3.90}$$

which can be solved by

$$P(\tilde{v}_r, \tilde{r}) = \frac{1}{\tilde{r}^{1/3}}p\left(\frac{\tilde{v}_r}{\tilde{r}^{1/3}}\right). \tag{3.91}$$

This is what is called a self-similar distribution and the corresponding self-similar probability distribution in the old length and time scales reads

$$p(v_r, r) = \frac{1}{(\langle\varepsilon\rangle r)^\zeta}p\left(\frac{\tilde{v}_r}{(\langle\varepsilon\rangle r)^\zeta}\right). \tag{3.92}$$

where ζ is a constant that remains to be determined.
In order to see the consequences for the n-th order moments of the longitudinal velocity increments we have to evaluate the following integral

$$\langle v_r(r)^n\rangle = \int \mathrm{d}v v^n p(v,r) = \frac{1}{(\langle\varepsilon\rangle r)^\zeta}\int \mathrm{d}v v^n p\left(\frac{\tilde{v}}{(\langle\varepsilon\rangle r)^\zeta}\right) = C_n(\langle\varepsilon\rangle r)^{n\zeta}, \tag{3.93}$$

where the coefficients are given by $C_n = \int \mathrm{d}\omega\omega^n p(\omega)$.
Since equation (3.93) also holds for $n = 3$, the exponents ζ are determined by Kolmogorov's 4/5 law. The hypothesis of self-similarity in addition to an exact relation which follows directly from the Navier-Stokes equation, therefore determines the scaling behavior of every longitudinal structure function according to

$$\langle v_r(r)^n\rangle = C_n(\langle\varepsilon\rangle r)^{\zeta_n}, \tag{3.94}$$

with the exponent $\zeta_n = \frac{n}{3}$.

3.5.1 The energy cascade in 3D hydrodynamic turbulence

As it was mentioned above, the extrapolation of (3.95) for the second order longitudinal structure function

$$D_{rr}(r) = C_2(\langle\varepsilon\rangle r)^{\frac{2}{3}}, \tag{3.95}$$

is of great importance for the kinetic energy of an eddy of size r. This scaling behavior implies an energy spectrum of the form

$$E(k) = C_K\langle\varepsilon\rangle^{\frac{2}{3}}k^{-\frac{5}{3}}, \tag{3.96}$$

for the inertial range. The energy spectrum is thereby defined as the Fourier transform of the auto-correlation function

$$E(k,t) = \frac{1}{2}\sum_{i=j}\langle\hat{u}_i(\mathbf{k},t)\hat{u}_j(\mathbf{k},t)\rangle, \tag{3.97}$$

where

$$\hat{\mathbf{u}}(\mathbf{k},t) = \frac{1}{2\pi^3}\int \mathrm{d}\mathbf{x}\ \mathbf{u}(\mathbf{x},t)e^{i\mathbf{k}\cdot\mathbf{r}}, \tag{3.98}$$

is the Fourier transform of the velocity field $\mathbf{u}(\mathbf{x},t)$.
This so-called *5/3 energy spectrum* was also reproduced by Heisenberg's heuristic formulation of statistically isotropic homogeneous hydrodynamic turbulence and can be observed in several experiments. A typical energy spectrum is depicted in fig. 5.7: The K41 phenomenology suggests a cascade process where energy is injected at large scales (thus small values of k).

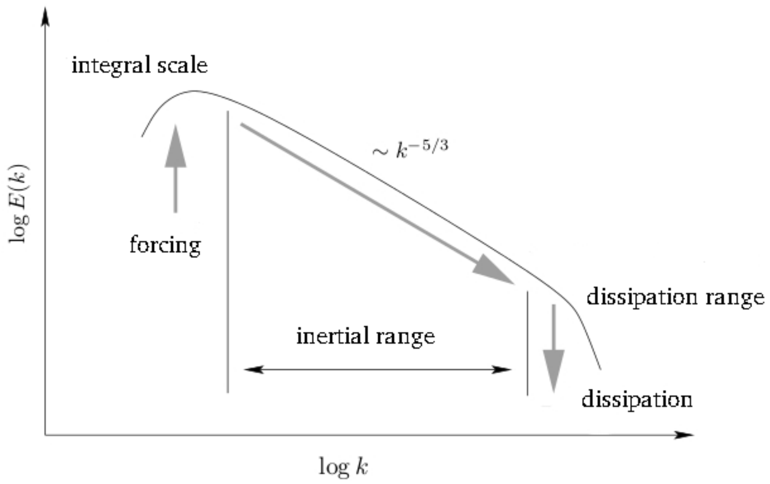


Figure 3.5: Schematic energy spectrum of 3D turbulence: The energy is injected at the integral scale L and cascades down towards small scales at the rate $\langle\varepsilon\rangle$ until it is removed by dissipation.

This picture changes dramatically if one considers two-dimensional flows: As it was mentioned in the section about the vorticity equation 2.1.3, the vorticity tends to organize itself into bigger clusters. Therefore, the energy transfer does not proceed from larger to smaller scales anymore, but propagates inversely from smaller to bigger scales. The cascade is therefore termed an inverse cascade.

3.5.2 The phenomenon of intermittency

In the preceding section, the significance of the assumption of self-similarity represented by equation (3.91), was underlined. In experiments with hydrodynamical flows however, one observes deviations from the self-similarity in small scale turbulence [Ren01].

Whereas at larger scales, the probability density functions $p(v, r)$ follow a normal distribution, the smaller scales distinguish themselves by a more frequent occurrence of small and big velocity increments v, like it is depicted in fig. 3.6. This phenomenon is called intermittency, and there exist several multi-fractal models that take the breaking of the self-similarity into account. Thereby, the scaling exponents ζ_n in equation (3.95) become nonlinear functions of n and an increasing deviation from the K41 prediction of $\zeta_n = \frac{n}{3}$ with increasing n agrees better with experimental and numerical data.

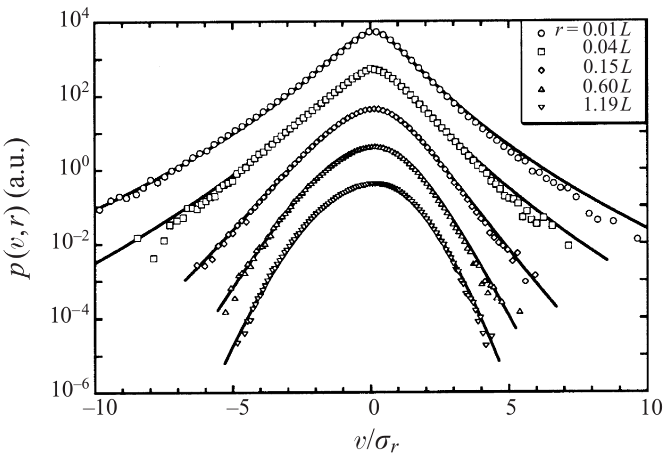


Figure 3.6: Probability density functions of the velocity increments v from experimental data [Ren01]. Deviations from Gaussianity is attained at small scales r given in unities of the integral length scale L.

3.6 The energy cascade in MHD turbulence

We have seen from the von Kármán-Howarth equation of MHD turbulence (3.65), and its connection to the equation of the two-point magnetic field correlation function (3.77), that we have to deal with the possibility of energy cascades in the magnetic energy in addition to the usual energy cascade in hydrodynamic turbulence.
It seems therefore more complicated to deduce a phenomenological approach similar to that mentioned for the hydrodynamic case and to identify mechanisms responsible for the energy transfer. Nevertheless, there exist two different theories for the energy cascade in MHD turbulence, the Iroshnikov-Kraichnan energy cascade [Iro64], [Kra65], and the Goldreich-Sridhar energy cascade [Gol94]:

i.) The Iroshnikov-Kraichnan energy cascade:

This theory takes into account a modification of the inertial range scaling based on the Alfvén effect. Its main assumption is that the large-scale magnetic field makes the small-scale fluctuations to behave approximately as Alfvén waves. Therefore, Alfvén waves which move in opposite direction along the lines of force, interact with each other and their correlation time is given by $\tau_A = r/v_A$, where r is the typical size of a wave packet and v_A is the Alfvén velocity from 2.3. The energy transfer time is thus increased by a factor τ_r/τ_A, where τ_r is the energy transfer time from ordinary turbulence, and we get

$$\tau = \frac{\tau_r^2}{\tau_A}. \tag{3.99}$$

Let us denote δv_r and δb_r as the velocity and magnetic field fluctuations in an 'eddy'. Since the Alfvén waves are an equipartition solution, we get $\delta v_r \sim \delta b_r$ and this yields

$$\delta v_r^4 \tau / r^2 \sim \epsilon \tag{3.100}$$

The requirement of constant energy flux over scales leads immediately to

$$\delta v_r \sim (\epsilon v_a)^{1/4} r^{1/4}. \tag{3.101}$$

It is important to stress that a wave packet has to experience many uncorrelated interactions with oppositely moving wave packets and that the large scale magnetic field is not necessarily a mean field, which is excluded by the assumption of isotropy, but is rather a macro-state which has a non-negligible influence on the small scale fluctuations. The scaling behavior of equation (3.101) corresponds to an energy spectrum

$$E(k) = C_{IK} (\epsilon v_a)^{1/2} k^{-3/2}, \tag{3.102}$$

where C_{IK} is expected to differ from the Kolmogorov constant C_K. This spectrum is less steep than the Kolmogorov spectrum, but the difference between the two is rather small, which makes it difficult to distinguish them in numerical simulations and experiments. An interesting fact of this energy spectrum is that it formally coincides with the energy spectrum derived by Kraichnan in his direct-interaction approximation (DIA) for hydrodynamic turbulence [Kra59]. This theory does not properly distinguish between the advection and the stretching of eddies and small-scale fluctuations are therefore not independent from the large-scale velocity field, which means that the localness of interactions assumed in K41 is not fulfilled anymore. isotropic turbulence the large-scale velocity field can be removed by a Belinicher-L'vov transformation to a co-moving reference frame and therefore the small scales should not be influenced by a large-scale field. In a later chapter we will explicitly see that a nonlocal interaction between the large-scale magnetic field and the small-scale fluctuations can be introduced by the cross helicity.

ii.) The Goldreich-Sridhar energy cascade

Up to know, we have only considered isotropic MHD turbulence. A theory for a strong external field, however seems to contradict the assumption of isotropy, since we now have to deal with a distinguished direction. Therefore, Goldreich and Sridhar assumed that 'eddies' are strongly anisotropic and elongated in the direction of the large-scale magnetic field. Their central statement is the so-called critical balance condition which relates the field-parallel size $r_\parallel$ and the field-perpendicular $r_\perp$ size of the 'eddy'. The turbulent transfer to small scales in the field-perpendicular direction is not affected by the magnetic field and therefore $r_\perp$ decreases, which will elongate the eddy. The observed structures are then current sheets. This stretching process will however be limited by a propagation along the lines of force with the Alfvén velocity v_A. We therefore obtain the critical balance condition

$$\frac{\delta v_{r_\perp}}{r_\perp} \sim \frac{v_A}{r_\parallel}. \tag{3.103}$$

This can also be interpreted as a formal balance of the two advective terms in the MHD equations in terms of the Elsässer fields with an external field $\mathbf{H}$

$$\frac{\partial}{\partial t}\mathbf{z}^\pm + \mathbf{z}^\mp \cdot \nabla \mathbf{z}^\pm \mp \mathbf{v}_A \cdot \nabla_\parallel \mathbf{z}^\pm = -\frac{1}{\rho}\nabla\left(p + \frac{1}{2}\rho h^2\right), \tag{3.104}$$

where

$$\mathbf{v}_A = \left(\frac{\mu}{4\pi}\right)^{\frac{1}{2}} \mathbf{H}. \tag{3.105}$$

Under the assumption of constant energy flux, which takes mainly place in the $k_\perp$ direction we arrive at

$$E(k_\perp) \sim \epsilon^{2/3} k_\perp^{-5/3}. \tag{3.106}$$

The parallel spectrum can be obtained by making use of the critical balance condition, and we get

$$E(k_\parallel) \sim \epsilon^{3/2} k_\parallel^{-5/2}. \tag{3.107}$$

These two phenomenological theories and their applications are highly controversial to the present day. A model that takes into account both theories is given by Boldyrev [Bol06], and reproduces both spectra in their corresponding regimes, in focusing on the alignment characteristics of the MHD flow.

3.7 Dynamo theory

The process of an amplification or the maintenance of a magnetic field by the conversion from kinetic to magnetic energy is generally referred to as a *dynamo*. Since the theory of dynamos is crucial for the description of the persistent presence of magnetic fields in the universe, there exists a variety of contributions to this special field of magnetohydrodynamics. For an excellent overview of the topic, the reader is referred to [Bra05]. In the following we want to focus on the generation of a magnetic field due to the turbulent motion of the fluid. To this end we consider the turbulent motion as given and focus solely on the induction equation (2.8) neglecting any back-reaction of the magnetic field on the fluid by the Lorentz force in (2.11). This kind of dynamo is also called a *kinematic dynamo* and the turbulent motion has to lack mirror symmetry, since symmetric fields can not maintain an axisymmetric field, as it is shown in the following section.

3.7.1 Symmetric fields and dynamos

It was shown by Cowling [Cow76] that a magnetic field, which is symmetric about an axis, cannot be maintained by a symmetric motion. For a proof of this so-called anti-dynamo theorem, we divide the axisymmetric fields $\mathbf{u}(\mathbf{x}t)$ and $\mathbf{H}(\mathbf{x},t)$ into poloidal $\mathbf{u}_p(\mathbf{x},t)$, $\mathbf{H}_p(\mathbf{x},t)$ and toroidal or azimuthal parts $u_\varphi(\mathbf{x},t)\mathbf{e}_\varphi$, $H_\varphi(\mathbf{x},t)\mathbf{e}_\varphi$, according to fig. 3.7 a). The corresponding poloidal field $\mathbf{H}_\varphi(\mathbf{x},t)$ in a meridian plane is depicted in fig. 3.7 b): The lines of force are closed curves and they have two neutral points O and O', where the poloidal field vanishes. In order to maintain the magnetic field at these neutral points, an azimuthal current is required. Therefore, we need an azimuthal electric field component $E_\varphi(\mathbf{x}=\vec{O},t)$, which can not be guaranteed by $\mathbf{u}(\mathbf{x}=\vec{O},t)\times\mathbf{H}(\mathbf{x}=\vec{O},t)$, because the magnetic field reduces to $H_\varphi(\mathbf{x}=\vec{O},t)\mathbf{e}_\varphi$ at this point.

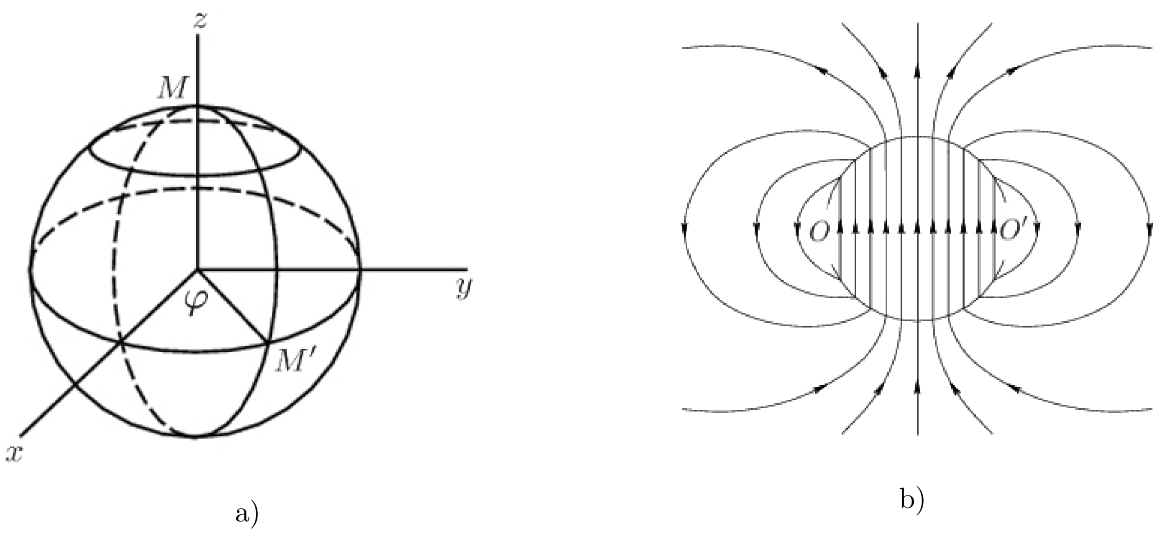


Figure 3.7: a) Spatial division of the fields $\mathbf{u}(\mathbf{x},t)$ and $\mathbf{H}(\mathbf{x},t)$ into poloidal and toroidal (azimuthal) components in a sphere. The circle MM' is called the meridian circle. b) Lines of force of the poloidal field $\mathbf{H}_p(\mathbf{x},t)$ in a meridian plane through the axis of symmetry. The points O and O' are neutral points.

3.7.2 Mean-field electrodynamics

As we have seen in the section about the Reynolds averaged MHD equations 3.3, the division of the magnetic and velocity field into mean and fluctuating parts according to (3.11) yields an induction equation for the mean magnetic field $\bar{\mathbf{h}}(\mathbf{x},t)$, namely (3.17). This induction equation that contains a supplementary mean electric force $\boldsymbol{\epsilon}=\langle\mathbf{u}'\times\mathbf{h}'\rangle$ resulting from the interaction of the turbulent motion and field, is of great interest in the realm of kinematic dynamo theory. A systematic approach to such a theory was first given by Steenbeck, Krause and Rädler in 1966 [Ste66], and is termed as mean-field electrodynamics. Since these works were only published in German, their significance was only recognized by few people like Parker [Par70] and Cowling [Cow81]. For a review of mean-field electrodynamics, the reader is referred to [Rae07].

In mean-field electrodynamics one is purely concerned to see if a given type of turbulent motion $\mathbf{u}(\mathbf{x},t)$ can generate and maintain a mean magnetic field $\bar{\mathbf{h}}(\mathbf{x},t)$. Thereby one assumes that the correlation time τ and scale length Λ of the velocity $\mathbf{u}(\mathbf{x},t)$ small compared to the scale time t_0 and scale length l_0 of the variation of the mean fields $\bar{\mathbf{u}}(\mathbf{x},t)$ and $\bar{\mathbf{h}}(\mathbf{x},t)$. The meaning of τ is thereby the time period in which the turbulent motion $\mathbf{u}'(\mathbf{x},t)$ is correlated with its initial value and Λ can be regarded as a scale length comparable to the mean eddy size.

In order to determine the mean electric field $\boldsymbol{\epsilon}(\mathbf{x},t)$ occurring in (3.17) we have to derive an evolution equation for the fluctuating part of the magnetic field $\mathbf{h}'(\mathbf{x},t)$. To this end we divide the evolution equation of $\mathbf{h}(\mathbf{x},t) = \bar{\mathbf{h}}(\mathbf{x},t) + \mathbf{h}'(\mathbf{x},t)$ into mean and fluctuating parts

$$\frac{\partial}{\partial t}\bar{\mathbf{h}}(\mathbf{x},t) = \nabla \times (\boldsymbol{\epsilon}(\mathbf{x},t) + \bar{\mathbf{u}}(\mathbf{x},t) \times \bar{\mathbf{h}}(\mathbf{x},t)) + \lambda \nabla^2 \bar{\mathbf{h}}(\mathbf{x},t), \tag{3.108}$$

$$\frac{\partial}{\partial t}\mathbf{h}'(\mathbf{x},t) = \nabla \times (\bar{\mathbf{u}}(\mathbf{x},t) \times \mathbf{h}'(\mathbf{x},t) + \mathbf{u}'(\mathbf{x},t) \times \bar{\mathbf{h}}(\mathbf{x},t) + \mathbf{g}(\mathbf{x},t)) + \lambda \nabla^2 \mathbf{h}'(\mathbf{x},t). \tag{3.109}$$

where $\mathbf{g}(\mathbf{x},t) = \mathbf{u}'(\mathbf{x},t) \times \mathbf{h}'(\mathbf{x},t) - \boldsymbol{\epsilon}(\mathbf{x},t)$. Obviously, the first equation is identically to (3.17) from the the section of the Reynolds averaged MHD equations.
In the following treatment of (3.109) we neglect $\bar{\mathbf{u}}(\mathbf{x},t)$ and restrict ourselves to an electric force $\boldsymbol{\epsilon}(\mathbf{x},t)$ generated purely by the turbulent motion. Furthermore we neglect $\mathbf{g}(\mathbf{x},t)$, which is known as the "first-order smoothing" approximation. This approximation is only valid if

1. Rm$= \frac{\sqrt{\langle \mathbf{u}'^2 \rangle}\Lambda}{\lambda} \ll 1$. This is the high-resistance case where $\nabla \times (\bar{\mathbf{u}}(\mathbf{x},t) \times \mathbf{h}'(\mathbf{x},t))$ is balanced by the resistive term.
2. $\frac{\partial}{\partial t}\mathbf{h}'(\mathbf{x},t)$ is balanced by $\nabla \times (\bar{\mathbf{u}}(\mathbf{x},t) \times \mathbf{h}'(\mathbf{x},t))$, which is the case if the fluctuations occur on a small time scale $\tau \ll \frac{\lambda}{\sqrt{\langle \mathbf{u}'^2 \rangle}}$.

The evolution equation for the turbulent magnetic field thus reads

$$\frac{\partial}{\partial t}\mathbf{h}'(\mathbf{x},t) - \lambda \nabla^2 \mathbf{h}'(\mathbf{x},t) = \nabla \times (\mathbf{u}'(\mathbf{x},t) \times \bar{\mathbf{h}}(\mathbf{x},t)). \tag{3.110}$$

Since we are only interested in the part of $\mathbf{h}''(\mathbf{x},t)$ of $\mathbf{h}'(\mathbf{x},t)$ that is correlated with $\mathbf{u}'(\mathbf{x},t)$, we integrate (3.110) from time $t-\tau$ to t with the initial condition $\mathbf{h}'(\mathbf{x},t-\tau) = 0$ and obtain

$$\mathbf{h}''(\mathbf{x},t) = \int_{t-\tau}^{t} \mathrm{d}t' \; \nabla \times (\mathbf{u}'(\mathbf{x},t') \times \bar{\mathbf{h}}(\mathbf{x},t')), \tag{3.111}$$

where resistive effects have been neglected.
Since τ is at most of the order of an eddy turnover time $\frac{\lambda}{\sqrt{\langle \mathbf{u}'^2 \rangle}}$, it is reasonable to consider the integrand in (3.111) as independent from t, which yields

$$\mathbf{h}''(\mathbf{x},t) = \tau \nabla \times (\mathbf{u}'(\mathbf{x}) \times \bar{\mathbf{h}}(\mathbf{x})). \tag{3.112}$$

In taking the vector product with $\mathbf{u}'(\mathbf{x},t)$ from the left and averaging, we obtain

$$\langle \mathbf{u}' \times \mathbf{h}'' \rangle = \langle \mathbf{u}' \times \mathbf{h}' \rangle = \tau \langle \mathbf{u}' \times [\nabla \times (\mathbf{u}' \times \bar{\mathbf{h}})] \rangle. \tag{3.113}$$

Therefore the i-th component of $\boldsymbol{\epsilon}$ reads

$$\begin{aligned}
\epsilon_i &= \tau \varepsilon_{ijk} \varepsilon_{kmn} \varepsilon_{nop} \langle u'_j \frac{\partial}{\partial x_m} u'_o \bar{h}_p \rangle \qquad (3.114) \\
&= \tau \varepsilon_{ijk} (\delta_{ko}\delta_{pm} - \delta_{kp}\delta_{om}) \left(\bar{h}_p \langle u'_j \frac{\partial}{\partial x_m} u'_o \rangle + \langle u'_j u'_o \rangle \frac{\partial}{\partial x_m} \bar{h}_p \right) \\
&= \tau \varepsilon_{ijk} \left(\bar{h}_m \langle u'_j \frac{\partial}{\partial x_m} u'_k \rangle - \langle u'_j u'_m \rangle \frac{\partial}{\partial x_m} \bar{h}_k \right),
\end{aligned}$$

where the incompressibility condition was used in the last step.
Under the assumption of isotropy, the following tensors can be inserted in (3.114)

$$\langle u'_j \frac{\partial}{\partial x_m} u'_k \rangle = \frac{1}{6} \langle \mathbf{u}' \cdot \nabla \times \mathbf{u}' \rangle \varepsilon_{jmk}, \tag{3.115}$$

$$\langle u'_j u'_m \rangle = \frac{1}{3} \langle \mathbf{u}' \cdot \mathbf{u}' \rangle \delta_{jm} \tag{3.116}$$

The mean electric field can thus be written as

$$\boldsymbol{\epsilon}(\mathbf{x},t) = \alpha(\mathbf{x},\tau)\bar{\mathbf{h}}(\mathbf{x},t) + \beta(\mathbf{x},\tau)\nabla\times\bar{\mathbf{h}}(\mathbf{x},t), \tag{3.117}$$

where

$$\alpha(\mathbf{x},\tau) = -\frac{\tau}{3}\langle\mathbf{u}'\cdot(\nabla\times\mathbf{u}')\rangle \qquad \text{and} \qquad \beta(\mathbf{x},\tau) = \frac{\tau}{3}\langle u'^2\rangle. \tag{3.118}$$

The quantity $\mathbf{u}'(\mathbf{x},t)\cdot(\nabla\times\mathbf{u}'(\mathbf{x},t))$ is the helicity of the flow which is conserved in inviscid fluid dynamics. If the mean helicity does not vanish, the vorticity $\boldsymbol{\omega}'(\mathbf{x},t) = \nabla\times\mathbf{u}'(\mathbf{x},t)$ precesses about the direction of the vorticity. Furthermore, non-zero values of the helicity are connected with the lack of mirror symmetry in the turbulent motion as discussed in section 3.1. The influence of the turbulent motion on the mean field $\bar{\mathbf{h}}(\mathbf{x},t)$ can be seen in inserting (3.117) into (3.108). This yields the mean-field dynamo equation

$$\frac{\partial}{\partial t}\bar{\mathbf{h}}(\mathbf{x},t) = \nabla\times(\alpha(\mathbf{x},\tau)\bar{\mathbf{h}}(\mathbf{x},t) + \bar{\mathbf{u}}(\mathbf{x},t)\times\bar{\mathbf{h}}(\mathbf{x},t)) - \nabla\times(\lambda+\beta(\mathbf{x},\tau))\nabla\times\bar{\mathbf{h}}(\mathbf{x},t). \tag{3.119}$$

The quantity $\beta(\mathbf{x},\tau)$ is therefore an eddy diffusivity that is similar to the Ohmic diffusivity λ. The term $\alpha(\mathbf{x},\tau)\bar{\mathbf{h}}(\mathbf{x},t)$, which consists of an electric field parallel to $\bar{\mathbf{h}}(\mathbf{x},t)$, is of great importance for the dynamo maintenance. Since it was shown in the previous section that an axisymmetric magnetic field cannot be maintained by an axisymmetric motion the so-called α-effect guarantees an electric field $\alpha(\mathbf{x},\tau)h_\varphi(r,\theta,t)\mathbf{e}_\varphi$ in azimuthal direction and therefore allows the mean magnetic field to escape from the anti-dynamo theorem.

3.7.3 $\alpha\omega$ dynamos

In order to discuss axisymmetric mean field dynamos of a fluid in a rotating sphere, we introduce cylindrical coordinates (r,φ,z). The axis of rotation is assumed to point in the $\mathbf{e}_z$-direction and the sphere is assumed to rotate with the (in general nonuniform) angular velocity $\omega(r,t)$. The azimuthal component of the velocity is therefore given as $u_\varphi = \omega(r,t)r$ and the poloidal magnetic field

$$\mathbf{h}_p(r,z,t) = \nabla\times(a_\varphi(r,z,t)\mathbf{e}_\varphi) = \left(-\frac{\partial a_\varphi(r,z,t)}{\partial z},\ 0, \frac{1}{r}\frac{\partial(ra_\varphi(r,z,t))}{\partial r}\right), \tag{3.120}$$

is determined by the toroidal vector potential $a_\varphi(r,z,t)\mathbf{e}_\varphi$. Inserting the toroidal and poloidal fields into the mean-field dynamo equation yields (3.119)

$$\begin{aligned}&\left[\frac{\partial}{\partial t} - \lambda_t\left(\nabla^2 - \frac{1}{r^2}\right)\right]h_\varphi(r,z,t) + h_\varphi(r,z,t)(\nabla\cdot\mathbf{u}_p(r,z,t)) + r(\mathbf{u}_p(r,z,t)\cdot\nabla)\frac{h_\varphi(r,z,t)}{r}\\ =\ & r(\mathbf{h}_p(r,z,t)\cdot\nabla)\omega(r,t) + [\nabla\times(\alpha(r,z,\tau)\mathbf{h}_p(r,z,t))]\cdot\mathbf{e}_\varphi,\end{aligned} \tag{3.121}$$

for the azimuthal component of $\mathbf{h}(r,z,t)$, where $\lambda_t = \lambda+\beta$ is the total diffusivity, which is assumed to be uniform.
The evolution equation for the poloidal part of the magnetic field $\mathbf{h}_p(r,z,t) = \nabla\times(a_\varphi(r,z,t)\mathbf{e}_\varphi)$ can be integrated in order to obtain an evolution equation for the azimuthal vector potential

$$\left[\frac{\partial}{\partial t} - \lambda_t\left(\nabla^2 - \frac{1}{r^2}\right)\right]a_\varphi(r,z,t) + \frac{1}{r}\mathbf{u}_p(r,z,t)\cdot\nabla(ra_\varphi) = \alpha(r,z,\tau)h_\varphi(r,z,t). \tag{3.122}$$

The first two terms on the right hand side of (3.121) can be considered as source terms that generate toroidal field from the poloidal field $\mathbf{h}_p(r,z,t)$. In most cosmical problems the first term that is due to nonuniform rotation is large compared to the α term. These kinds of dynamos are known as $\alpha\omega$ dynamos, whereas the neglect of the first term in (3.121) leads to α^2 dynamos.

The production of toroidal field $h_\varphi(r,z,t)\mathbf{e}_\varphi$ in an $\alpha\omega$ dynamo is depicted in fig. 3.8. Due to the nonuniform rotation, the outer equatorial regions rotate more slowly than the inner regions. According to Elsässer [Els46-50], the nonuniform rotation shears the magnetic field near the center and the arms of the lines of force are drawn out into the azimuthal direction where the regions farther from the axis rotate more slowly $\frac{\partial\omega(r,t)}{\partial r} < 0$.

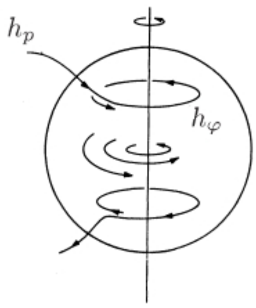


Figure 3.8: Generation of toroidal field $h_\varphi(r,z,t)\mathbf{e}_\varphi$ from poloidal field $\mathbf{h}_p(r,z,t)$ due to the nonuniform rotation of the sphere

The source term on the right hand side of (3.122) is the reaction of the toroidal field onto the poloidal field. The existence of a non-vanishing $\alpha(r,z,\tau)$ is therefore necessary to maintain the dynamo action. In neglecting the poloidal velocity component in (3.121) and (3.122), we arrive at the evolution equations for an $\alpha\omega$ dynamo in the form

$$\left[\frac{\partial}{\partial t} - \lambda_t\left(\nabla^2 - \frac{1}{r^2}\right)\right] h_\varphi(r,z,t) = r(\nabla\omega(r,t)\times\nabla a_\varphi(r,z,t)), \tag{3.123}$$

$$\left[\frac{\partial}{\partial t} - \lambda_t\left(\nabla^2 - \frac{1}{r^2}\right)\right] a_\varphi(r,z,t) = \alpha(r,z,\tau)h_\varphi(r,z,t). \tag{3.124}$$

These two equations determine the maintenance of dynamo action on nonuniform rotating spheres. Nonuniform rotation can be the result of Coriolis forces acting on convection cells, for instance in the earths core. However, one should not forget that the parameter $\alpha(r,z,\tau)$ retains the velocity field fluctuations and therefore all the uncertainties of a dynamo process. Therefore the generation of the helicity for the α-effect has been discussed by several authors [Bra05]. Parker explained this phenomenon by Coriolis forces that are acting on convection cells, like they occur in the Sun's convection zone. In a more general interpretation, a lack of symmetry between rising and descending motion is needed to create a non-vanishing α. An approximate expression for $\alpha(r,z,\tau)$ is given by Cowling [Cow81] as

$$\alpha(r,z,\tau) \sim \tau^2\langle u'^2\rangle\omega(r)\cos\theta\frac{\partial}{\partial r}\ln(\rho(r,z)\sqrt{\langle u'^2\rangle}), \tag{3.125}$$

where $\rho(r,z)$ is a changing density, which can change with height z as it is the case in most of the Sun's convection zone.
Although the properties of the factor $\alpha(r,z,\tau)$ are rather unknown, $\alpha\omega$ dynamos were successfully applied to the theory of the solar cycle, known as the Babcock-Leighton theory [Bab61], [Lei64].

Chapter 4

Hierarchy of structure function equations for locally isotropic MHD turbulence

In this chapter, Kolmogorov's theory of locally isotropic turbulence is generalized to the case of MHD turbulence. To this end, the tensor calculus introduced by Chandrasekhar in [Cha51] is used and a hierarchy of structure function relations is derived. The equation of energy balance of the velocity and magnetic increments is used to generalize the 4/5 law in the presence of a magnetic field. A second evolution equation for the cross helicity of the increments is obtained and discussed for the case of a homogeneous large-scale magnetic field that affects the small-scale fluctuations in a nonlocal manner.
The next order equation relates the third- and fourth-order velocity structure function and is the first order which provides a direct dependence of the longitudinal and the transversal structure function of fourth order based on the dynamics and not on the incompressibility condition. This procedure is a generalization of Hill's treatment [Hil01b] of structure function relations in hydrodynamic turbulence to MHD turbulence. Equations identical to that of Hill were derived by Yakhot [Yak02] using a generating function introduced by Polyakov [Pol95]. In this order one has to deal for the first time with correlations containing the pressure-gradient, which vanishes in the order below by the homogeneity assumption. It is therefore necessary to model the pressure-gradient and an interesting way to do this consist in a projection of the pressure-gradient on the quadratic longitudinal velocity increments based on the Bernoulli equation [Got02].
Furthermore, a rescaling relation between longitudinal and transversal structure function introduced by [Gra12] for hydrodynamic turbulence is discussed and used for an approximation of the relations between longitudinal and transverse structure functions in MHD turbulence.

4.1 Evolution equation for the magnetic and velocity increments

We consider the MHD equations in terms of the quantity

$$\mathbf{h} = \left(\frac{\mu}{4\pi\rho}\right)^{\frac{1}{2}} \mathbf{H} \ , \tag{4.1}$$

which has the dimension of a velocity.
The MHD equations thus read

$$\begin{aligned} \frac{\partial}{\partial t}u_i + u_n\frac{\partial}{\partial x_n}u_i - h_n\frac{\partial}{\partial x_n}h_i &= -\frac{1}{\rho}\frac{\partial}{\partial x_i}\left(p+\frac{1}{2}\rho|\mathbf{h}|^2\right)+\nu\nabla^2_{\mathbf{x}}u_i, \quad (4.2)\\ \frac{\partial}{\partial t}h_i + u_n\frac{\partial}{\partial x_n}h_i - h_n\frac{\partial}{\partial x_n}u_i &= \lambda\nabla^2_{\mathbf{x}}h_i, \quad (4.3)\end{aligned}$$

where summation over equal indices is implied, and $\nabla^2_{\mathbf{x}} = \frac{\partial}{\partial x_n}\frac{\partial}{\partial x_n}$ is the Laplace operator in $\mathbf{x}$ space.
We introduce the velocity and magnetic increments

$$\begin{aligned} v_i(\mathbf{x},\mathbf{x}',t) &= u_i(\mathbf{x},t)-u_i(\mathbf{x}',t), \quad (4.4)\\ b_i(\mathbf{x},\mathbf{x}',t) &= h_i(\mathbf{x},t)-h_i(\mathbf{x}',t). \quad (4.5)\end{aligned}$$

For brevity, let $u_i(\mathbf{x}',t)=u_i'$, $h_i(\mathbf{x}',t)=h_i'$ etc. Furthermore it is required that $\mathbf{x}$ and $\mathbf{x}'$ have no relative motion, so that terms like $\frac{\partial}{\partial x_n}u_i'$ and $\frac{\partial}{\partial x_n'}u_i$ vanish.
Subtracting (4.2) at point $\mathbf{x}'$ from (4.2) at point $\mathbf{x}$ and performing the same procedure for (4.3) yields

$$\begin{aligned} \frac{\partial}{\partial t}v_i + u_n\frac{\partial}{\partial x_n}v_i + u_n'\frac{\partial}{\partial x_n'}v_i - h_n\frac{\partial}{\partial x_n}b_i - h_n'\frac{\partial}{\partial x_n'}b_i &= -P_i+\nu(\nabla^2_{\mathbf{x}}+\nabla^2_{\mathbf{x}'})v_i, \quad (4.6)\\ \frac{\partial}{\partial t}b_i + u_n\frac{\partial}{\partial x_n}b_i + u_n'\frac{\partial}{\partial x_n'}b_i - h_n\frac{\partial}{\partial x_n}v_i - h_n'\frac{\partial}{\partial x_n'}v_i &= \lambda(\nabla^2_{\mathbf{x}}+\nabla^2_{\mathbf{x}'})b_i, \quad (4.7)\end{aligned}$$

where

$$P_i(\mathbf{x},\mathbf{x}',t) = \frac{1}{\rho}\frac{\partial}{\partial X_i}\left[p(\mathbf{x},t)-p(\mathbf{x}',t)+\frac{1}{2}\rho(|\mathbf{h}(\mathbf{x},t)|^2-|\mathbf{h}(\mathbf{x}',t)|^2)\right]. \quad (4.8)$$

was introduced for the total pressure increment.
The equations of motion for the velocity and the magnetic increment read

$$\frac{\partial}{\partial t}v_i + v_n\frac{\partial}{\partial r_n}v_i + U_n\frac{\partial}{\partial X_n}v_i - b_n\frac{\partial}{\partial r_n}b_i - H_n\frac{\partial}{\partial X_n}b_i = -\frac{1}{\rho}\frac{\partial}{\partial X_i}P+\nu(\nabla^2_{\mathbf{x}}+\nabla^2_{\mathbf{x}'})v_i, \quad (4.9)$$

$$\frac{\partial}{\partial t}b_i + v_n\frac{\partial}{\partial r_n}b_i + U_n\frac{\partial}{\partial X_n}b_i - b_n\frac{\partial}{\partial r_n}v_i - H_n\frac{\partial}{\partial X_n}v_i = \lambda(\nabla^2_{\mathbf{x}}+\nabla^2_{\mathbf{x}'})b_i\ , \quad (4.10)$$

where we have introduced relative and center coordinates

$$\begin{aligned} \mathbf{r} &= \mathbf{x}-\mathbf{x}', \quad (4.11)\\ \mathbf{X} &= \frac{\mathbf{x}+\mathbf{x}'}{2}. \quad (4.12)\end{aligned}$$

Furthermore, we have introduced the quantities[1]

$$\mathbf{U}(\mathbf{x},\mathbf{x}',t) = \frac{\mathbf{u}(\mathbf{x},t)+\mathbf{u}(\mathbf{x}',t)}{2}, \quad (4.13)$$

[1] In this case $\mathbf{H}(\mathbf{x},\mathbf{x}',t)$ is not to be confused with a strong uniform magnetic field $\mathbf{H}$, as it has been discussed in the section about the Alfvén waves 2.3. It can rather be interpreted as a large-scale field, that can strongly affect the small-scale fluctuations.

$$\mathbf{H}(\mathbf{x},\mathbf{x}',t) = \frac{\mathbf{h}(\mathbf{x},t)+\mathbf{h}(\mathbf{x}',t)}{2}. \tag{4.14}$$

By this change of variables, the implication of homogeneity for the statistical interpretation of the increment equations (4.9) and (4.10) can be seen directly: If $\frac{\partial}{\partial X_n}$ acts on a statistical quantity, this gives zero, since this quantity should be independent of the rate of change with respect for the place where the measurement is performed [Hil01b].

4.2 Structure functions of second order

The symmetry of the MHD equations provides the evolution equation of two different symmetric structure functions of second order. These are $\langle v_i v_j + b_i b_j\rangle$ and $\langle v_i b_j + v_j b_i\rangle$. The evolution equation for the antisymmetric tensor $\langle v_i b_j - v_j b_i\rangle$ can not be written in a closed form but the case of a uniform field $\mathbf{H}(\mathbf{x},\mathbf{x}',t)$ is discussed in section 4.2.2.

4.2.1 The equation of energy balance in MHD turbulence

Let us multiply (4.9) by v_j and then do the same procedure for interchanged indices. This yields

$$\frac{\partial}{\partial t}v_iv_j + \frac{\partial}{\partial r_n}v_nv_iv_j + \frac{\partial}{\partial X_n}U_nv_iv_j - b_nv_j\frac{\partial}{\partial r_n}b_i - b_nv_i\frac{\partial}{\partial r_n}b_j - H_nv_j\frac{\partial}{\partial X_n}b_i - H_nv_i\frac{\partial}{\partial X_n}b_j$$
$$= -v_jP_i - v_iP_j + \nu v_j(\nabla^2_{\mathbf{x}}+\nabla^2_{\mathbf{x}'})v_i + \nu v_i(\nabla^2_{\mathbf{x}}+\nabla^2_{\mathbf{x}'})v_j, \tag{4.15}$$

where we have made use of the incompressibility condition for v_n and U_n. We now multiply (4.10) by b_j and again interchange the indices, which leads to

$$\frac{\partial}{\partial t}b_ib_j + \frac{\partial}{\partial r_n}v_nb_ib_j + \frac{\partial}{\partial X_n}U_nb_ib_j - b_nb_i\frac{\partial}{\partial r_n}v_j - b_nb_j\frac{\partial}{\partial r_n}v_i - H_nb_i\frac{\partial}{\partial X_n}v_j - H_nb_j\frac{\partial}{\partial X_n}v_i$$
$$= \lambda b_j(\nabla^2_{\mathbf{x}}+\nabla^2_{\mathbf{x}'})b_i + \lambda b_i(\nabla^2_{\mathbf{x}}+\nabla^2_{\mathbf{x}'})b_j. \tag{4.16}$$

It can readily be seen that equation (4.15) and (4.16) are linked together by terms in (4.9) and (4.10) that are advected by either $\mathbf{b}(\mathbf{x},\mathbf{x}',t)$ or $\mathbf{H}(\mathbf{x},\mathbf{x}',t)$, since we are not able to write the balance equation for the kinetic and magnetic increments in a closed form separately. This can be seen as a significant feature of locally isotropic MHD turbulence.
Adding equation (4.15) to (4.16) and taking the averages, gives an evolution equation for the symmetric tensor $\langle v_iv_j + b_ib_j\rangle$ in MHD turbulence.

$$\begin{aligned} &\frac{\partial}{\partial t}\langle v_iv_j+b_ib_j\rangle + \frac{\partial}{\partial r_n}\langle v_n(v_iv_j+b_ib_j)\rangle - \frac{\partial}{\partial r_n}\langle b_n(v_ib_j+v_jb_i)\rangle \\ &\qquad + \frac{\partial}{\partial X_n}\langle U_n(v_iv_j+b_ib_j)\rangle - \frac{\partial}{\partial X_n}\langle H_n(v_ib_j+v_jb_i)\rangle \\ = \quad & -\langle v_jP_i+v_iP_j\rangle + 2\nu\left[\left(\nabla^2_{\mathbf{r}}+\frac{1}{4}\nabla^2_{\mathbf{X}}\right)\langle v_iv_j\rangle - \langle\varepsilon^{\mathbf{uu}}_{ij}\rangle\right] + 2\lambda\left[\left(\nabla^2_{\mathbf{r}}+\frac{1}{4}\nabla^2_{\mathbf{X}}\right)\langle b_ib_j\rangle - \langle\varepsilon^{\mathbf{hh}}_{ij}\rangle\right], \end{aligned} \tag{4.17}$$

where we have introduced

$$\varepsilon^{\mathbf{uu}}_{ij} = \left(\frac{\partial u_i}{\partial x_n}\right)\left(\frac{\partial u_j}{\partial x_n}\right) + \left(\frac{\partial u'_i}{\partial x'_n}\right)\left(\frac{\partial u'_j}{\partial x'_n}\right), \tag{4.18}$$

$$\varepsilon^{\mathbf{hh}}_{ij} = \left(\frac{\partial h_i}{\partial x_n}\right)\left(\frac{\partial h_j}{\partial x_n}\right) + \left(\frac{\partial h'_i}{\partial x'_n}\right)\left(\frac{\partial h'_j}{\partial x'_n}\right). \tag{4.19}$$

The treatment of the viscous terms is described in the appendix B.2.3. Under the assumption of homogeneity, the terms that stand behind the center derivative $\frac{\partial}{\partial X_n}$ can be neglected. Furthermore the pressure term vanishes on the basis of local homogeneity [Hil97].
Summing over equal indices i and j in equation (4.17) leads to the equation of energy balance of MHD turbulence in a briefer form

$$\frac{1}{2}\frac{\partial}{\partial t}\left\langle v^2(\mathbf{r},t)+b^2(\mathbf{r},t)\right\rangle+\nabla_{\mathbf{r}}\cdot\left\langle \mathbf{v}(\mathbf{r},t)\frac{v^2(\mathbf{r},t)+b^2(\mathbf{r},t)}{2}\right\rangle-\nabla_{\mathbf{r}}\cdot\langle\mathbf{b}(\mathbf{r},t)\,\mathbf{v}(\mathbf{r},t)\cdot\mathbf{b}(\mathbf{r},t)\rangle$$
$$=\nu\nabla^2_{\mathbf{r}}\left\langle v^2(\mathbf{r},t)\right\rangle+\lambda\nabla^2_{\mathbf{r}}\left\langle b^2(\mathbf{r},t)\right\rangle-2\langle\varepsilon^{\mathbf{v}}(\mathbf{x})+\varepsilon^{\mathbf{b}}(\mathbf{x})\rangle+Q(\mathbf{r},t)\ , \tag{4.20}$$

where $\langle\varepsilon^{\mathbf{v}}(\mathbf{x},t)\rangle$ and $\langle\varepsilon^{\mathbf{b}}(\mathbf{x},t)\rangle$ denote the corresponding local energy dissipation rates and $Q(\mathbf{r},t)=\langle\mathbf{v}(\mathbf{r},t)\cdot\mathbf{F}(\mathbf{r},t)\rangle+\langle\mathbf{b}(\mathbf{r},t)\cdot\mathbf{G}(\mathbf{r},t)\rangle$ takes into account a forcing procedure.

4.2.1.1 The 4/5 law in MHD turbulence

In the following we introduce the tensors

$$D^{\mathbf{vv}}_{ij}(\mathbf{r},t) = \langle v_iv_j\rangle \tag{4.21}$$
$$D^{\mathbf{bb}}_{ij}(\mathbf{r},t) = \langle b_ib_j\rangle \tag{4.22}$$
$$D^{\mathbf{vvv}}_{ijn}(\mathbf{r},t) = \langle v_iv_jv_n\rangle \tag{4.23}$$
$$D^{\mathbf{bbv}}_{ijn}(\mathbf{r},t)-D^{\mathbf{vbb}}_{ijn}(\mathbf{r},t) = \langle b_ib_jv_n\rangle-\langle(v_ib_j+v_jb_i)b_n\rangle\ . \tag{4.24}$$

where the subscripts denote the corresponding increments.
The last tensor (4.24) decomposes in terms of the corresponding correlation functions according to

$$D^{\mathbf{bbv}}_{ijn}(\mathbf{r},t)-D^{\mathbf{vbb}}_{ijn}(\mathbf{r},t)= +\ 2(\langle h_jh_nu_i'\rangle+\langle h_ih_nu_j'\rangle+\langle h_ih_ju_n'\rangle)$$
$$-\ 2(\langle(u_nh_j-u_jh_n)h_i'\rangle+\langle(u_nh_i-u_ih_n)h_j'\rangle-\langle(u_jh_i+u_ih_j)h_n'\rangle), \tag{4.25}$$

where terms like $\langle u_ih_jh_n\rangle-\langle u_i'h_j'h_n'\rangle$ vanish under the assumption of homogeneity. It can readily be seen, that the structure function does not decompose into the corresponding correlation functions like the structure function in the hydrodynamic case (4.23), considered in the appendix B.2.2. This is crucial for the appearance of the antisymmetric tensor

$$\langle(h_ju_n-u_jh_n)h_i'\rangle=C^{\mathbf{uhh}}(r,t)\left(\frac{r_j}{r}\delta_{in}-\frac{r_n}{r}\delta_{ij}\right), \tag{4.26}$$

whose defining scalar $C^{\mathbf{uhh}}(r,t)$ is not touched by the incompressibility condition. We already know this tensor and its properties from section 3.4.4.
The equation of energy balance in spherical coordinates can now be written in the form

$$\frac{1}{2}\frac{\partial}{\partial t}\left(D^{\mathbf{vv}}(\mathbf{r},t)+D^{\mathbf{bb}}(\mathbf{r},t)\right)+\frac{1}{2r^2}\frac{\partial}{\partial r}\left(r^2D^{\mathbf{vvv}}(r,t)+r^2(D^{\mathbf{bbv}}(r,t)-D^{\mathbf{vbb}}(r,t))\right)$$
$$=\nu\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}D^{\mathbf{vv}}(r,t)\right)+\lambda\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}D^{\mathbf{bb}}(r,t)\right)-2\langle\varepsilon^{\mathbf{v}}+\varepsilon^{\mathbf{b}}\rangle+Q(r,t)\ ,$$

where we have inserted the following functions, which can be expressed in terms of longitudinal structure functions and correlation functions as derived in the appendix B.2.1.1 and B.2.2.

$$\begin{aligned}
D^{\mathbf{vv}}(r,t) &= \langle v^2(r,t)\rangle = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^3 D^{\mathbf{vv}}_{r\,r}(r,t)\right) && (4.27)\\
\frac{\partial}{\partial r}D^{\mathbf{vv}}(r,t) &= \frac{1}{r^3}\frac{\partial}{\partial r}\left(r^4\frac{\partial}{\partial r}D^{\mathbf{vv}}_{r\,r}(r,t)\right) && (4.28)\\
D^{\mathbf{bb}}(r,t) &= \langle b^2(r,t)\rangle = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^3 D^{\mathbf{bb}}_{r\,r}(r,t)\right) \\
\frac{\partial}{\partial r}D^{\mathbf{bb}}(r,t) &= \frac{1}{r^3}\frac{\partial}{\partial r}\left(r^4\frac{\partial}{\partial r}D^{\mathbf{b}}_{r\,r}(r,t)\right) && (4.29)\\
D^{\mathbf{vvv}}(r,t) &= \langle v_r(r,t)\mathbf{v}(r,t)^2\rangle \\
&= \frac{1}{3r^3}\frac{\partial}{\partial r}\left(r^4 D^{\mathbf{vvb}}_{r\,r\,r}(r,t)\right) && (4.30)\\
D^{\mathbf{bbv}}(r,t) - D^{\mathbf{vbb}}(r,t) &= \langle v_r(r,t)\mathbf{b}(r,t)^2\rangle - 2\langle b_r(r,t)\mathbf{v}(r,t)\cdot\mathbf{b}(r,t)\rangle \\
&= -\frac{4}{r^3}\frac{\partial}{\partial r}\left(r^4 C^{\mathbf{hhu}}_{t\,t\,r}(r,t)\right) - 8C^{\mathbf{uhh}}(r,t)\ . && (4.31)
\end{aligned}$$

For stationary turbulence the fields are driven by a time independent source term $Q(r,t) = Q(r)$, and we obtain

$$\begin{aligned}
&\frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r^4\left\{\frac{1}{6}D^{\mathbf{vvv}}_{r\,r\,r}(r) - 2C^{\mathbf{hhu}}_{t\,t\,r}(r) - \nu\frac{\partial}{\partial r}D^{\mathbf{vv}}_{r\,r}(r) - \lambda\frac{\partial}{\partial r}D^{\mathbf{bb}}_{r\,r}(r)\right\}\right)\right)\\
&-\frac{4}{r^2}\left(\frac{\partial}{\partial r}r^2 C^{\mathbf{uhh}}(r)\right) = -2\langle\varepsilon^{\mathbf{v}} + \varepsilon^{\mathbf{b}}\rangle + Q(r)\ , && (4.32)
\end{aligned}$$

Two integrations with respect to r yield

$$\begin{aligned}
D^{\mathbf{vvv}}_{r\,r\,r}(r) - 12C^{\mathbf{hhu}}_{t\,t\,r}(r) - \frac{24}{r^4}\int_0^r \mathrm{d}r'\ r'^3 C^{\mathbf{uhh}}(r') = &-\ \frac{4}{5}\langle\varepsilon^{\mathbf{v}} + \varepsilon^{\mathbf{b}}\rangle r\\
&+\ 6\nu\frac{\partial}{\partial r}D^{\mathbf{vv}}_{r\,r}(r) + 6\lambda\frac{\partial}{\partial r}D^{\mathbf{bb}}_{r\,r}(r) + q(r).
\end{aligned} \qquad (4.33)$$

where the source $q(r)$ is given by

$$q(r) = \frac{6}{r^4}\int_0^r \mathrm{d}r' r' \int_0^{r'} \mathrm{d}r'' r''^2 Q(r'') \qquad (4.34)$$

Equation (4.33) is the generalization of the 4/5-law from hydrodynamic turbulence in the presence of a magnetic field. The 4/5 law in hydrodynamic turbulence is an exact relation between the second and third order longitudinal velocity structure function and the energy dissipation rate $\langle\varepsilon^{\mathbf{v}}\rangle$. However, in the case of MHD turbulence this relation is not closed, since the source term from $C^{\mathbf{uhh}}(r)$ doesn't vanish in the inertial range. In the following we discuss the implications of (4.33).

4.2.1.2 Dissipation range

As it has been discussed in section 3.4.4, the third order longitudinal velocity structure function $D^{\mathbf{vvv}}_{r\,r\,r}(r)$ scales as $\sim r^3$ for small r, whereas the mixed correlation functions only scale as r

$$\begin{aligned}
C^{\mathbf{hhu}}_{t\,t\,r}(r) &= -2C_0 r,\\
C^{\mathbf{uhh}}(r) &= 5C_0 r, && (4.35)
\end{aligned}$$

where C_0 is taken from (3.60) and can be seen as the contribution to the magnetic energy from the stretching of the lines of force by the velocity field. The source term can thus be expressed in terms of the defining scalar $C^{\mathbf{hhu}}_{t\,t\,r}(r)$, namely $C^{\mathbf{uhh}}(r) = -\frac{5}{2}C^{\mathbf{hhu}}_{t\,t\,r}(r)$.
In the dissipation range $r \ll \min(\eta_{\mathbf{v}}, \eta_{\mathbf{b}})$[1], $D^{\mathbf{vvv}}_{r\,r\,r}(r)$ can thus be neglected. Furthermore, by inserting (4.35) into (4.33), one can readily see that the two mixed correlation terms exactly cancel each other, which implies that energy is only lost due to dissipative effects. In the dissipation range (4.33) thus reads

$$6\nu\frac{\partial}{\partial r}D^{\mathbf{vv}}_{r\,r}(r) + 6\lambda\frac{\partial}{\partial r}D^{\mathbf{bb}}_{r\,r}(r) = \frac{4}{5}\left\langle \varepsilon^{\mathbf{v}} + \varepsilon^{\mathbf{b}}\right\rangle r, \tag{4.36}$$

where the forcing was assumed to take place on larger scales, so that $q(r) = 0$. An integration with respect to r yields

$$D^{\mathbf{vv}}_{r\,r}(r) + \frac{1}{\mathrm{Pm}}D^{\mathbf{bb}}_{r\,r}(r) = \frac{\left\langle \varepsilon^{\mathbf{v}} + \varepsilon^{\mathbf{b}}\right\rangle}{15}r^2. \tag{4.37}$$

4.2.1.3 Inertial range behavior

The inertial range behavior of (4.33) is obtained by neglecting the dissipative terms which yields

$$D^{\mathbf{vvv}}_{r\,r\,r}(r) - 12C^{\mathbf{hhu}}_{t\,t\,r}(r) - \frac{24}{r^4}\int_0^r \mathrm{d}r'\ r'^3 C^{\mathbf{uhh}}(r') = -\frac{4}{5}\langle \varepsilon^{\mathbf{v}} + \varepsilon^{\mathbf{b}}\rangle r. \tag{4.38}$$

The left-hand side of (4.38) therefore has to scale as r. The source term is somehow not closed since it involves an integral relation and $C^{\mathbf{uhh}}(r)$ can not be written in terms of $C^{\mathbf{hhu}}_{t\,t\,r}(r)$, as in the preceding section.
It would be necessary to find an expression for the defining scalar of the antisymmetric tensor (4.26), which can already be seen from the equation of energy balance (4.20), where the cross helicity term $\langle \mathbf{b}(\mathbf{r},t)\,\mathbf{v}(\mathbf{r},t)\cdot\mathbf{b}(\mathbf{r},t)\rangle$ is not closed. We will come back to this issue in section 4.2.3.
Focusing on the local behavior of (4.38), we conclude that in regions $\mathbf{x}$ where there is alignment between the velocity and the magnetic field, we have no contributions from the source term, since $\langle (h_j u_n - u_j h_n)h'_i\rangle$ vanishes in these regions. Therefore we obtain

$$D^{\mathbf{vvv}}_{r\,r\,r}(r) - 12C^{\mathbf{hhu}}_{t\,t\,r}(r) = -\frac{4}{5}\left\langle \varepsilon^{\mathbf{v}} + \varepsilon^{\mathbf{b}}\right\rangle r. \tag{4.39}$$

In these regions, (4.39) is an exact relation and one concludes that the velocity and magnetic increments scale as $r^{\frac{1}{3}}$.

4.2.1.4 The 4/5-law in terms of the vector potential a

The source term in (4.33) can be written in another form by introducing the vector potential $\mathbf{a}$, which is defined by

$$\mathbf{h} = \nabla \times \mathbf{a}. \tag{4.40}$$

We replace h'_i in (4.26) by the vector potential and get

$$\langle h'_i(u_j h_n - u_n h_j)\rangle = \varepsilon_{ikm}\frac{\partial}{\partial r_k}\langle a'_m(u_j h_n - u_n h_j)\rangle\ . \tag{4.41}$$

[1] These are the corresponding Kolmogorov length scales introduced in chapter 3.

The defining tensor is skew, solenoidal in i and antisymmetric in j and n. It can be written in analogy to equation (3.83), namely

$$\langle a_i'(u_j h_n - u_n h_j)\rangle = A(r,t)\varepsilon_{ijn} + \frac{r}{2}\frac{\partial}{\partial r}A(r,t)\left(\frac{r_j}{r}\varepsilon_{inl}\frac{r_l}{r} - \frac{r_n}{r}\varepsilon_{ijl}\frac{r_l}{r}\right) \ . \tag{4.42}$$

We insert this tensor into (4.41) and get by comparison an expression for the scalar $C^{\mathbf{uhh}}(r,t)$ in terms of the scalar $A(r,t)$,

$$C^{\mathbf{uhh}}(r,t) = -\frac{r}{2}\frac{\partial^2}{\partial r^2}A(r,t) + 2\frac{\partial}{\partial r}A(r,t) \ . \tag{4.43}$$

The source term in (4.33) thus can be rewritten and we get

$$D^{\mathbf{vvv}}_{r\,r\,r}(r) - 12C^{\mathbf{hhu}}_{t\,t\,r}(r) = -\frac{4}{5}\left\langle \varepsilon^{\mathbf{v}} + \varepsilon^{\mathbf{b}}\right\rangle r - 12\frac{\partial}{\partial r}A(r). \tag{4.44}$$

Therefore, the integral relation that enters in (4.33) is rewritten in terms of a vector potential correlation. However, the correlation function (4.42) has no direct physical meaning since it depends on a gauge field for **a** and the inertial range behavior of (4.44) has to be discussed in including the evolution equation of the cross helicity, discussed in the following section.

4.2.2 The evolution equation of the cross helicity for the velocity and magnetic increments

The cross helicity $\langle v_i b_j\rangle$ in MHD turbulence is determined by the evolution equation of the second symmetric tensor $\langle v_i b_j + v_j b_i\rangle$ and the antisymmetric tensor $\langle v_i b_j - v_j b_i\rangle$. These equations can be obtained in multiplying (4.9) by b_j and (4.10) for the index j by v_i.
This yields

$$\begin{aligned}&\frac{\partial}{\partial t}v_i b_j + \frac{\partial}{\partial r_n}v_n v_i b_j + \frac{\partial}{\partial X_n}U_n v_i b_j - b_j b_n\frac{\partial}{\partial r_n}b_i - v_i b_n\frac{\partial}{\partial r_n}v_j - b_j H_n\frac{\partial}{\partial X_n}b_i - v_i H_n\frac{\partial}{\partial X_n}v_j\\ &= -b_j P_i + \nu b_j(\nabla^2_{\mathbf{x}} + \nabla^2_{\mathbf{x}'})v_i + \lambda v_i(\nabla^2_{\mathbf{x}} + \nabla^2_{\mathbf{x}'})b_j.\end{aligned}$$

From this equation, we are able to calculate the symmetric and antisymmetric tensor of the cross helicity. The equations read

$$\begin{aligned}&\frac{\partial}{\partial t}\langle v_i b_j + v_j b_i\rangle - \frac{\partial}{\partial r_n}\langle b_n b_i b_j\rangle - \frac{\partial}{\partial r_n}\langle b_n v_i v_j\rangle + \frac{\partial}{\partial r_n}\langle v_n(v_i b_j + v_j b_i)\rangle\\ =\ & -\frac{1}{\rho}\langle b_j\frac{\partial}{\partial X_i}P + b_i\frac{\partial}{\partial X_j}P\rangle\\ &+\nu\langle b_j(\nabla^2_{\mathbf{x}} + \nabla^2_{\mathbf{x}'})v_i + b_i(\nabla^2_{\mathbf{x}} + \nabla^2_{\mathbf{x}'})v_j\rangle + \lambda\langle v_j(\nabla^2_{\mathbf{x}} + \nabla^2_{\mathbf{x}'})b_i + v_i(\nabla^2_{\mathbf{x}} + \nabla^2_{\mathbf{x}'})b_j\rangle.\end{aligned} \tag{4.45}$$

and

$$\begin{aligned}&\frac{\partial}{\partial t}\langle v_i b_j - v_j b_i\rangle + \frac{\partial}{\partial r_n}\langle v_n(v_i b_j - v_j b_i)\rangle\\ &-\langle b_n(b_j\frac{\partial}{\partial r_n}b_i - b_i\frac{\partial}{\partial r_n}b_j)\rangle - \langle H_n(b_j\frac{\partial}{\partial X_n}b_i - b_i\frac{\partial}{\partial X_n}b_j)\rangle\\ &-\langle b_n(v_j\frac{\partial}{\partial r_n}v_i - v_i\frac{\partial}{\partial r_n}v_j)\rangle - \langle H_n(v_j\frac{\partial}{\partial X_n}v_i - v_i\frac{\partial}{\partial X_n}v_j)\rangle\\ =\ & -\frac{1}{\rho}\langle\frac{\partial}{\partial X_i}P - b_i\frac{\partial}{\partial X_j}P\rangle\\ &+\nu\langle b_j(\nabla^2_{\mathbf{x}} + \nabla^2_{\mathbf{x}'})v_i - b_i(\nabla^2_{\mathbf{x}} + \nabla^2_{\mathbf{x}'})v_j\rangle + \lambda\langle v_j(\nabla^2_{\mathbf{x}} + \nabla^2_{\mathbf{x}'})b_i - v_i(\nabla^2_{\mathbf{x}} + \nabla^2_{\mathbf{x}'})b_j\rangle.\end{aligned} \tag{4.46}$$

Since the tensor of the cross helicity is skew, it can be written as

$$\langle v_i b_j \rangle = S(r,t)\varepsilon_{ijk}\frac{r_k}{r}, \tag{4.47}$$

and the symmetric tensor $\langle v_i b_j + v_j b_i \rangle$ vanishes in equation (4.45).
The tensors $\langle v_i v_j b_n \rangle$ and $\langle (v_i b_j + v_j b_i) v_n \rangle$ are skew tensors, symmetric in i, j. They can be written as

$$\begin{aligned} \langle v_i v_j b_n \rangle &= B(r,t)\left(\frac{r_i}{r}\varepsilon_{jnk}\frac{r_k}{r} + \frac{r_j}{r}\varepsilon_{ink}\frac{r_k}{r}\right), & (4.48)\\ \langle (v_i b_j + v_j b_i) v_n \rangle &= V(r,t)\left(\frac{r_i}{r}\varepsilon_{jnk}\frac{r_k}{r} + \frac{r_j}{r}\varepsilon_{ink}\frac{r_k}{r}\right). & (4.49) \end{aligned}$$

Since $\langle b_i b_j b_n \rangle$ is skew and symmetric in i, j and n, it follows that it must vanish, thus

$$\langle b_i b_j b_n \rangle = 0. \tag{4.50}$$

The same argument holds for the viscous and pressure terms, which are skew and symmetric in two indices. Furthermore, the calculation of the divergence of the tensors (4.48) and (4.49) gives zero

$$\frac{\partial}{\partial r_n}\langle v_i v_j b_n \rangle - \frac{\partial}{\partial r_n}\langle (v_i b_j + v_j b_i) v_n \rangle = 0, \tag{4.51}$$

and the whole equation (4.45) vanishes. This is an interesting result and an implication of the fact that skew tensors can not be symmetric in all their indices.
Turning to equation (4.46), the terms which cause problems, are obviously the terms that are advected by the large scale magnetic field $\mathbf{H}(\mathbf{x}, \mathbf{x}', t)$. A mean field approach for a homogeneous magnetic field $\langle H_n \rangle \gg \langle b_n \rangle$ yields

$$\left\langle H_n \left(b_j \frac{\partial}{\partial X_n} b_i - b_i \frac{\partial}{\partial X_n} b_j \right) \right\rangle = \langle H_n \rangle \left\langle b_j \frac{\partial}{\partial X_n} b_i - b_i \frac{\partial}{\partial X_n} b_j \right\rangle, \tag{4.52}$$

and

$$\left\langle H_n \left(v_j \frac{\partial}{\partial X_n} v_i - v_i \frac{\partial}{\partial X_n} v_j \right) \right\rangle = \langle H_n \rangle \left\langle v_j \frac{\partial}{\partial X_n} v_i - v_i \frac{\partial}{\partial X_n} v_j \right\rangle. \tag{4.53}$$

Under the assumption of isotropy the following tensors can be inserted, in analogy to equation (4.62) from the section about mean field electrodynamics 3.7.2

$$\begin{aligned} \langle b_i \frac{\partial}{\partial X_j} b_n \rangle &= \frac{1}{6}\varepsilon_{ijn}\langle \mathbf{b} \cdot \nabla_{\mathbf{x}} \times \mathbf{b} \rangle, & (4.54)\\ \langle v_i \frac{\partial}{\partial X_j} v_n \rangle &= \frac{1}{6}\varepsilon_{ijn}\langle \mathbf{v} \cdot \nabla_{\mathbf{x}} \times \mathbf{v} \rangle. & (4.55) \end{aligned}$$

One obtains

$$\langle H_n \rangle \left\langle b_j \frac{\partial}{\partial X_n} b_i - b_i \frac{\partial}{\partial X_n} b_j \right\rangle = \langle H_n \rangle \left\langle b_j \left(\frac{\partial}{\partial X_n} b_i - \frac{\partial}{\partial X_i} b_n \right) \right\rangle = -\frac{1}{3}\varepsilon_{ijn}\langle H_n \rangle \langle \mathbf{b} \cdot \nabla_{\mathbf{x}} \times \mathbf{b} \rangle, \tag{4.56}$$

$$\langle H_n \rangle \left\langle v_j \frac{\partial}{\partial X_n} v_i - v_i \frac{\partial}{\partial X_n} v_j \right\rangle = \langle H_n \rangle \left\langle v_j \left(\frac{\partial}{\partial X_n} v_i - \frac{\partial}{\partial X_i} v_n \right) \right\rangle = -\frac{1}{3}\varepsilon_{ijn}\langle H_n \rangle \langle \mathbf{v} \cdot \nabla_{\mathbf{x}} \times \mathbf{v} \rangle. \tag{4.57}$$

The contribution from $\langle \mathbf{b} \cdot \nabla_{\mathbf{X}} \times \mathbf{b} \rangle = 4\pi \langle \mathbf{b} \cdot \mathbf{J} \rangle$ has the meaning of a turbulent resistivity, whereas the contribution from $\langle \mathbf{v} \cdot \nabla_{\mathbf{X}} \times \mathbf{v} \rangle = \langle \mathbf{u} \cdot \mathbf{\Omega} \rangle$ is the kinetic helicity of the velocity field.

Under the assumption that the small scale terms in (4.46) are much smaller than the terms that are advected by $\mathbf{H}(\mathbf{x}, \mathbf{x}', t)$, the evolution equation for the cross helicity now reads

$$\begin{aligned} & \frac{\partial}{\partial t}\langle v_i b_j - v_j b_i \rangle + \frac{1}{3}\varepsilon_{ijn}\langle H_n \rangle \langle \mathbf{b} \cdot \nabla_{\mathbf{X}} \times \mathbf{b} \rangle + \frac{1}{3}\varepsilon_{ijn}\langle H_n \rangle \langle \mathbf{v} \cdot \nabla_{\mathbf{X}} \times \mathbf{v} \rangle \\ = \; & \nu \langle b_j(\nabla^2_{\mathbf{x}} + \nabla^2_{\mathbf{x}'})v_i - b_i(\nabla^2_{\mathbf{x}} + \nabla^2_{\mathbf{x}'})v_j \rangle + \lambda \langle v_j(\nabla^2_{\mathbf{x}} + \nabla^2_{\mathbf{x}'})b_i - v_i(\nabla^2_{\mathbf{x}} + \nabla^2_{\mathbf{x}'})b_j \rangle . \end{aligned} \tag{4.58}$$

In the following, we assume that the turbulent resistivity is rather small and conclude that the source term for the cross helicity only exists if the large scale vorticity precises around the small scale velocity vector so that the kinetic helicity does not vanish. Furthermore, due to the large scale magnetic field $\mathbf{H}(\mathbf{x}, \mathbf{x}', t)$, the cross helicity introduces a nonlocal character into the equation of energy balance (4.20). The implications of this fact are discussed in the next section.

4.2.3 An approximation for the source term

As it has been discussed in the two preceding sections, the equation system, consisting of the equation of energy balance (4.20) and the equation of the cross helicity (4.46), is not closed. It would be necessary to consider a third equation, namely the evolution equation for the large-scale magnetic field $\mathbf{H}(\mathbf{x}, \mathbf{x}', t)$

$$\frac{\partial}{\partial t}H_i + v_n \frac{\partial}{\partial r_n}H_i + U_n \frac{\partial}{\partial X_n} - b_n \frac{\partial}{\partial r_n}U_i - H_n \frac{\partial}{\partial X_n}U_i = \lambda(\nabla^2_{\mathbf{x}} + \nabla^2_{\mathbf{x}'})H_i. \tag{4.59}$$

However, if we assume that the large-scale magnetic field doesn't change that much over time, we can directly integrate (4.58)

$$\langle v_i b_j - v_j b_i \rangle \approx \frac{1}{3}\varepsilon_{ijn}\langle H_n \rangle \int_{-\infty}^{t} \mathrm{d}t' \langle \mathbf{v} \cdot \nabla_{\mathbf{X}} \times \mathbf{v} \rangle, \tag{4.60}$$

where we have neglected the viscous and the resistive parts.
The cross helicity tensor can be simplified according to

$$\langle v_i b_j \rangle = \langle (u_i - u'_i)(h_i - h'_i) \rangle = 2\langle u_i h_j \rangle - \langle u_i h'_j \rangle - \langle u'_i h'_j \rangle = 2\langle u_i h_j \rangle, \tag{4.61}$$

where in the last step it was used that $\langle u_i h'_j \rangle = -\langle u'_i h_j \rangle$, due to the lack of mirror symmetry.
Furthermore, we introduce the tensor

$$\alpha_{ijn} = \frac{1}{6}\varepsilon_{ijn} \int_{-\infty}^{t} \mathrm{d}t' \langle \mathbf{v} \cdot \nabla_{\mathbf{X}} \times \mathbf{v} \rangle, \tag{4.62}$$

along the lines of the α-effect from mean field electrodynamics. This yields

$$\langle u_i h_j - u_j h_i \rangle = \alpha_{ijn} \langle H_n \rangle. \tag{4.63}$$

Making use of the first order smoothing approximation from section 3.119 and multiplying by $h'_k = h_k(\mathbf{x} + \mathbf{r})$, and again taking the average yields

$$\langle (u_i h_j - u_j h_i) h'_k \rangle = \langle H_n \rangle \langle \alpha_{ijn} h'_k \rangle. \tag{4.64}$$

For the case where the source term is the dominant term in equation (4.33), we thus obtain the scaling

$$\langle \mathbf{H} \rangle \langle \alpha \mathbf{h}' \rangle \sim r, \tag{4.65}$$

where α is a matrix in this case.
The preceding discussion shows similarities to the Iroshnikov-Kraichnan phenomenology from section 3.101. First-of-all, the large-scale magnetic field $\mathbf{H}(\mathbf{x},\mathbf{x}',t)$ introduces a nonlocal character to the equation of energy balance (4.20). Therefore the Alfvén effect can occur solely by the presence of $\mathbf{H}(\mathbf{x},\mathbf{x}',t)$, without an additional external magnetic field. In addition, equation (4.65) predicts the scaling of a moment of fourth order, since the matrix α is a moment of second order. The interpretation in the sense of the Iroshnikov-Kraichnan phenomenology would be that the magnetic and velocity increments scale as $\frac{r}{4}$, since the vorticity and the velocity in (4.62) have the same dimension. However, it should be stressed that the above model takes into account several approximations and is far from being exact.

4.3 Next order symmetric structure function

In the next order of the hierarchy we have to deal the first time with structure functions containing the pressure gradient. There exist two obvious structure functions of third order which are symmetric in ijk. We consider first the structure function which involves the triple velocity increments $v_iv_jv_k$. By multiplying (4.9) and (4.10) by the corresponding increments we obtain a first evolution equation for the triple velocity structure function $\langle v_iv_jv_k\rangle$, which also involves the dynamics of three terms similar to $\langle v_ib_jb_k\rangle$ (indices interchanged). Under the assumption of homogeneity we obtain

$$\begin{aligned}
& \frac{\partial}{\partial t}\left\langle v_iv_jv_k + v_ib_jb_k + b_iv_jb_k + b_ib_jv_k\right\rangle + \frac{\partial}{\partial r_n}\left\langle v_kv_iv_jv_n - b_ib_jb_kb_n\right\rangle \\
& +\frac{\partial}{\partial r_n}\langle v_n(v_ib_jb_k + b_iv_jb_k + b_ib_jv_k)\rangle - \frac{\partial}{\partial r_n}\langle b_n(v_iv_jb_k + b_iv_jv_k + v_ib_jv_k)\rangle \\
= \; & -\langle (v_iv_j + b_ib_j)P_k + (v_iv_k + b_ib_k)P_j + (v_jv_k + b_jb_k)P_i\rangle \\
& +\nu\left\langle v_iv_j(\nabla_x^2 + \nabla_{x'}^2)v_k + v_jv_k(\nabla_x^2 + \nabla_{x'}^2)v_i + v_iv_k(\nabla_x^2 + \nabla_{x'}^2)v_j\right\rangle \\
& +\nu\left\langle b_ib_j(\nabla_x^2 + \nabla_{x'}^2)v_k + b_jb_k(\nabla_x^2 + \nabla_{x'}^2)v_i + b_ib_k(\nabla_x^2 + \nabla_{x'}^2)v_j\right\rangle \\
& +\lambda\left\langle v_ib_j(\nabla_x^2 + \nabla_{x'}^2)b_k + v_kb_j(\nabla_x^2 + \nabla_{x'}^2)b_i + v_kb_i(\nabla_x^2 + \nabla_{x'}^2)b_j\right\rangle \\
& +\lambda\left\langle v_jb_i(\nabla_x^2 + \nabla_{x'}^2)b_k + v_jb_k(\nabla_x^2 + \nabla_{x'}^2)b_i + v_ib_k(\nabla_x^2 + \nabla_{x'}^2)b_j\right\rangle .
\end{aligned} \tag{4.66}$$

Another evolution equation can be derived for the triple magnetic structure function $\langle b_ib_jb_k\rangle$:

$$\begin{aligned}
& \frac{\partial}{\partial t}\langle b_ib_jb_k\rangle + \frac{\partial}{\partial t}\langle v_iv_jb_k + v_ib_jv_k + b_iv_jv_k\rangle \\
& +\frac{\partial}{\partial r_k}\langle v_n(v_iv_jb_k + v_ib_jv_k + b_iv_jv_k + b_ib_jb_k)\rangle - \frac{\partial}{\partial r_k}\langle b_n(b_ib_jv_k + b_iv_jb_k + v_ib_jb_k + v_iv_jv_k)\rangle \\
= \; & -\langle (v_ib_j + v_jb_i)P_k + (v_ib_k + v_kb_j)P_j + (v_kb_j + v_jb_k)P_i\rangle \\
& +\nu\left\langle v_ib_j(\nabla_x^2 + \nabla_{x'}^2)v_k + v_jb_k(\nabla_x^2 + \nabla_{x'}^2)v_i + v_kb_i(\nabla_x^2 + \nabla_{x'}^2)v_j\right\rangle \\
& +\nu\left\langle v_jb_i(\nabla_x^2 + \nabla_{x'}^2)v_k + v_kb_j(\nabla_x^2 + \nabla_{x'}^2)v_i + v_ib_k(\nabla_x^2 + \nabla_{x'}^2)v_j\right\rangle \\
& +\lambda\left\langle v_iv_j(\nabla_x^2 + \nabla_{x'}^2)b_k + v_jv_k(\nabla_x^2 + \nabla_{x'}^2)b_i + v_iv_k(\nabla_x^2 + \nabla_{x'}^2)b_j\right\rangle \\
& +\lambda\left\langle b_ib_j(\nabla_x^2 + \nabla_{x'}^2)b_k + b_jb_k(\nabla_x^2 + \nabla_{x'}^2)b_i + b_ib_k(\nabla_x^2 + \nabla_{x'}^2)b_j\right\rangle .
\end{aligned} \tag{4.67}$$

In the following, we focus on the first equation (4.66). The tensors are written in the following form

$$
\begin{aligned}
D^{\mathbf{vvv}}_{ijk}(\mathbf{r},t) &= \langle v_i v_j v_k\rangle, && (4.68)\\
D^{\mathbf{vbb}}_{ijk}(\mathbf{r},t) &= \langle v_i b_j b_k + b_i v_j b_k + b_i b_j v_k\rangle, && (4.69)\\
D^{\mathbf{vvvv}}_{ijkn}(\mathbf{r},t) &= \langle v_i v_j v_k v_n\rangle, && (4.70)\\
D^{\mathbf{bbbb}}_{ijkn}(\mathbf{r},t) &= \langle b_i b_j b_k b_n\rangle, && (4.71)\\
D^{\mathbf{vvbb}}_{ij,kn}(\mathbf{r},t) &= \langle v_i v_j b_k b_n - b_i b_j v_k v_n\rangle, && (4.72)\\
T_{ijk}(\mathbf{r},t) &= \langle (v_i v_j + b_i b_j)P_k + (v_i v_k + b_i b_k)P_j + (v_j v_k + b_j b_k)P_i\rangle\,. && (4.73)
\end{aligned}
$$

For the viscous terms, we restrict ourselves to the case where the magnetic Prandtl number Pm= $\frac{\nu}{\lambda}$ is unity, which simplifies the treatment. This restriction seems somehow arbitrary, but for later times we are only interested in the inertial range behavior of the forth order structure functions, where it was shown for the hydrodynamic case [Hil01a], that the viscous terms should have no contribution. We arrive at the following equation

$$
\begin{aligned}
&\frac{\partial}{\partial t}\left(D^{\mathbf{vvv}}_{ijk}(\mathbf{r},t) + D^{\mathbf{vbb}}_{ijk}(\mathbf{r},t)\right) + \frac{\partial}{\partial r_n}\left(D^{\mathbf{vvvv}}_{ijkn}(\mathbf{r},t) - D^{\mathbf{bbbb}}_{ijkn}(\mathbf{r},t)\right) \qquad (4.74)\\
&-\frac{\partial}{\partial r_n}\left(D^{\mathbf{vvbb}}_{ij,kn}(\mathbf{r},t) + D^{\mathbf{vvbb}}_{kj,in}(\mathbf{r},t) + D^{\mathbf{vvbb}}_{ik,jn}(\mathbf{r},t)\right)\\
=\;& -T_{ijk}(\mathbf{r},t) + 2\nu\left[\nabla^2_{\mathbf{r}}\left(D^{\mathbf{vvv}}_{ijk}(\mathbf{r},t) + D^{\mathbf{vbb}}_{ijk}(\mathbf{r},t)\right) - Z^{\mathbf{vvv}}_{ijk}(\mathbf{r},t) - Z^{\mathbf{vbb}}_{ijk}(\mathbf{r},t)\right].
\end{aligned}
$$

where we have introduced

$$
\begin{aligned}
Z^{\mathbf{vvv}}_{ijk}(\mathbf{r},t) &= \langle v_i\varepsilon^{\mathbf{uu}}_{jk} + v_j\varepsilon^{\mathbf{uu}}_{ki} + v_k\varepsilon^{\mathbf{uu}}_{ij}\rangle,\\
Z^{\mathbf{vbb}}_{ijk}(\mathbf{r},t) &= \langle v_i\varepsilon^{\mathbf{hh}}_{jk} + v_j\varepsilon^{\mathbf{uu}}_{ki} + v_k\varepsilon^{\mathbf{uu}}_{ij} + b_i(\varepsilon^{\mathbf{uh}}_{jk} + \varepsilon^{\mathbf{uh}}_{kj}) + b_j(\varepsilon^{\mathbf{uh}}_{ki} + \varepsilon^{\mathbf{uh}}_{ik}) + b_k(\varepsilon^{\mathbf{uh}}_{ij} + \varepsilon^{\mathbf{uh}}_{ji})\rangle,
\end{aligned}
\qquad (4.75)
$$

and

$$
\begin{aligned}
\varepsilon^{\mathbf{uu}}_{ij} &= \left(\frac{\partial u_i}{\partial x_l}\right)\left(\frac{\partial u_j}{\partial x_l}\right) + \left(\frac{\partial u_i'}{\partial x_l'}\right)\left(\frac{\partial u_j'}{\partial x_l'}\right) && (4.76)\\
\varepsilon^{\mathbf{uh}}_{ij} &= \left(\frac{\partial u_i}{\partial x_l}\right)\left(\frac{\partial h_j}{\partial x_l}\right) + \left(\frac{\partial u_i'}{\partial x_l'}\right)\left(\frac{\partial h_j'}{\partial x_l'}\right). && (4.77)
\end{aligned}
$$

The treatment of the viscous terms can be found in [Hil01a].
In the inertial range (4.74) reads

$$
\begin{aligned}
&\frac{\partial}{\partial t}\left(D^{\mathbf{vvv}}_{ijk}(\mathbf{r},t) + D^{\mathbf{vbb}}_{ijk}(\mathbf{r},t)\right)\\
+\;&\frac{\partial}{\partial r_n}\left(D^{\mathbf{vvvv}}_{ijkn}(\mathbf{r},t) - D^{\mathbf{bbbb}}_{ijkn}(\mathbf{r},t) - D^{\mathbf{vvbb}}_{ij,kn}(\mathbf{r},t) + D^{\mathbf{vvbb}}_{kj,in}(\mathbf{r},t) + D^{\mathbf{vvbb}}_{ik,jn}(\mathbf{r},t)\right) = T_{ijk}(\mathbf{r},t)
\end{aligned}
\qquad (4.78)
$$

The tensors in the first bracket are symmetric in all four indices. The tensorial form of such a tensor and the calculation of its divergence is given in the appendix B.2.4. Only the tensor $D^{\mathbf{vvbb}}_{ij,kn}(\mathbf{r},t)$ has to be determined. Since it is antisymmetric in exchanging ij against kn, its tensorial form is

$$
D^{\mathbf{vvbb}}_{ij,kn}(\mathbf{r},t) = D^{\mathbf{vvbb}}_{rr,tt}(r,t)\left(\frac{r_i r_j}{r^2}\delta_{kn} - \frac{r_k r_n}{r^2}\delta_{ij}\right). \qquad (4.79)
$$

The calculation is exerted in the appendix B.2.4. Under the assumption of stationary turbulence, we obtain

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\left(D^{\mathbf{vvvv}}_{rrrr}(r)-D^{\mathbf{bbbb}}_{rrrr}(r)\right)\right]-\frac{6}{r}\left(D^{\mathbf{vvvv}}_{rrtt}(r)-D^{\mathbf{bbbb}}_{rrtt}(r)-D^{\mathbf{vvbb}}_{rr,tt}(r)\right) \quad = \quad -T_{rrr}(r), \tag{4.80}$$

for the longitudinal structure functions and

$$\frac{1}{r^4}\frac{\partial}{\partial r}\left[r^4\left(D^{\mathbf{vvvv}}_{rrtt}(r)-D^{\mathbf{bbbb}}_{rrtt}(r)\right)\right]-\frac{4}{3r}\left(D^{\mathbf{vvvv}}_{tttt}(r)-D^{\mathbf{bbbb}}_{tttt}(r)\right)+\frac{\partial}{\partial r}D^{\mathbf{vvbb}}_{rr,tt}(r) \quad = \quad -T_{rtt}(r), \tag{4.81}$$

for the mixed structure functions.
This can be written in a more investigative form as,

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\left(\langle v_rv_rv_rv_r\rangle-\langle b_rb_rb_rb_r\rangle\right)\right]-\frac{6}{r}(\langle v_rv_rv_tv_t\rangle-\langle b_rb_rb_tb_t\rangle-\langle v_rv_rb_tb_t-v_tv_tb_rb_r\rangle)=-T_{rrr} \tag{4.82}$$

and

$$\frac{1}{r^4}\frac{\partial}{\partial r}\left[r^4\left(\langle v_rv_rv_tv_t\rangle-\langle b_rb_rb_tb_t\rangle\right)\right]-\frac{4}{3r}(\langle v_tv_tv_tv_t\rangle-\langle b_tb_tb_tb_t\rangle)+\frac{\partial}{\partial r}\langle v_rv_rb_tb_t-v_tv_tb_rb_r\rangle=-T_{rtt}. \tag{4.83}$$

For strong turbulence, the inertial range scaling of the fourth order structure functions derived by Yakhot [Yak02] and Hill [Hil01a] is recovered.

4.3.1 Rescaling relations between longitudinal and transverse structure functions

The basic idea of this section was first developed by Siefert and Peinke [Sie04]. They start with the Kármán-Howarth relation that relates the transverse structure function to the longitudinal structure function, according to

$$D^{\mathbf{vv}}_{tt}(r)=\frac{1}{2r}\frac{\partial}{\partial r}\left(r^2D^{\mathbf{vv}}_{rr}(r,t)\right). \tag{4.84}$$

This relation is an implication of the incompressibility condition of the velocity field and is derived in the appendix B.2.1.1. Based on the observation that $D^{\mathbf{vv}}_{rr}(r)$ is a smooth function of r we can interpret the right-hand side of (4.84) as the first two terms of a Taylor expansion for small r[1] of the form

$$D^{\mathbf{vv}}_{rr}\left(r+\frac{r}{2}\right)\approx D^{\mathbf{vv}}_{rr}(r)+\frac{r}{2}\frac{\partial}{\partial r}D^{\mathbf{vv}}_{rr}(r). \tag{4.85}$$

Under this approximation, the transverse structure function is simply the longitudinal structure function rescaled by the factor $\frac{3}{2}$

$$D^{\mathbf{vv}}_{tt}(r)\approx D^{\mathbf{vv}}_{rr}\left(\frac{3}{2}r\right). \tag{4.86}$$

Grauer et al. [Gra12] generalized this procedure for structure functions of the order n in hydrodynamic turbulence. The procedure is described at the example of the fourth order equations

[1]With small we mean small compared to the integral scale L.

(4.80) and (4.81)

$$\begin{aligned}\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2 D^{\mathbf{vvvv}}_{r\,r\,r\,r}(r)\right]-\frac{6}{r}D^{\mathbf{vvvv}}_{r\,r\,t\,t}(r) &= -T_{rrr}(r),\\ \frac{1}{r^4}\frac{\partial}{\partial r}\left[r^4 D^{\mathbf{vvvv}}_{r\,r\,t\,t}(r)\right]-\frac{4}{3r}D^{\mathbf{vvvv}}_{t\,t\,t\,t}(r) &= -T_{rtt}(r).\end{aligned}$$

These two equations are solely related by the mixed term $D^{\mathbf{vvvv}}_{r\,r\,t\,t}(r)$, contrary to the MHD equations, where the antisymmetric mixed terms $D^{\mathbf{vvbb}}_{r\,r,t\,t}(r)$ appear. The rescaling properties can be repeated under the neglect of the pressure terms, so that we obtain

$$3D^{\mathbf{vvvv}}_{r\,r\,t\,t}(r) \approx D^{\mathbf{vvvv}}_{r\,r\,r\,r}(r)+\frac{r}{2}\frac{\partial}{\partial r}D^{\mathbf{vvvv}}_{r\,r\,r\,r}(r)\approx D^{\mathbf{vvvv}}_{r\,r\,r\,r}\left(\frac{3}{2}r\right), \tag{4.87}$$

$$\frac{1}{3}D^{\mathbf{vvvv}}_{t\,t\,t\,t}(r) \approx D^{\mathbf{vvvv}}_{r\,r\,t\,t}(r)+\frac{r}{2}\frac{\partial}{\partial r}D^{\mathbf{vvvv}}_{r\,r\,t\,t}(r)\approx D^{\mathbf{vvvv}}_{r\,r\,t\,t}\left(\frac{3}{2}r\right), \tag{4.88}$$

(4.89)

which yields the rescaling between fourth order transverse and longitudinal structure function

$$D^{\mathbf{vvvv}}_{t\,t\,t\,t}(r)\approx D^{\mathbf{vvvv}}_{r\,r\,r\,r}\left(\frac{3}{2}\frac{5}{4}r\right). \tag{4.90}$$

In general a transverse structure function of order n can be rescaled according to

$$D^{n\mathbf{v}}_{nt}(r)\approx D^{n\mathbf{v}}_{nr}\left(\frac{3}{2}\frac{5}{4}...\frac{n+1}{n}\right)=D^{n\mathbf{v}}_{nr}\left(\frac{\Gamma(n+2)}{2^n\Gamma^2(n/2+1)}r\right) \tag{4.91}$$

In this context, two things are worthwhile noticing: First-of-all, the mapping of longitudinal onto transverse structure functions is generally more complicated then the linear rescaling. This is already the case for the approximation of the Kármán-Howarth relation. Secondly, in contrary to the relation for the second order structure functions, the pressure term in the higher order relations introduces significant differences between the longitudinal and transverse structure functions, which have to be interpreted in the realm of intermittency.
The rescaling relation for the fourth order structure functions in MHD turbulence can only be used if the antisymmetric tensor (4.79) and the pressure correlations are neglected. Under these approximations, one obtains

$$D^{\mathbf{vvvv}}_{t\,t\,t\,t}(r)-D^{\mathbf{bbbb}}_{t\,t\,t\,t}(r)\approx D^{\mathbf{vvvv}}_{r\,r\,r\,r}\left(\frac{3}{2}\frac{5}{4}r\right)-D^{\mathbf{bbbb}}_{r\,r\,r\,r}\left(\frac{3}{2}\frac{5}{4}r\right). \tag{4.92}$$

This procedure will be adapted for the two-dimensional case in chapter 5. However, the rescaling relation (4.92) for MHD turbulence should show significant differences to the hydrodynamic rescaling relation from [Gra12], since the antisymmetric tensor (4.79) is neglected. Furthermore the neglected magnetic pressure in $T_{ijk}(\mathbf{r},t)$ should manifest itself in addition to the neglected hydrodynamic pressure term.

4.3.2 A remark on the implications of a Gaussian distribution of the velocity field for the Yakhot-Hill equations

The idea of expressing fourth order moments in terms of second order moments traces back to the first half of the last century [Mil41]. The simplest way to do so, is to assume a Gaussian distribution for the velocity moments. In the following the implications of such a distribution in addition to the Kármán-Howarth relation are discussed at the example of the Yakhot-Hill equations ((4.82) and (4.83) in the hydrodynamic limit), namely

$$\frac{1}{r^2}\frac{\partial}{\partial r}[r^2\langle v_r v_r v_r v_r\rangle] - \frac{6}{r}\langle v_r v_r v_t v_t\rangle = -T_{rrr}, \tag{4.93}$$

and

$$\frac{1}{r^4}\frac{\partial}{\partial r}[r^4\langle v_r v_r v_t v_t\rangle] - \frac{4}{3r}\langle v_t v_t v_t v_t\rangle = -T_{rtt}. \tag{4.94}$$

If the velocity increments follow a Gaussian distribution, the fourth order moments read

$$\langle v_r v_r v_r v_r\rangle = 3\langle v_r v_r\rangle^2, \tag{4.95}$$
$$\langle v_t v_t v_t v_t\rangle = 3\langle v_t v_t\rangle^2, \tag{4.96}$$
$$\langle v_r v_r v_t v_t\rangle = \langle v_r v_r\rangle\langle v_t v_t\rangle. \tag{4.97}$$

In replacing the transverse structure function by the Kármán-Howarth relation

$$\langle v_t v_t\rangle = \frac{1}{2r}\frac{\partial}{\partial r}(r^2\langle v_r v_r\rangle), \tag{4.98}$$

and inserting the expressions into (4.93), we observe that the terms on the left-hand side cancel each other, so that

$$T_{rrr}(r) = 0. \tag{4.99}$$

Non-vanishing longitudinal pressure correlations $T_{rrr}(r)$ therefore stand in contrast to a Gaussian distribution of the velocity moments. The equation for the mixed structure functions (4.94) reads

$$\frac{r}{2}\left(\langle v_r v_r\rangle\frac{\partial^2}{\partial r^2}\langle v_r v_r\rangle + \frac{1}{r}\langle v_r v_r\rangle\frac{\partial}{\partial r}\langle v_r v_r\rangle - \left(\frac{\partial}{\partial r}\langle v_r v_r\rangle\right)^2\right) = -T_{rtt}(r), \tag{4.100}$$

which is a second order differential equation for $\langle v_r v_r\rangle$ with an inhomogeneity $T_{rtt}(r)$. In summary it can be said that the Gaussian approximation is not appropriate to describe the statistical behavior of the longitudinal structure functions, however the mixed and transverse structure function can be described by a Gaussian distribution and further investigation of equation (4.100) seems necessary. For the case of two spatial dimension, Yakhot [Yak99] derived that the transverse structure functions follow a Gaussian statistics that is governed by a corresponding Langevin equation.

Chapter 5

Two dimensional MHD turbulence

The relations for the structure functions in 3D MHD turbulence derived in the preceding chapter are checked by numerical simulations of forced 2D MHD turbulence. Therefore, the tensor calculus described in the appendix B.1 has to be translated for case of two dimensions. Relations similar to the 3D MHD structure function relations are obtained. The chapter begins with a description of the numerical methods used for the numerical simulations that were performed on a graphic card. To this end, a pseudo-spectral code that was written for the Electron-MHD equations by Martin Rieke [Rie10] was rewritten for the purpose of solving the 2D MHD equations. A forcing procedure that conserves the total energy was applied to the evolution equation of the vorticity, in order to guarantee statistical isotropy and stationarity. The characteristic length scales from section 3.4 are calculated to make qualitative statements about the obtained velocity and magnetic structure functions. The rescaling relation from section 4.3.1 is applied to the 2D case and checked for the second and fourth order moments. Furthermore, the kinetic and magnetic energy spectra are calculated and compared to the Iroshnikov-Kraichnan phenomenology.

5.1 The 2D MHD equations

The evolution equation for the magnetic field (2.14) in 2D MHD turbulence reduces to a scalar advection-diffusion equation for the flux function $\psi(\mathbf{x}, t)$.

$$\frac{\partial}{\partial t}\psi(\mathbf{x}, t) + \mathbf{u}(\mathbf{x}, t) \cdot \nabla\psi(\mathbf{x}, t) = \lambda\nabla^2\psi(\mathbf{x}, t). \tag{5.1}$$

The relation to the magnetic field is thereby given as

$$\mathbf{h}(\mathbf{x}, t) = \mathbf{e}_z \times \nabla\psi(\mathbf{x}, t). \tag{5.2}$$

There exists now an analogy between the current density $j(\mathbf{x}, t)$ and the vorticity $\omega(\mathbf{x}, t)$, which both reduce to a pseudo-scalar in 2D MHD. The current density is related to the flux function by

$$j(\mathbf{x}, t) = \nabla^2\psi(\mathbf{x}, t). \tag{5.3}$$

The vorticity equation reads

$$\frac{\partial}{\partial t}\omega(\mathbf{x}, t) + \mathbf{u}(\mathbf{x}, t) \cdot \nabla\omega(\mathbf{x}, t) - \mathbf{h}(\mathbf{x}, t) \cdot \nabla j(\mathbf{x}, t) = \nu\nabla^2\omega(\mathbf{x}, t)\ . \tag{5.4}$$

5.2 Numerical methods

In the following, the numerical methods used for the purpose of solving the 2D MHD equations within a pseudo-spectral code on a periodic box of the length 2π are discussed.
Since the 2D MHD turbulence exhibits a direct energy cascade, in contrast to ordinary 2D turbulence which exhibits an inverse energy cascade, one has to deal with high velocity and magnetic field gradients imposed by the formation and decay of current sheets, as mentioned in section 2.1.3. In the regime of quasi-stationary sheets, where the spectral energy density scales as $E(k) \sim k^{-2}$, one would not speak of turbulence at all and the generation of turbulence in its proper sense is dependent on three important factors

- initial conditions: As it has been shown in [Bis89], the generation of small scale turbulence arises from non-symmetric initial conditions. In this work we consider two kinds of initial conditions.

  A-type initial conditions:

  The A-type initial condition from [Bis89] is given by

  $$\begin{aligned} \phi_{BW}(x,y) &= \cos(x+1.4)+\cos(y+0.5), \\ \psi_{BW}(x,y) &= \frac{1}{3}(\cos(2x+2.3)+\cos(y+4.1)), \end{aligned} \tag{5.5}$$

  which constitutes a generalization of the Orszag-Tang vortex

  $$\begin{aligned} \phi_{OT}(x,y) &= \cos(x)+\cos(y), \\ \psi_{OT}(x,y) &= \frac{1}{2}\cos(2x)+\cos(y). \end{aligned} \tag{5.6}$$

  This large-scale initial condition has a ratio of kinetic to magnetic energy of $E_{kin}/E_{mag} = 0.4$ and one observes the formation of four current sheets which become unstable via a so-called tearing instability [Bis03].

- forcing procedure: In order to guarantee statistical isotropy, a forcing procedure based on the Diploma thesis of Anton Daitche [Dai09] was chosen. It consists of an energy conserving scheme applied to the large scales of the vorticity. This avoids the buildup of correlated structures and is important for the statistical investigation of different realizations of the fields. We can therefore take time averaged mean values, provided that the ergodicity hypothesis holds.

- hyperviscosity: To allow a clearer separation from the dissipative scale and the inertial range, more general diffusion terms were used and the simulated evolution equations for the vorticity and the magnetic potential finally read

  $$\frac{\partial}{\partial t}\omega(\mathbf{x},t)+\mathbf{u}(\mathbf{x},t)\cdot\nabla\omega(\mathbf{x},t)-\mathbf{h}(\mathbf{x},t)\cdot\nabla j(\mathbf{x},t) = (-1)^{\gamma-1}\nu^{(\gamma)}\nabla^{2\gamma}\omega(\mathbf{x},t)+F(\mathbf{x},t), \tag{5.7}$$

  $$\frac{\partial}{\partial t}\psi(\mathbf{x},t)+\mathbf{u}(\mathbf{x},t)\cdot\nabla\psi(\mathbf{x},t) = (-1)^{\gamma-1}\lambda^{(\gamma)}\nabla^{2\gamma}\psi(\mathbf{x},t), \tag{5.8}$$

  where $\gamma = 1$ would correspond to ordinary diffusion. $F(\mathbf{x},t)$ represents the applied forcing mechanism mentioned above.

The choice of such a hyperviscosity guarantees that small scale structures are damped more efficiently (in Fourier space by $\exp[-\nu k^4 t]$). In the following $\gamma = 2$ was chosen and a restriction to unity magnetic Prandtl numbers, Pm= $\nu^{(\gamma)}/\lambda^{(\gamma)} = 1$, was made.

In the next section the basic ideas of a pseudo-spectral code for the numerical integration of a partial differential equation are discussed.

5.2.1 The idea of a pseudo-spectral code

The pseudo-spectral code treats the MHD equations in Fourier space. The convolutions arising from the nonlinearities are evaluated in real space and then transformed into Fourier space again. To this end, the CUDA FFT library[1] is used in order to calculate the Fourier transforms of the physical fields, for example

$$\mathbf{u}(\mathbf{x}, t) = \sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}} \hat{\mathbf{u}}_{\mathbf{k}}(t). \tag{5.9}$$

The Fourier coefficients $\hat{\mathbf{u}}_{\mathbf{k}}(t)$ are calculated on a $N \times N$ square grid, where N is the spatial resolution.
In the numerical simulations, the evolution equation for $j(\mathbf{x}, t)$ was used instead of the evolution equation for $\psi(\mathbf{x}, t)$ (5.8), since the continued multiplication with k^2 rendered the code instable. The evolution equation for $j(\mathbf{x}, t)$ is obtained in applying the Laplacian to (5.8). Therefore, the simulated equations in Fourier space read

$$\frac{\partial}{\partial t}\hat{\omega}_{\mathbf{k}}(t) + \nu^{(\gamma)} k^{2\gamma} \hat{\omega}_{\mathbf{k}}(t) = i\mathbf{k} \cdot \mathcal{F}[\mathbf{u}(\mathbf{x}, t)\omega(\mathbf{x}, t) - \mathbf{h}(\mathbf{x}, t) j(\mathbf{x}, t)]_{\mathbf{k}} \tag{5.10}$$

$$\frac{\partial}{\partial t}\hat{j}_{\mathbf{k}}(t) + \lambda^{(\gamma)} k^{2\gamma} \hat{j}_{\mathbf{k}}(t) = -ik^2\mathbf{k} \cdot \mathcal{F}[\mathbf{u}(\mathbf{x}, t) j(\mathbf{x}, t)]_{\mathbf{k}} \tag{5.11}$$

Since $\mathbf{u}(\mathbf{x}, t)$ and $\mathbf{h}(\mathbf{x}, t)$ are related to $\omega(\mathbf{x}, t)$ and $j(\mathbf{x}, t)$ by the Biot-Savart law, mentioned in A.1, they can be obtained in Fourier space according to

$$\hat{\mathbf{u}}_{\mathbf{k}}(t) = \frac{i\mathbf{k} \times \hat{\omega}_{\mathbf{k}}(t)}{k^2} \quad \text{and} \quad \hat{\mathbf{h}}_{\mathbf{k}}(t) = \frac{i\mathbf{k} \times \hat{j}_{\mathbf{k}}(t)}{k^2}. \tag{5.12}$$

5.2.1.1 De-aliasing

The destabilizing effect of aliasing for the pseudo-spectral code arises from the finite resolution in real space and its translation into Fourier space. Since the simulated physical fields are only known at certain grid points in real space, there exists a smallest structure that can be resolved properly. The characteristic length λ_c of this structure corresponds to a maximal wave number $k_{max} = \frac{2\pi}{\lambda_c}$ in Fourier space.
If a wave number $k > k_{max}$ occurs during the numerical simulations, this wave number is misinterpreted as a smaller wave number leading to an energy transfer from an unresolved domain to smaller wave numbers leading to a blow-up of the solution for the physical fields. These higher wave numbers can not arise from the linear terms in (5.10) and (5.11), and are restricted to the nonlinearities on the right-hand side of the equations. A convenient mean to avoid Fourier coefficients originating from the nonlinearities is to set all Fourier coefficients with wave numbers $|\mathbf{k}| \geqslant \frac{2}{3} k_{max}$ to zero. For further discussion, see for instance [Wil07].

[1] https://developer.nvidia.com/cufft

5.2.2 Time integration

The time integration used in the simulations consists of Heun's scheme of second order. The linear terms in the MHD equations are solved analytically. This is demonstrated at the example of the vorticity equation of 2D fluid mechanics

$$\left(\frac{\partial}{\partial t} - (-1)^{\gamma-1}\nu^{(\gamma)}\nabla^{2\gamma}\right)\omega(\mathbf{x},t) = N[\omega(\mathbf{x},t)] \tag{5.13}$$

where the nonlinearity was written as

$$N[\omega(\mathbf{x},t)] = -\mathbf{u}(\mathbf{x},t)\cdot\nabla\omega(\mathbf{x},t). \tag{5.14}$$

In Fourier space this reads

$$\left(\frac{\partial}{\partial t} + \nu^{(\gamma)}\nabla^{2\gamma}\right)\hat{\omega}_{\mathbf{k}}(t) = \hat{N}[\hat{\omega}_{\mathbf{k}}(t)], \tag{5.15}$$

We make the substitution

$$\tilde{\omega}_{\mathbf{k}}(t) = \hat{\omega}_{\mathbf{k}}(t)e^{-\nu^{(\gamma)}k^{2\gamma}t}, \tag{5.16}$$

which leads to the equation

$$\frac{\partial}{\partial t}\tilde{\omega}_{\mathbf{k}}(t) = \hat{N}[\tilde{\omega}_{\mathbf{k}}(t)e^{-\nu^{(\gamma)}k^{2\gamma}t}]e^{\nu^{(\gamma)}k^{2\gamma}t}. \tag{5.17}$$

The viscous terms are therefore treated exactly within every specific time step. The time stepping consists in two steps, according to

$$\begin{aligned}\hat{\omega}_1 &= \hat{\omega}_0 + \mathrm{d}t\hat{N}[\hat{\omega}_0], \\ \hat{\omega}_2 &= \frac{\hat{\omega}_1+\hat{\omega}_0}{2} + \frac{\mathrm{d}t}{2}N[\hat{\omega}_1].\end{aligned} \tag{5.18}$$

The whole scheme involving the viscous steps (5.17) thus reads

$$\begin{aligned}\hat{\omega}_1 &= \left(\hat{\omega}_0 + \mathrm{d}t\hat{N}[\hat{\omega}_0]\right)e^{-\nu^{(\gamma)}k^2\mathrm{d}t}, \\ \hat{\omega}_2 &= \left(\frac{\hat{\omega}_1+\hat{\omega}_0}{2} + \frac{\mathrm{d}t}{2}\hat{N}[\hat{\omega}_1]\right)e^{-\nu^{(\gamma)}k^2\mathrm{d}t}.\end{aligned} \tag{5.19}$$

5.2.3 The CFL condition

In order to guarantee the stability of the numerical integration, the time step $\mathrm{d}t$ has to be chosen in such a way that an information, for instance the value of the vorticity ω at a point $\mathbf{x}_i$, is not advected farther then one grid point per time step. This manifests itself in the so-called Courant-Friedrichs-Levy condition

$$\mathrm{d}t = \frac{\mathrm{d}x\lambda_c}{u_{max}} = \frac{2\pi\lambda_c}{u_{max}N}, \tag{5.20}$$

where λ_c is the Courant number and $\lambda_c < 1$ and N is the resolution of the simulation. This procedure takes into account the correct advection of an information. A similar condition exist for a diffusive transport due to the viscous terms. However, one can show [Rie10] that the time-stepping scheme introduced in the previous section, removes the condition for the diffusive process.

5.2.4 Energy-conserving forcing procedure

The applied forcing procedure conserves the total energy

$$E_{tot} = E_{kin} + E_{mag} = \int_V \mathrm{d}\mathbf{x} \left(\frac{\mathbf{u}^2}{2} + \frac{\mathbf{h}^2}{2} \right) = \frac{1}{2} \sum_{\mathbf{k}} \left(|\hat{\mathbf{u}}_{\mathbf{k}}|^2 + |\hat{\mathbf{h}}_{\mathbf{k}}|^2 \right). \tag{5.21}$$

In applying the vector product with i$\mathbf{k}$ to (5.12) and making use of $\mathbf{k} \cdot \hat{\omega}_{\mathbf{k}} = 0$ and $\mathbf{k} \cdot \hat{\jmath}_{\mathbf{k}} = 0$, we can write the total energy according to

$$E_{tot} = \frac{1}{2} \sum_{\mathbf{k}} \left(\frac{|\hat{\omega}_{\mathbf{k}}|^2}{k^2} + \frac{|\hat{\jmath}_{\mathbf{k}}|^2}{k^2} \right). \tag{5.22}$$

In order to conserve the total energy, Fourier modes of $\hat{\omega}_{\mathbf{k}}$ that lie within the so-called forcing band B are multiplied with a factor f. Thereby, the forcing is only applied on large-scales of the vorticity, and $B = \{k \in [0, 3]\}$ was chosen.
The forcing factor is determined by the condition that the total energy after a time step $E_{tot}^{(n+1)}$ is equal to the energy before the time step $E_{tot}^{(n)}$, namely

$$E_{tot}^{(n+1)} = f^2 E_{band}^{(n)} + E_{rest}^{(n)} = E_{tot}^{(n)}, \tag{5.23}$$

where

$$E_{band}^{(n)} = \frac{1}{2} \sum_{\mathbf{k} \in B} \frac{|\hat{\omega}_{\mathbf{k}}|^2}{k^2}, \tag{5.24}$$

is the total energy that is contained in the forcing band, and

$$E_{rest}^{(n)} = \frac{1}{2} \sum_{\mathbf{k} \notin B} \left(\frac{|\hat{\omega}_{\mathbf{k}}|^2}{k^2} + \frac{|\hat{\jmath}_{\mathbf{k}}|^2}{k^2} \right), \tag{5.25}$$

is the total energy that is contained in the other Fourier modes.
Since during a time step the total energy is only dissipated, so that $E_{tot}^{(n+1)} > E_{tot}^{(n)}$, the forcing factor is determined by

$$f = \sqrt{\frac{E_{tot}^{(n+1)} - E_{rest}^{(n)}}{E_{band}^{(n)}}}. \tag{5.26}$$

If only the vorticity equation is forced, one can simply multiply the Fourier coefficients $\hat{\omega}_{\mathbf{k}} \in B$ by f. Therefore the following scheme is obtained

$$\hat{\omega}_{\mathbf{k}}^{(n+1)} = \begin{cases} \hat{\omega}_{\mathbf{k}}^{(n)} & \text{if } \mathbf{k} \notin B \\ f \hat{\omega}_{\mathbf{k}}^{(n)} & \text{if } \mathbf{k} \in B \end{cases} \tag{5.27}$$

and

$$\hat{\jmath}_{\mathbf{k}}^{(n+1)} = \hat{\jmath}_{\mathbf{k}}^{(n)}. \tag{5.28}$$

For the discussion of statistical isotropy of the applied forcing procedure or alternative forcing procedures, the reader is referred to [Dai09].

5.3 Analysis of the numerical simulations

High-resolution numerical simulations of 2D MHD turbulence were performed in a periodic box with a resolution of 3072×3072 collocation points.
A typical time series for the current density $j(\mathbf{x}, t)$ originating from A type initial conditions is depicted in fig. 5.2. One clearly observes the emergence of four stretched monopoles at t=1.6. As it can be seen from the time series of $\omega(\mathbf{x}, t)$, these sheets are surrounded by a quadrupole distribution of the vorticity.
These structures are the so-called current sheets, regions of strong current density and vorticity. In the following these sheets are stretched, which is known as thinnig, and eventually become unstable and decay. An interesting effect in this context, is the effect of magnetic reconnection, which can be observed in the left upper corner of the plot for $j(\mathbf{x}, t)$ at $t = 2.1$: In the regions of the black ovals, an exchange of magnetic field lines occurs between different magnetic domains, initially separated by the current sheet. This process that can be observed in highly conducting plasmas, is thus a rearrangement of the magnetic topology, whereby a great amount of magnetic energy is converted into kinetic energy. Since the total energy is conserved due to the forcing, it is interesting to investigate the temporal evolution of the kinetic and magnetic energy, as depicted in fig. 5.1.

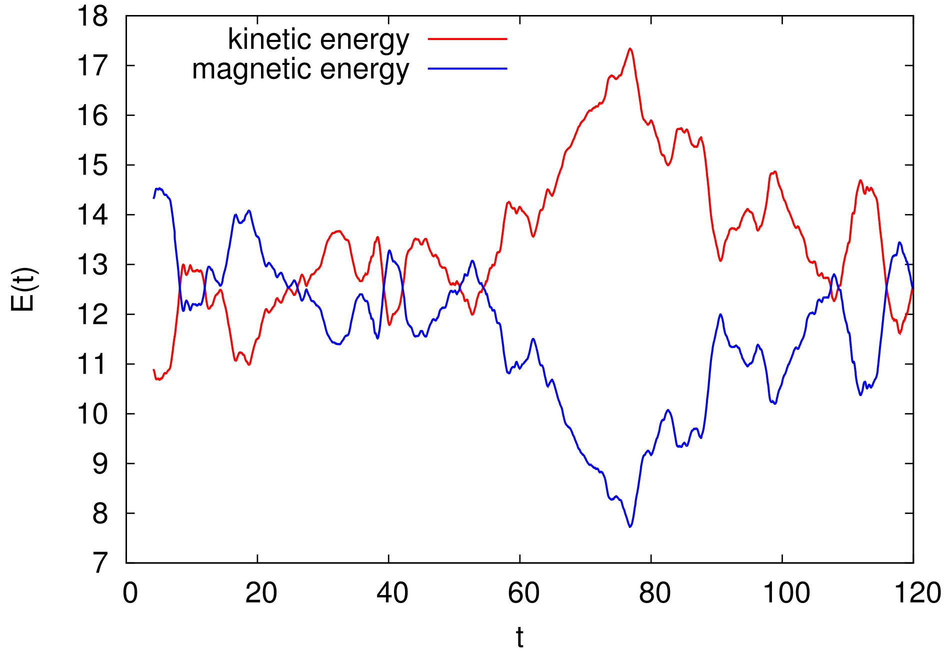
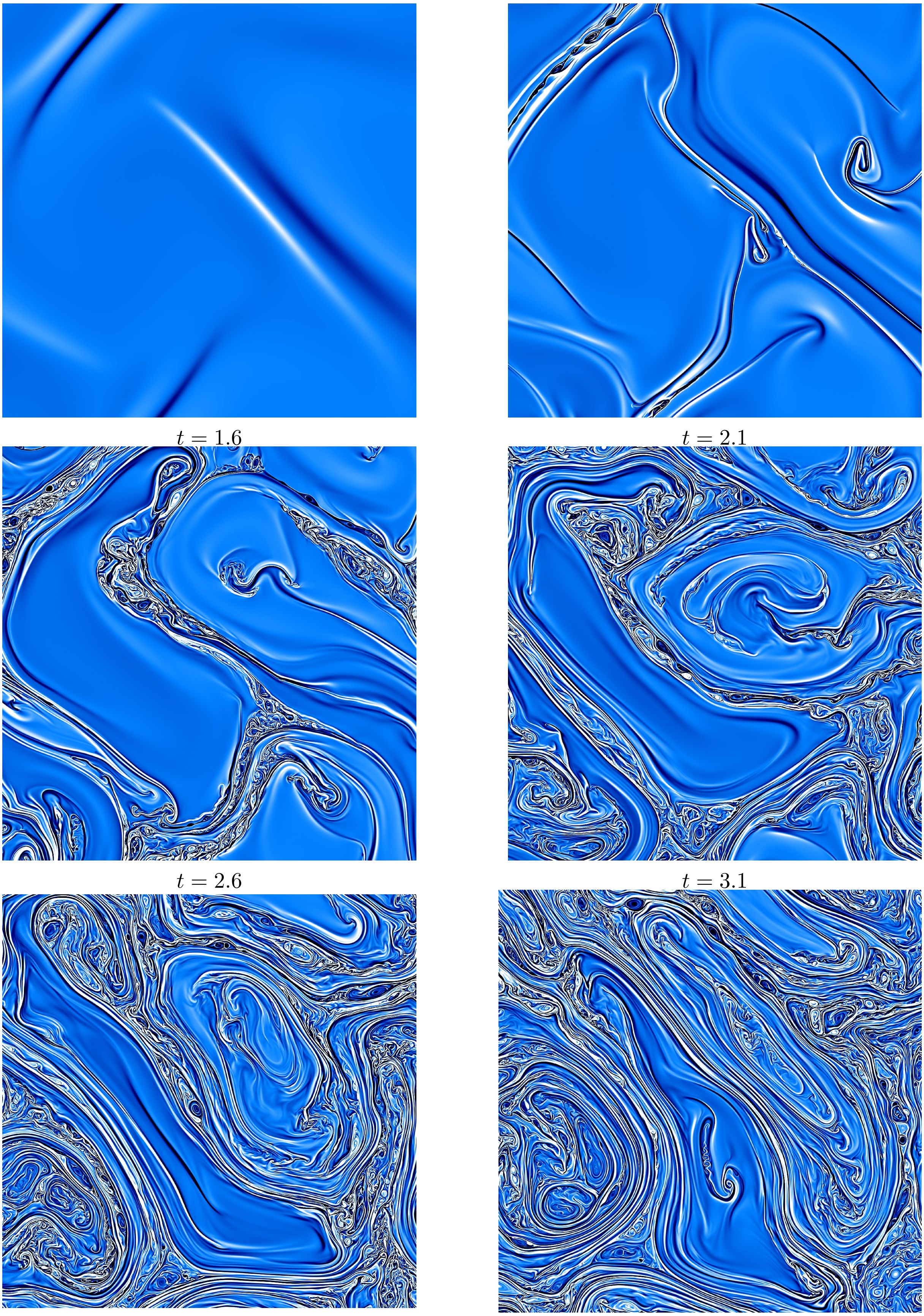


Figure 5.1: The temporal evolution of the kinetic and magnetic energy.

The observed curves reveal a chaotic behavior and the occurrence of reversals, where either the kinetic or the magnetic energy becomes dominant. In total, 10 reversal events can be counted. Furthermore an extreme increase of the kinetic energy can be seen after $t \approx 55$. Whether this increase is caused by the forcing in the vorticity equation, or if it is a fundamental signature of 2D MHD turbulence, remains an open question. Furthermore, the chaotic time series has to be evaluated in the context of dynamic systems theory.
In the following, the calculation methods for the energy spectra and the characteristic scales in the simulations are described.



$t = 1.6$ $t = 2.1$

$t = 2.6$ $t = 3.1$

$t = 3.6$ $t = 4.1$

Figure 5.2: Temporal evolution of the current density $j(\mathbf{x}, t)$ for six times t.

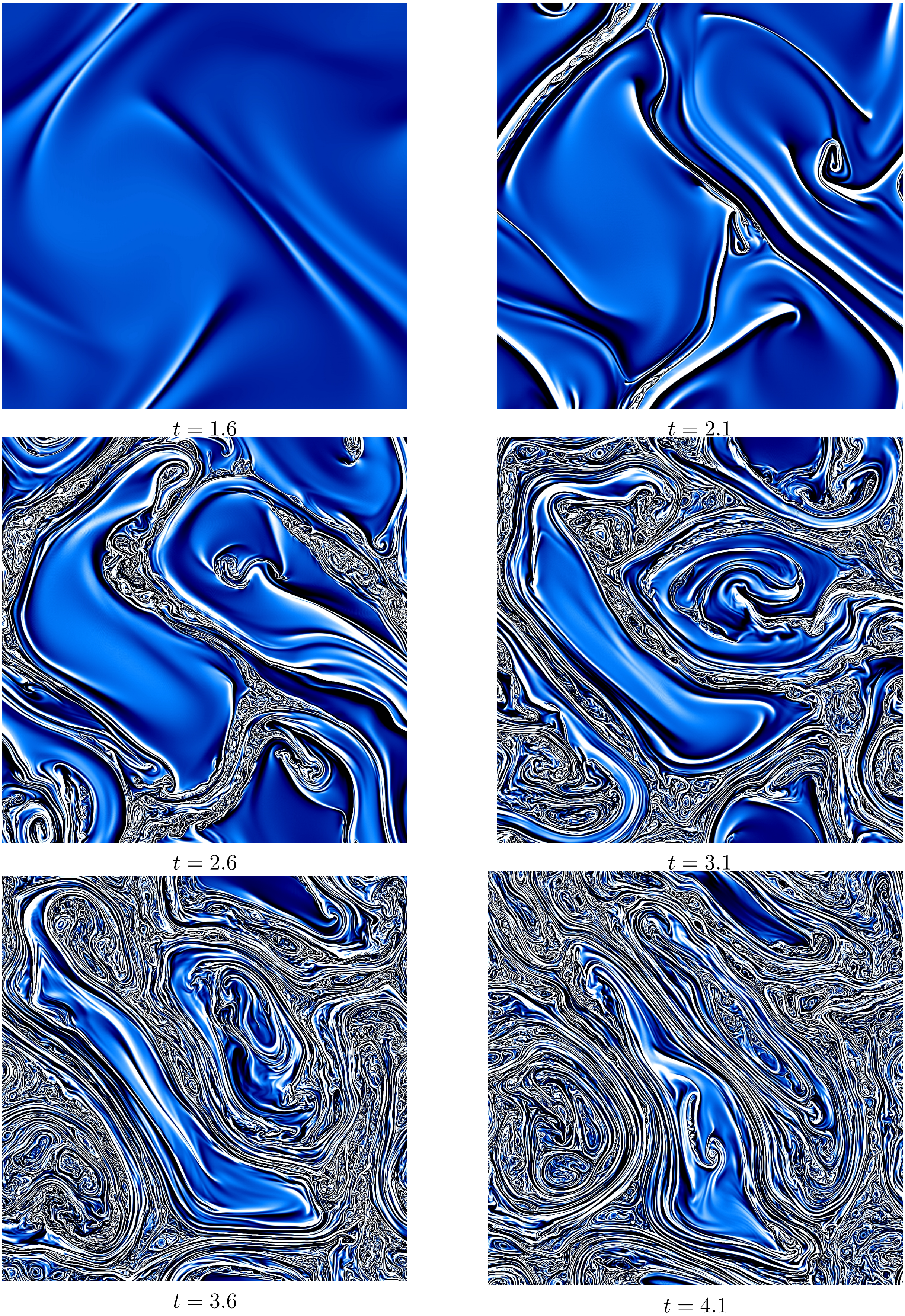


Figure 5.3: Temporal evolution of the vorticity $\omega(\mathbf{x}, t)$ for six times t.

5.3.1 Energy spectra and energy dissipation rate

Under the assumption of isotropy, the radial energy spectra are calculated in a spherical shell of the energy sphere according to

$$E_{kin}(k,t) = \frac{1}{2} \sum_{k-\frac{1}{2}<k'<k+\frac{1}{2}} k'^2 |\phi(k',t)|^2, \tag{5.29}$$

$$E_{mag}(k,t) = \frac{1}{2} \sum_{k-\frac{1}{2}<k'<k+\frac{1}{2}} k'^2 |\psi(k',t)|^2. \tag{5.30}$$

The averaged rate of energy dissipation in the case of hyperviscosity is

$$\langle \varepsilon_{kin} \rangle = \nu^{(2)} \langle (\nabla^2 \mathbf{u})^2 \rangle \qquad \text{and} \qquad \langle \varepsilon_{mag} \rangle = \lambda^{(2)} \langle (\nabla^2 \mathbf{H})^2 \rangle. \tag{5.31}$$

These quantities were calculated in Fourier space as

$$\langle \varepsilon_{kin} \rangle = \frac{\nu^{(2)}}{A} \sum_{\mathbf{k}} (k^2 \mathbf{u}(\mathbf{k},t))^2 \qquad \text{and} \qquad \langle \varepsilon_{kin} \rangle = \frac{\lambda^{(2)}}{A} \sum_{\mathbf{k}} (k^2 \mathbf{H}(\mathbf{k},t))^2, \tag{5.32}$$

where A is the area of the 2D simulation, in that case $A = 4\pi^2$.

5.3.2 Characteristic length scales and hyperviscosity

In order to obtain useful statistical statements from the direct numerical simulations, the calculation of the characteristic length scales mentioned in 3.4 is indispensable. Since we deal with hyperviscosity and hyperresistivity, the unities of $\nu^{(\gamma)}$ and $\lambda^{(\gamma)}$ are somewhat different. For our case, $\gamma = 2$, we get

$$[\nu^{(2)}] = \frac{\mathrm{m}^4}{\mathrm{s}} \qquad \text{and} \qquad [\lambda^{(2)}] = \frac{\mathrm{m}^4}{\mathrm{s}}. \tag{5.33}$$

The unity of the averaged rate of energy dissipation remains $\frac{\mathrm{m}^2}{\mathrm{s}^3}$. As a consequence, the quantities in chapter 3.4 that depend on $\nu^{(1)}$ and $\lambda^{(1)}$, have a different form for hyperviscosity. The Kolmogorov scales for $\gamma = 2$ and Pm= 1 thus read

$$\eta = \left(\frac{(\nu^{(2)})^3}{\langle \varepsilon \rangle} \right)^{\frac{1}{10}}, \qquad u_\eta = \left(\langle \varepsilon \rangle^3 \nu^{(2)} \right)^{\frac{1}{10}}, \qquad \text{and} \quad \tau_\eta = \left(\frac{\nu^{(2)}}{\langle \varepsilon \rangle^2} \right)^{\frac{1}{5}}, \tag{5.34}$$

where $\langle \varepsilon \rangle = \langle \varepsilon_{kin} \rangle + \langle \varepsilon_{mag} \rangle$ is the total energy dissipation.
The Taylor scale λ is given as

$$\lambda = \left(\frac{\nu^{(2)} E}{\langle \varepsilon \rangle} \right)^{\frac{1}{4}}. \tag{5.35}$$

For Pm= 1, these length scales can be considered as universal length scales in MHD turbulence, since they depend essentially on the total energy E and the corresponding energy dissipation rate $\langle \varepsilon \rangle = \langle \varepsilon_{kin} \rangle + \langle \varepsilon_{kin} \rangle$.
Therefore the integral length scale reads

$$L = \frac{\left(\frac{1}{2} (u_{rms}^2 + h_{rms}^2) \right)^{\frac{3}{2}}}{\langle \varepsilon \rangle}, \tag{5.36}$$

where u_{rms} and h_{rms} are the root mean squared velocity and magnetic field.
The so-called large-eddy turn-over time can thus be defined as

$$T_L = \frac{L}{u_{rms}}. \tag{5.37}$$

Furthermore, the corresponding macro- and microscale Reynolds numbers read

$$\mathrm{Re}_L = \frac{u_{rms}L^3}{\nu^{(2)}} \qquad \text{and} \qquad \mathrm{Re}_\lambda = \frac{u_{rms}\lambda^3}{\nu^{(2)}}. \tag{5.38}$$

5.3.3 Structure functions in 2D MHD turbulence

The structure functions from the numerical simulations discussed in the preceding section are evaluated over several large-eddy turn-over times (≈ 19). It is necessary to average over such long time spans, in order to include such rare events, as the strong increase of the kinetic energy that is observed in fig. 5.1. Statistical stationarity is thereby attained by the energy conserving forcing mechanism, described in section 5.2.4. The characteristic parameters of the simulations are listed in table 5.1.

Re_λ	u_{rms}	h_{rms}	$\nu^{(2)} = \lambda^{(2)}$	dx	η	τ_η	L	T_L	N^2
110-170	0.831	0.759	$2 \cdot 10^{-10}$	$2,05 \cdot 10^{-3}$	$1,64 \cdot 10^{-3}$	0.037	4.741	5.705	$(3072)^2$

Table 5.1: Characteristic parameters of the numerical simulations: Taylor-Reynolds number $\mathrm{Re}_\lambda = \frac{u_{rms}\lambda^3}{\nu^{(2)}}$, root mean square velocity $u_{rms} = \sqrt{\langle \mathbf{u} \rangle^2}$, root mean square magnetic field $h_{rms} = \sqrt{\langle \mathbf{h} \rangle^2}$, dissipation length $\eta = \left(\frac{(\nu^{(2)})^3}{\langle \varepsilon \rangle}\right)^{\frac{1}{10}}$, dissipation time $\tau_\eta = \left(\frac{\nu^{(2)}}{\langle \varepsilon \rangle^2}\right)^{\frac{1}{5}}$, integral length scale $L = \frac{\left(\frac{1}{2}(u_{rms}^2 + h_{rms}^2)\right)^{\frac{3}{2}}}{\langle \varepsilon \rangle}$ and large-eddy turn-over time $T_L = \frac{L}{u_{rms}}$.

The structure functions of even order up to the 8th order are depicted in fig 5.4. The curves in blue stand for the magnetic and the curves in red stand for the velocity structure functions. Focusing on the second order structure functions, one observes that $D^{\mathbf{bb}}_{tt}(r)$ overlaps with $D^{\mathbf{vv}}_{rr}(r)$ in a great part of the inertial range. This is an interesting result, since for now no relation between these two functions could be derived from the MHD equations.
Furthermore, one observes that longitudinal and transverse magnetic structure function approach each other at large r. This is an effect that can be observed at all even orders. In the fourth order, longitudinal magnetic and velocity structure functions overlap each other in the dissipation range. The same is true for the transverse structure functions. In the inertial range the functions cross each other two times, and the fourth order longitudinal (transverse) magnetic structure functions is for the first time larger then the corresponding velocity structure function. Compared to the second order, the magnetic structure functions are thus shifted to the top. This effect increases at the sixth and eighth order. However, the transverse structure functions become rather unsmooth and a more accurate evaluation of the structure functions seems necessary. In the following the scaling behavior in the inertial range for the structure functions up to the sixth order is discussed.

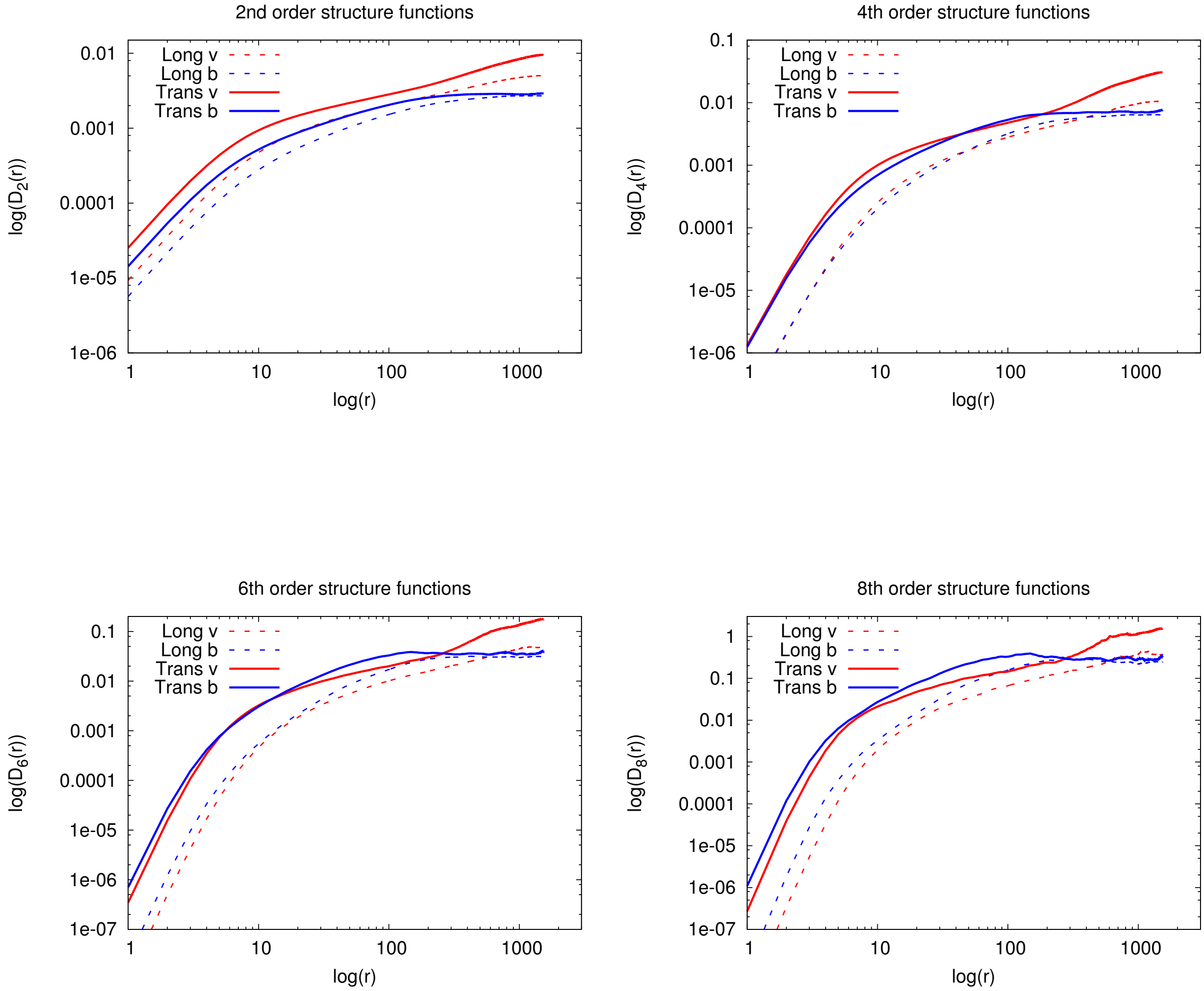


Figure 5.4: Structure functions of even order up to the 8th order. The transverse structure functions are always above the corresponding longitudinal structure functions. Blue curves represent the magnetic structure functions and red curves represent the velocity structure functions.

5.3.4 Scaling behavior of the structure functions

From the Taylor-Proudman theorem of MHD turbulence, discussed in chapter 2.4.3, it follows that 2D MHD turbulence can be considered as a limit case of 3D MHD turbulence in the presence of an external magnetic field. Therefore, the Iroshnikov-Kraichnan scaling seems to be the appropriate choice for the scaling behavior and it was already reported for the case of 2D decaying MHD turbulence [Bis01]. To this end, the longitudinal structure functions of order n up to order six were compensated by a factor r^{ζ_n}. Thereby, a factor $\zeta_n = \frac{n}{4}$ would correspond to the Iroshnikov-Kraichnan phenomenology. The corresponding functions are plotted in fig. 5.5 a) and fig. 5.5 b). Furthermore a 'guessed' power law for each function is plotted. In order to simplify the plots, no functions that are compensated by r^{ζ_n} with $\zeta_n = \frac{n}{3}$ were plotted. It will be seen in the next section that the K 41 / Goldreich-Sridhar phenomenology can be excluded for this simulation.

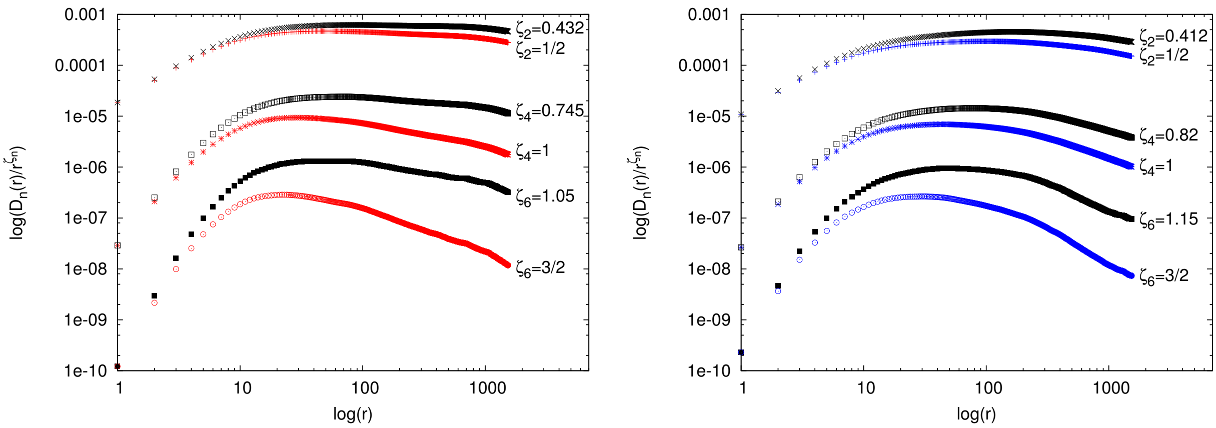


Figure 5.5: Compensated longitudinal a) velocity and b) magnetic structure functions of even order up to the 6th order. The functions were vertically shifted for clarity.

The corresponding guessed exponents are plotted in fig. 5.6. It becomes obvious that they are always inferior to the Iroshnikov-Kraichnan scaling. Whether this is an effect similar to the intermittency that leads to deviations from K41 in hydrodynamic turbulence, or if it is an effect related to the proposed Alfvén wave picture in the Iroshnikov-Kraichnan phenomenology, remains an open question. Furthermore, the difference between the magnetic and velocity structure function exponents seems to increase with the order n, and the magnetic exponents correspond better to the Iroshnikov-Kraichnan scaling for larger n.

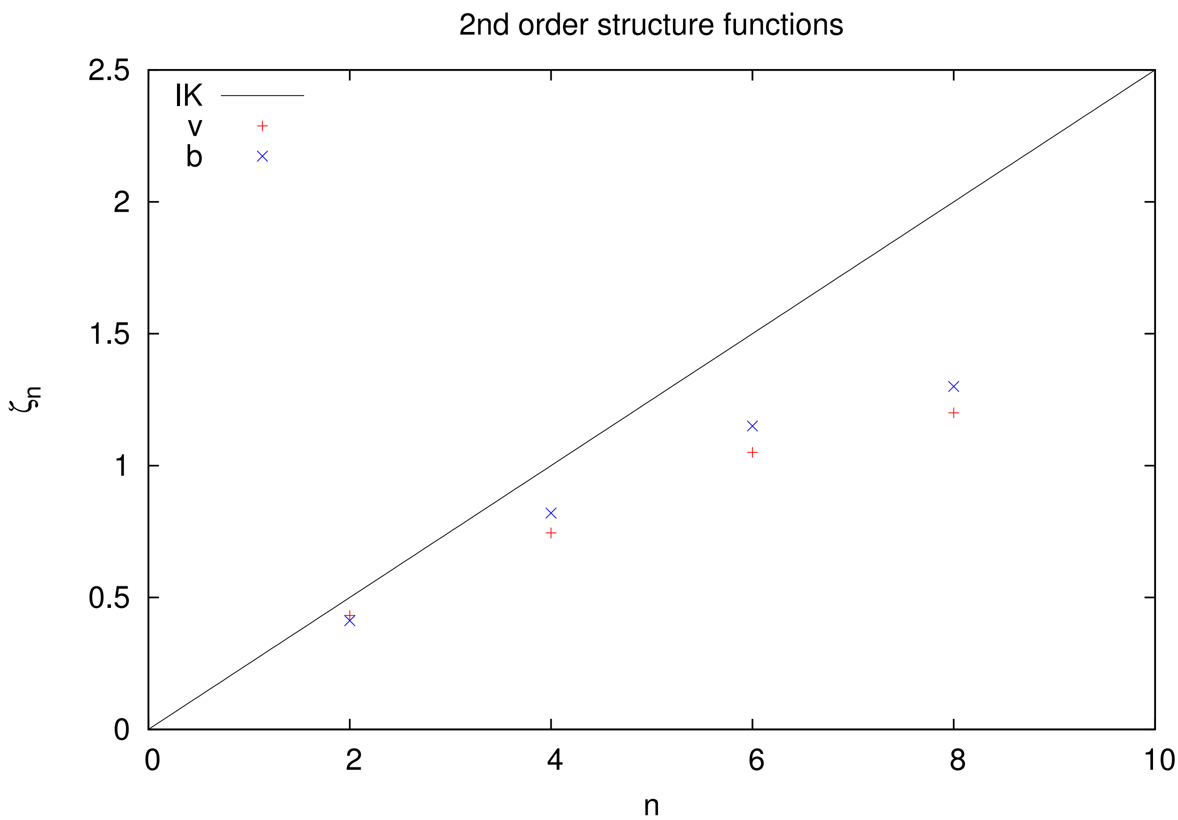


Figure 5.6: Velocity (red) and magnetic (blue) structure function exponents of order n against the n. The straight line corresponds to the Iroshnikov-Kraichnan prediction $\zeta = \frac{n}{4}$.

5.3.5 Evaluation of the energy spectra

The kinetic and magnetic energy spectra were calculated according to (5.29) and (5.30). The time averaged spectra are depicted in fig. 5.7 a) and 5.7 b), where the average was taken over the same time span as for the structure functions in section 5.3.3.

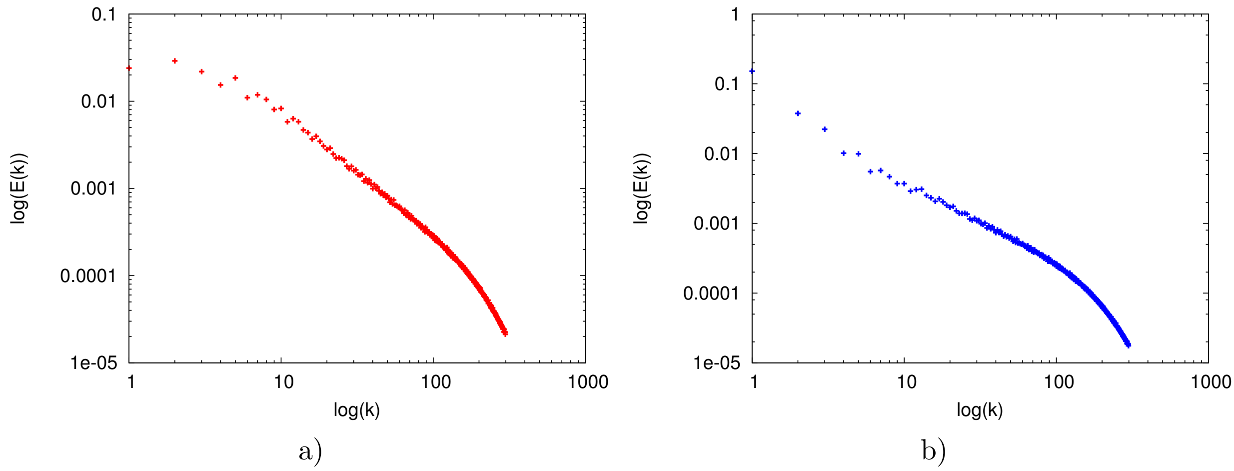


Figure 5.7: The kinetic a) and magnetic b) time averaged energy spectra from the numerical simulations according to table 5.1. A clear power law behavior can be observed in the inertial range.

In order to determine the power law behavior of the energy spectra, the spectra depicted in fig. 5.8 a) and 5.8 b) are compensated by a factor k^a. The exponent $a = -\frac{5}{3}$ thereby corresponds to the Goldreich-Sridhar scaling and the exponent $a = -\frac{3}{2}$ represents the Iroshnikov-Kraichnan scaling. The compensated spectra indicate that the Iroshnikov-Kraichnan theory is more appropriate to explain the spectral properties of this simulation of forced 2D MHD turbulence, then the Goldreich-Sridhar theory does. However, the exponent of $\frac{3}{2}$ is not the best fit and a third plot for a guessed exponent a is depicted in each of the two figures. Furthermore, it seems that the magnetic spectra differs more from the $\frac{3}{2}$ value than the velocity field does. This can depend on the forcing mechanism which is solely applied to the vorticity equation. Therefore, further investigations are necessary, in order to determine if there exists a difference between the scaling exponents in the power law of the kinetic and magnetic energy.

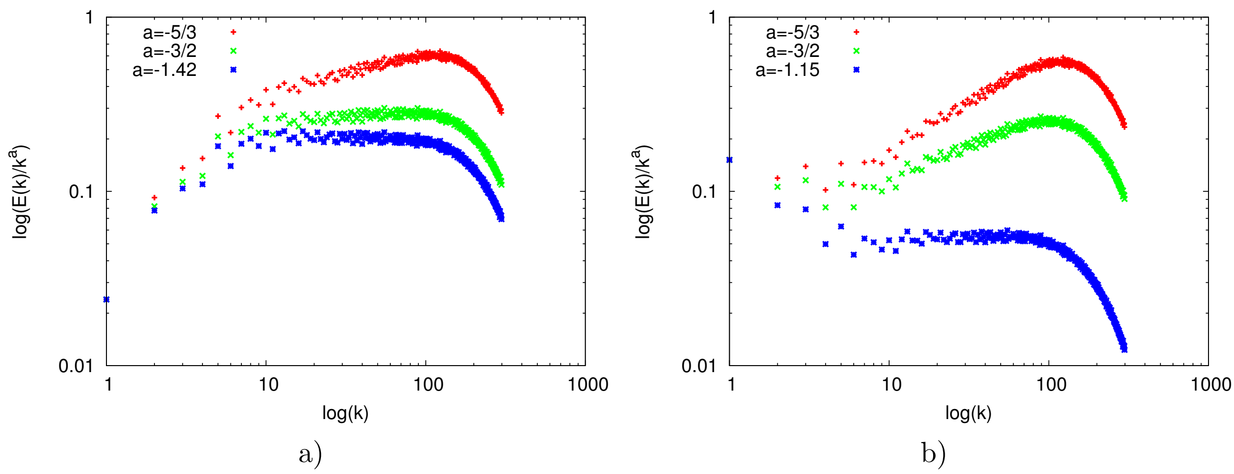


Figure 5.8: The compensated kinetic a) and magnetic b) time averaged energy spectra from the numerical simulations according to table 5.1. The blue curves correspond to guessed exponents a.

5.3.6 The Kármán-Howarth relation in 2D MHD turbulence

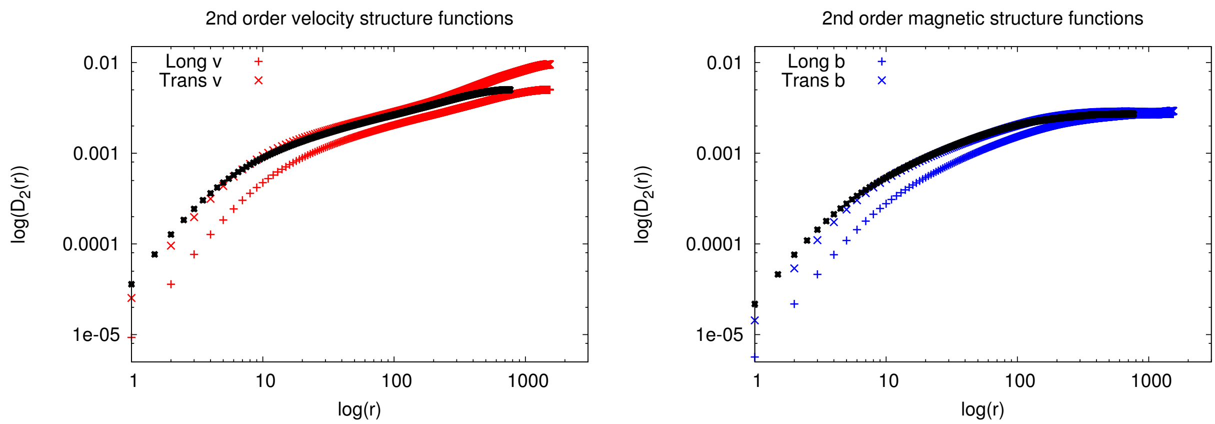
The Kármán-Howarth relations in 2D for the second order velocity and magnetic field structure functions follow from the incompressibility condition in analogy to the 3D case. The procedure is described in the appendix B.2.5.1.

$$D^{\mathbf{vv}}_{t\,t}(r,t) \;=\; \frac{\partial}{\partial r}(rD^{\mathbf{vv}}_{r\,r}(r,t)), \tag{5.39}$$
$$D^{\mathbf{bb}}_{t\,t}(r,t) \;=\; \frac{\partial}{\partial r}(rD^{\mathbf{bb}}_{r\,r}(r,t)). \tag{5.40}$$

Following Siefert and Peinke [Sie04], the right-hand side of (5.39) can be interpreted as the Taylor expansion of the function $D^{\mathbf{vv}}_{r\,r}(r+r,t)$. In this approximation the transverse structure can be expressed as a spatial shift of the longitudinal structure function

$$D^{\mathbf{vv}}_{t\,t}(r,t) \;\approx\; D^{\mathbf{vv}}_{r\,r}(2r,t), \tag{5.41}$$
$$D^{\mathbf{bb}}_{t\,t}(r,t) \;\approx\; D^{\mathbf{bb}}_{r\,r}(2r,t). \tag{5.42}$$

The rescaling relation is now checked for the second order velocity and magnetic structure functions.
In fig. 5.9 a) the rescaled longitudinal velocity structure function $D^{\mathbf{vv}}_{r\,r}(2r,t)$ is depicted in black. It coalesces with the transverse structure function $D^{\mathbf{vv}}_{t\,t}(r,t)$. The same is true for the rescaled longitudinal magnetic structure function in fig. 5.9 b).



a) b)

Figure 5.9: Rescaled a) velocity and b) magnetic structure functions: The black curves are the rescaled longitudinal structure functions according to equation (5.41).

The rescaling relation following from the 2D Kármán-Howarth relation, is therefore appropriate to map the longitudinal functions onto the transverse function. However, one observes small deviations between the black and red curves in 5.9, which can be the effect of higher order terms in the Taylor expansions. In the following the implications for the fourth order structure functions are discussed.

5.3.7 Fourth order structure functions in 2D MHD turbulence

The equations for the fourth order structure functions (4.80) and (4.81) are derived for two dimensions in the appendix B.2.5.4 according to

$$\frac{1}{r}\frac{\partial}{\partial r}\left[r\left(D^{\mathbf{vvvv}}_{rrrr}(r)-D^{\mathbf{bbbb}}_{rrrr}(r)\right)\right]-\frac{3}{r}\left(D^{\mathbf{vvvv}}_{rrtt}(r)-D^{\mathbf{bbbb}}_{rrtt}(r)-D^{\mathbf{vvbb}}_{rr,tt}(r)\right) = -T_{rrr}(r), \quad (5.43)$$

$$\frac{1}{r^3}\frac{\partial}{\partial r}\left[r^3\left(D^{\mathbf{vvvv}}_{rrtt}(r)-D^{\mathbf{bbbb}}_{rrtt}(r)\right)\right]-\frac{1}{r}\left(D^{\mathbf{vvvv}}_{tttt}(r)-D^{\mathbf{bbbb}}_{tttt}(r)\right)+\frac{\partial}{\partial r}D^{\mathbf{vvbb}}_{rr,tt}(r) = -T_{rtt}(r), \quad (5.44)$$

or in other form as

$$\frac{1}{r}\frac{\partial}{\partial r}\left[r\left(\langle v_r v_r v_r v_r\rangle-\langle b_r b_r b_r b_r\rangle\right)\right]-\frac{3}{r}\left(\langle v_r v_r v_t v_t\rangle-\langle b_r b_r b_t b_t\rangle-\langle v_r v_r b_t b_t - v_t v_t b_r b_r\rangle\right) = -T_{rrr}, \quad (5.45)$$

and

$$\frac{1}{r^3}\frac{\partial}{\partial r}\left[r^3\left(\langle v_r v_r v_t v_t\rangle-\langle b_r b_r b_t b_t\rangle\right)\right]-\frac{1}{r}\left(\langle v_t v_t v_t v_t\rangle-\langle b_t b_t b_t b_t\rangle\right)+\frac{\partial}{\partial r}\langle v_r v_r b_t b_t - v_t v_t b_r b_r\rangle = -T_{rtt} \quad (5.46)$$

In the following section, the rescaling relation for 2D hydrodynamics are derived.

5.3.7.1 The Yakhot-Hill equations and the rescaling relation for 2D hydrodynamic turbulence

If there is no magnetic field present, the equations (5.43) and (5.44) read

$$\frac{1}{r}\frac{\partial}{\partial r}\left[rD^{\mathbf{vvvv}}_{rrrr}(r)\right]-\frac{3}{r}D^{\mathbf{vvvv}}_{rrtt}(r) = -T_{rrr}(r), \quad (5.47)$$

$$\frac{1}{r^3}\frac{\partial}{\partial r}\left[r^3 D^{\mathbf{vvvv}}_{rrtt}(r)\right]-\frac{1}{r}D^{\mathbf{vvvv}}_{tttt}(r) = -T_{rtt}(r). \quad (5.48)$$

These equations are the Yakhot-Hill equations for 2D hydrodynamic turbulence. Following [Gra12], we are able to derive a rescaling relation for 2D in analogy to the

$$3D^{\mathbf{vvvv}}_{rrtt}(r) \approx D^{\mathbf{vvvv}}_{rrrr}(r)+r\frac{\partial}{\partial r}D^{\mathbf{vvvv}}_{rrrr}(r)\approx D^{\mathbf{vvvv}}_{rrrr}(2r), \quad (5.49)$$

$$\frac{1}{3}D^{\mathbf{vvvv}}_{tttt}(r) \approx D^{\mathbf{vvvv}}_{rrtt}(r)+\frac{r}{3}\frac{\partial}{\partial r}D^{\mathbf{vvvv}}_{rrtt}(r)\approx D^{\mathbf{vvvv}}_{rrtt}\left(\frac{4}{3}r\right), \quad (5.50)$$

(5.51)

which yields the rescaling between fourth order transverse and longitudinal structure function

$$D^{\mathbf{vvvv}}_{tttt}(r)\approx D^{\mathbf{vvvv}}_{rrrr}\left(2\frac{4}{3}r\right). \quad (5.52)$$

It is tempting to introduce the rescaling relation for a transverse structure function of order n in 2D according to

$$D^{n\mathbf{v}}_{nt}(r)\approx D^{n\mathbf{v}}_{nr}\left(2\frac{4}{3}\frac{6}{5}\cdots\frac{n}{n-1}\right)=D^{n\mathbf{v}}_{nr}\left(\frac{2^n\Gamma^2(n/2+1)}{\Gamma(n+1)}r\right) \quad (5.53)$$

5.3.7.2 The implications of the fourth order equations and the rescaling relation for 2D MHD turbulence

In analogy to the previous section, one can derive a rescaling relation for the fourth order moments in MHD turbulence. This is done in neglecting the antisymmetric tensor $D^{\mathbf{vvbb}}_{rr,tt}(r)$ and the pressure contributions in (5.43) and (5.44). One therefore obtains the rescaling relation

$$D^{\mathbf{vvvv}}_{tttt}(r) - D^{\mathbf{bbbb}}_{tttt}(r) = D^{\mathbf{vvvv}}_{rrrr}\left(2\frac{4}{3}r\right) - D^{\mathbf{bbbb}}_{rrrr}\left(2\frac{4}{3}r\right). \tag{5.54}$$

However, in considering the fourth order structure functions depicted in fig. 5.10, one concludes that magnetic and structure function overlap in a great part of the inertial range.

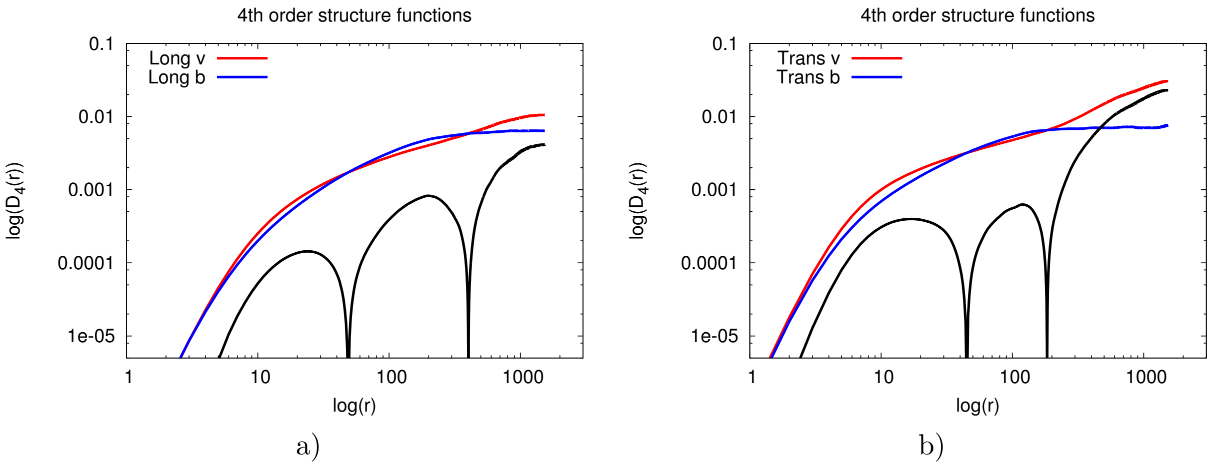


Figure 5.10: a) Fourth order longitudinal structure functions. $D^{\mathbf{vvvv}}_{rrrr}(r)$ is plotted in red, $D^{\mathbf{bbbb}}_{rrrr}(r)$ in blue and the difference $\left|D^{\mathbf{vvvv}}_{rrrr}(r) - D^{\mathbf{bbbb}}_{rrrr}(r)\right|$ in black.
b) Fourth order transverse structure functions. $D^{\mathbf{vvvv}}_{tttt}(r)$ is plotted in red, $D^{\mathbf{bbbb}}_{tttt}(r)$ in blue and the difference $\left|D^{\mathbf{vvvv}}_{tttt}(r) - D^{\mathbf{bbbb}}_{tttt}(r)\right|$ in black.

As it can be seen, the differences $D^{\mathbf{vvvv}}_{rrrr}(r) - D^{\mathbf{bbbb}}_{rrrr}(r)$ that are depicted in black, tend to zero for two times in the inertial range. This is an interesting result, since it implies an equipartition solution for the fourth order moments

$$\langle v_r v_r v_r v_r\rangle \approx \langle b_r b_r b_r b_r\rangle \qquad \text{and} \qquad \langle v_t v_t v_t v_t\rangle \approx \langle b_t b_t b_t b_t\rangle. \tag{5.55}$$

Furthermore, as it was shown in section 2.57 of chapter 2, the pressure contributions vanish for the equipartition solution and one might conclude that $T_{rrr}(r)$ and $T_{rtt}(r)$ are zero. In these regions of the inertial range, the equipartition solution of the fourth order moments can therefore be considered as a trivial solution of the equations (5.43) and (5.44).
However, the physical reasons for the observed equipartition solution remain uncertain and further investigations mainly of 3D MHD turbulence seem necessary. To this end, the meaning of the antisymmetric tensor $D^{\mathbf{vvbb}}_{rr,tt}(r)$ and its relation to the magnetic pressure has to be understood, since it introduces a new scaling behavior in equation (5.43) and (5.44).
In order to check the rescaling relation from the previous section, we focus on the magnetic and velocity structure functions only, and not on their differences, so that

$$D^{\mathbf{vvvv}}_{tttt}(r) = D^{\mathbf{vvvv}}_{rrrr}\left(2\frac{4}{3}r\right) \qquad \text{and} \qquad D^{\mathbf{bbbb}}_{tttt}(r) = D^{\mathbf{bbbb}}_{rrrr}\left(2\frac{4}{3}r\right). \tag{5.56}$$

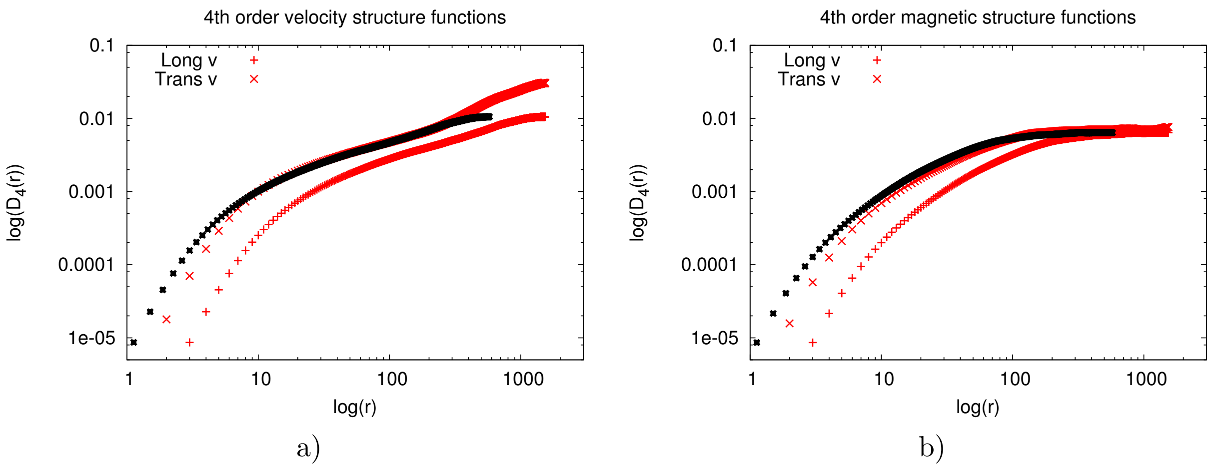


Figure 5.11: a)$D^{\mathbf{vvvv}}_{r\,r\,r\,r}(r)$ and $D^{\mathbf{vvvv}}_{t\,t\,t\,t}(r)$ and the rescaled curve in black according to equation (5.56). b) $D^{\mathbf{vbbb}}_{r\,r\,r\,r}(r)$ and $D^{\mathbf{bbbb}}_{t\,t\,t\,t}(r)$ and the rescaled curve in black according to equation (5.56)

The rescaling relation seems therefore to be fulfilled independently for each of the functions in equation (5.54).
In order to determine the influence of the pressure terms, each of the terms on the left -hand side in (5.43) and (5.44) are investigated. The plot for the longitudinal equation (5.43) is depicted in fig. 5.12 a). The longitudinal pressure contributions $rT_{rrr}(r)$ are represented by the black curve and reveal the question of a scaling behavior. The mixed pressure contributions $rT_{rtt}(r)$ in5.12 b) do not show a particular scaling behavior and saturate for larger r.

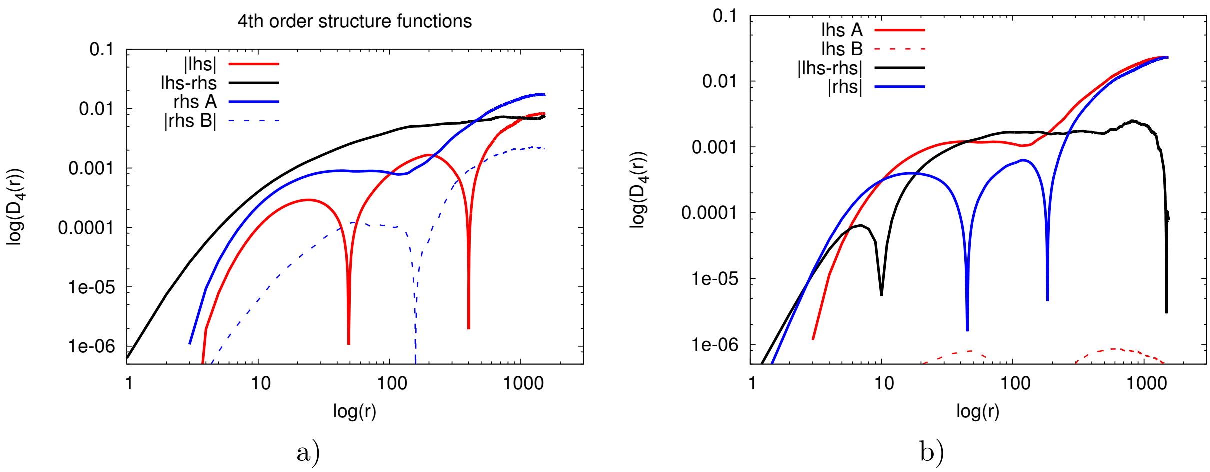


Figure 5.12: a) Longitudinal equation (5.43) in the form $\frac{\partial}{\partial r}\left[r\left(D^{\mathbf{vvvv}}_{r\,r\,r\,r}(r) - D^{\mathbf{bbbb}}_{r\,r\,r\,r}(r)\right)\right] = 3\left(D^{\mathbf{vvvv}}_{r\,r\,t\,t}(r) - D^{\mathbf{bbbb}}_{r\,r\,t\,t}(r) - D^{\mathbf{vvbb}}_{r\,r,t\,t}(r)\right)$. The left-hand side of this equation is thereby depicted in red. The right-hand side A corresponds to $D^{\mathbf{vvvv}}_{r\,r\,t\,t}(r) - D^{\mathbf{bbbb}}_{r\,r\,t\,t}(r)$ and the right-hand side B to the antisymmetric contribution $D^{\mathbf{vvbb}}_{r\,r,t\,t}(r)$. The pressure contributions $rT_{rrr}(r)$ can be read off from the black curve that stands for the difference between the left-hand side and the right-hand side.
b) Transverse equation (5.44) in the form $\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^3 D^{\mathbf{vvvv}}_{r\,r\,t\,t}(r)\right] = D^{\mathbf{vvvv}}_{t\,t\,t\,t}(r)$. The left-hand side A thereby corresponds to the term $\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^3 D^{\mathbf{vvvv}}_{r\,r\,t\,t}(r)\right]$ and the left-hand side B to the antisymmetric contribution $r\frac{\partial}{\partial r}D^{\mathbf{vvbb}}_{r\,r,t\,t}(r)$. The pressure contributions $rT_{rtt}(r)$ can be seen from the black curve.

These plots reveal the question whether the longitudinal pressure contribution can be modeled in a similar way as in [Got02]. This would be of great interest for the magnetic pressure contributions, since they already possess a quadratic form. In the last section, we want to focus on a relation for the third order longitudinal velocity structure function.

5.3.8 The 3/2 law in 2D MHD turbulence

In analogy to the 4/5 law in 3D MHD turbulence, a relation for the third order velocity structure function can be derived for 2D MHD turbulence, which is done in the appendix B.2.5.3. In the inertial range one obtains

$$D^{\mathbf{vvv}}_{r\,r\,r}(r) - 12C^{\mathbf{hhu}}_{t\,t\,r}(r) - \frac{24}{r^3}\int_0^r \mathrm{d}r'\; r'^2 C^{\mathbf{uhh}}(r') = \frac{3}{2}\left\langle \varepsilon^{\mathbf{v}} + \varepsilon^{\mathbf{b}} \right\rangle .$$

The third order longitudinal velocity structure function is shown in fig. 5.13. In contrast to the even order moments, discussed in the previous sections, the odd order moments are close to zero, and it becomes difficult to obtain useful statistical data. This is due to the fact, that all moments have only small deviations from Gaussianity.

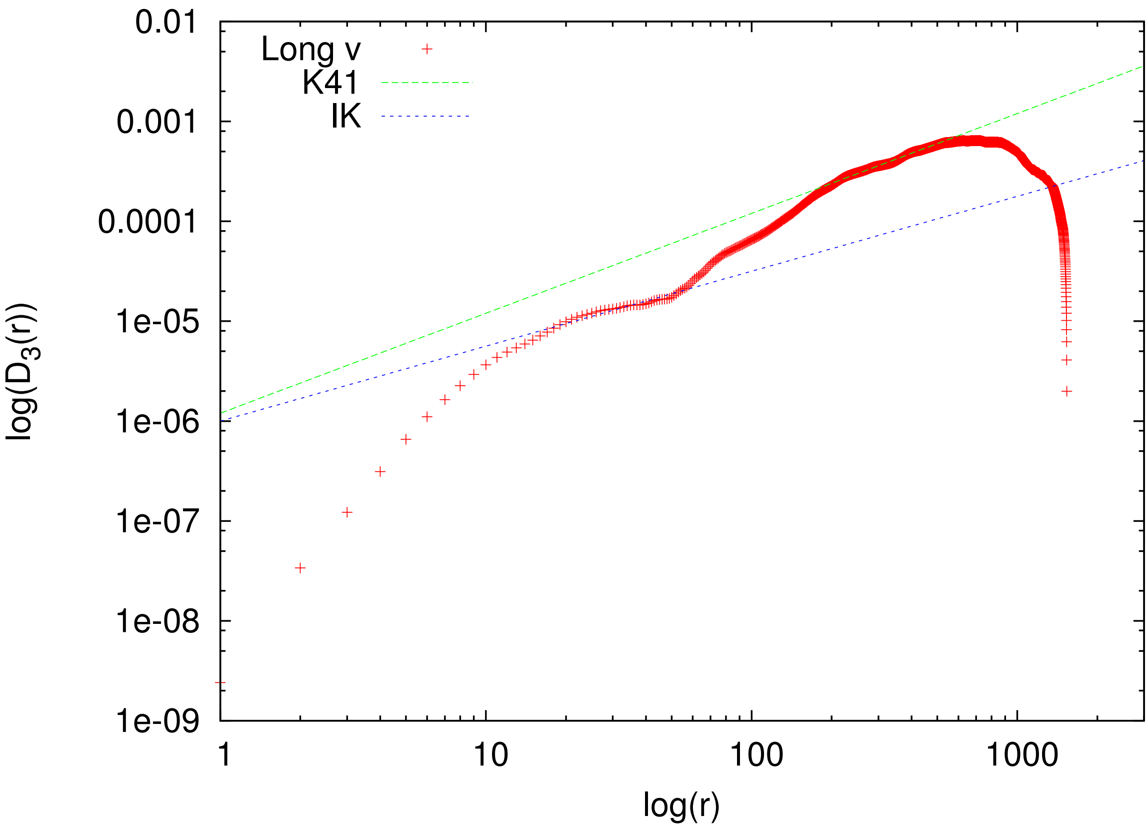


Figure 5.13: The third order longitudinal velocity structure function from the simulations in table 5.1. The lines $\sim r$ and $\sim r^{\frac{3}{4}}$ are plotted as a comparison. The former one corresponds to the K41 phenomenology that coincides with the Goldreich-Sridhar scaling, while the latter one corresponds to the Iroshnikov-Kraichnan phenomenology.

The interpretation of (5.57) is therefore rather complicated: First-of-all, we notice that no clear power-law behavior can be observed in the inertial range, since the slope increases from $\approx r^{3/4}$ to r in the middle of the inertial range. However, it is not clear if this is a characteristic feature of 2D MHD turbulence, or if it is a limitation of the numerical simulations. It is therefore necessary to investigate the third order structure function over longer time intervals and with

a higher spatial resolution. If these investigations reveal similar results, one might argue that the correlation functions that enter in (5.57), mainly $C^{\mathbf{uhh}}(r)$ are dominant on small scales and lead to deviations from the K 41 scaling.
In any case it seems satisfying to obtain a third order longitudinal velocity structure function that shows some kind of scaling, in contrast to the third order longitudinal magnetic structure function that is depicted in fig. C.1 in the appendix C.1. This might result from the fact that no 3/2 law exists for a third order longitudinal magnetic structure function. In addition, it might be argued that such a function can even not exist, since the third order magnetic moment lacks mirror symmetry and therefore $\langle b_i b_j b_k \rangle = 0$.

Chapter 6

Conclusion

An extension of the statistical description of the MHD equations was presented in this thesis. To this end, the tensor calculus introduced by Chandrasekhar [Cha51] for isotropic MHD turbulence was generalized to the case of locally isotropic MHD turbulence. For this purpose, Chandrasekhar's paper about the invariant theory of isotropic MHD turbulence was discussed from a modern point of view in chapter 3.4.4.

A hierarchy of structure function equations was derived in chapter 4. The hierarchy starts with the evolution equations for the energy and the cross helicity tensor. The 4/5 law for MHD turbulence was derived in section 4.2.1.1 and the implications of an additional term were discussed. It was stressed that the theory of locally isotropic MHD turbulence can show similarities to the Iroshnikov-Kraichnan theory. However, further investigations in this direction seem necessary to obtain more information about the nonlocal influence of the large-scale magnetic field on the small scale fluctuations. A starting point would be to write the evolution equation of the increments in terms of the Elsässer fields. In analogy to [Bel87], a change of variables to a frame of reference that moves either parallel *or* anti-parallel to the large scale magnetic field $\mathbf{H}(\mathbf{x}, \mathbf{x}', t)$ leads to four different evolution equations for the Elsässer fields. The resulting moment equations should show different features than the moment equations derived in this work. The derivation of exact relations from this approach would mean a step forward in the direction of a mean field theory for MHD turbulence, similar to the K41 theory.

Turning to the next order equations derived in section 4.3, it can be generalized to moment equations of order n. The influence of the antisymmetric tensor, especially on the rescaling relation remains to be evaluated. However, one can conclude from the numerical simulations that its contributions should remain rather small. Furthermore, the implications of the magnetic pressure should be discussed in analogy to the role of pressure in hydrodynamic turbulence [Got02].

The implications of a Gaussian distribution of the velocity moments for the next order equation in hydrodynamic turbulence were discussed in section 4.3.2. Whereas the description is not appropriate for the longitudinal velocity structure functions, it seems tempting to search for a Gaussian description of the mixed and transverse structure functions. A good starting point for this purpose seems to be the case of 2D turbulence in the context of the works of [Yak99] and [Frie10]. Therefore a modification of the Biot-Savart law seems to be appropriate to take the deviations of the longitudinal structure functions into account.

The other emphasis of this thesis was the numerical simulation of 2D MHD turbulence. For this purpose, a pseudo-spectral code was rewritten for the 2D MHD equations. The forcing mechanism that conserves the total energy seems to be appropriate to attain statistical isotropy of the turbulent flow. However, it remains to check if the structure functions are really invariant under rotations or if there exists a distinguished direction, probably introduced by the magnetic field.

Since the 2D MHD equations are characterized by the thinning of current sheets, hyperviscosity was chosen in order to damp the high gradients that occur. This extended the inertial range in a significant way. The obtained energy spectra pointed in the direction of the Iroshnikov-Kraichnan theory. However, the scaling exponents of the structure functions did not reproduce exactly the predicted scaling exponents. This is not astonishing, since the Iroshnikov-Kraichnan itself is not an exact theory.
The translation of the results from the section 4.3 to two spatial dimensions was performed and checked by the numerical data. The rescaling relation were used to map longitudinal structure functions onto transverse structure functions. However, it remains an open question how the pressure contribution can be modeled in the fourth order equation, and further investigations seem necessary.

Appendix A

A.1 The Biot-Savart law

Due to the incompressibility condition (2.48), we can express the velocity field $\mathbf{u}(\mathbf{x}, t)$ as the sum of a gradient field and the curl of a vector potential $\mathbf{A}(\mathbf{x}, t)$[1]

$$\mathbf{u}(\mathbf{x}, t) = \nabla\phi(\mathbf{x}, t) + \nabla \times \mathbf{A}(\mathbf{x}, t), \tag{A.1}$$

where the scalar field $\phi(\mathbf{x}, t)$ has to satisfy the Laplace equation

$$\nabla^2\phi(\mathbf{x}, t) = 0, \tag{A.2}$$

and the vector potential has to satisfy the Poisson equation[2]

$$\nabla^2\mathbf{A}(\mathbf{x}, t) = -\boldsymbol{\omega}(\mathbf{x}, t). \tag{A.3}$$

We can formally solve this equation by making use of the Green's function of the Laplacian $G(\mathbf{x} - \mathbf{x}')$, which satisfies

$$\nabla^2 G(\mathbf{x} - \mathbf{x}') = -\delta(\mathbf{x} - \mathbf{x}'). \tag{A.4}$$

The integrated form of (A.3) therefore reads

$$\mathbf{A}(\mathbf{x}, t) = \int \mathrm{d}\mathbf{x}' G(\mathbf{x} - \mathbf{x}')\boldsymbol{\omega}(\mathbf{x}', t). \tag{A.5}$$

The Green's functions of the Laplacian in an infinite two-, respectively three-dimensional space read

$$G(\mathbf{x} - \mathbf{x}') = \begin{cases} -\frac{1}{2\pi} \ln|\mathbf{x} - \mathbf{x}'| & \text{for } \mathbf{x} \in \mathbb{R}^2 \\ \frac{1}{4\pi}\frac{1}{|\mathbf{x}-\mathbf{x}'|} & \text{for } \mathbf{x} \in \mathbb{R}^3 \end{cases} \tag{A.6}$$

The velocity field in equation (A.1) thus reads

$$\mathbf{u}(\mathbf{x}, t) = \nabla_{\mathbf{x}}\phi(\mathbf{x}, t) + \nabla_{\mathbf{x}} \times \int \mathrm{d}\mathbf{x}' G(\mathbf{x} - \mathbf{x}')\boldsymbol{\omega}(\mathbf{x}', t). \tag{A.7}$$

If there is no gradient field present, we can express the velocity field by Biot-Savart's law as

$$\mathbf{u}(\mathbf{x}, t) = \int \mathrm{d}\mathbf{x}'\ \boldsymbol{\omega}(\mathbf{x}', t) \times \mathbf{K}(\mathbf{x} - \mathbf{x}'), \tag{A.8}$$

[1] This vector potential is not to be confused with the vector potential $\mathbf{A}(\mathbf{x}, t)$ defining the magnetic field $\mathbf{H}(\mathbf{x}, t) = \nabla \times \mathbf{A}(\mathbf{x}, t)$.

[2] This can be seen by taking the curl of (A.1)
$\nabla \times [\nabla \times \mathbf{A}(\mathbf{x}, t)] = \nabla(\nabla \cdot \mathbf{A}(\mathbf{x}, t)) - \nabla^2\mathbf{A}(\mathbf{x}, t) = \omega(\mathbf{x}, t)$. Since we possess a certain liberty in the choice of the vector potential, it can be assumed for convenience that it is incompressible and we arrive at (A.3).

where we have introduced the quantity

$$\mathbf{K}(\mathbf{x}-\mathbf{x}') = -\nabla_{\mathbf{x}} G(\mathbf{x}-\mathbf{x}'). \tag{A.9}$$

According to (A.6), we obtain

$$\mathbf{K}(\mathbf{x}-\mathbf{x}') = \begin{cases} \frac{1}{2\pi}\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^2} & \text{for } \mathbf{x}\in\mathbb{R}^2 \\ \frac{1}{4\pi}\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3} & \text{for } \mathbf{x}\in\mathbb{R}^3 \end{cases} \tag{A.10}$$

A.2 Balance equations for the conserved quantities in MHD turbulence

In the following we derive the balance equations, used in the section about conserved quantities in MHD turbulence.

A.2.1 The balance equation for the total energy

We multiply (2.13) by $\mathbf{u}(\mathbf{x},t)$ for $\rho = 1$ and (2.14) by $\mathbf{h}(\mathbf{x},t)$. This yields

$$\begin{aligned} & \frac{1}{2}\frac{\partial}{\partial t}\mathbf{u}^2(\mathbf{x},t) + \mathbf{u}(\mathbf{x},t)\cdot[\mathbf{u}(\mathbf{x},t)\cdot\nabla\mathbf{u}(\mathbf{x},t)] - \mathbf{u}(\mathbf{x},t)\cdot[\mathbf{h}(\mathbf{x},t)\cdot\nabla\mathbf{h}(\mathbf{x},t)] \\ = \; & -\mathbf{u}(\mathbf{x},t)\cdot\nabla\left(p(\mathbf{x},t)+\frac{\mathbf{h}^2(\mathbf{x},t)}{2}\right) + \nu\mathbf{u}(\mathbf{x},t)\cdot\nabla^2\mathbf{u}(\mathbf{x},t), \end{aligned} \tag{A.11}$$

and

$$\begin{aligned} & \frac{1}{2}\frac{\partial}{\partial t}\mathbf{h}^2(\mathbf{x},t) + \mathbf{h}(\mathbf{x},t)\cdot[\mathbf{u}(\mathbf{x},t)\cdot\nabla\mathbf{h}(\mathbf{x},t)] - \mathbf{h}(\mathbf{x},t)\cdot[\mathbf{h}(\mathbf{x},t)\cdot\nabla\mathbf{u}(\mathbf{x},t)] \\ = \; & +\lambda\mathbf{h}(\mathbf{x},t)\cdot\nabla^2\mathbf{h}(\mathbf{x},t), \end{aligned} \tag{A.12}$$

Adding these two equations to one another yields the evolution equation for total energy density $e_{tot} = (\mathbf{x},t)$. The following terms are evaluated by making use of the incompressibility of the velocity and magnetic field

$$\begin{aligned} \mathbf{u}(\mathbf{x},t)\cdot[\mathbf{u}(\mathbf{x},t)\cdot\nabla\mathbf{u}(\mathbf{x},t)] &= \nabla\cdot\left(\mathbf{u}(\mathbf{x},t)\frac{\mathbf{u}^2(\mathbf{x},t)}{2}\right), \\ \mathbf{h}(\mathbf{x},t)\cdot[\mathbf{u}(\mathbf{x},t)\cdot\nabla\mathbf{h}(\mathbf{x},t)] &= \nabla\cdot\left(\mathbf{u}(\mathbf{x},t)\frac{\mathbf{h}^2(\mathbf{x},t)}{2}\right) \\ \mathbf{u}(\mathbf{x},t)\cdot[\mathbf{h}(\mathbf{x},t)\cdot\nabla\mathbf{h}(\mathbf{x},t)], & \\ +\mathbf{h}(\mathbf{x},t)\cdot[\mathbf{h}(\mathbf{x},t)\cdot\nabla\mathbf{u}(\mathbf{x},t)] &= \nabla\cdot\left(\mathbf{u}(\mathbf{x},t)\mathbf{u}(\mathbf{x},t)\cdot\mathbf{h}(\mathbf{x},t)\right), \\ \mathbf{u}(\mathbf{x},t)\cdot\nabla\left(p+\frac{\mathbf{h}^2(\mathbf{x},t)}{2}\right) &= \nabla\cdot\left(\mathbf{u}(\mathbf{x},t)\left(p(\mathbf{x},t)+\frac{\mathbf{h}^2(\mathbf{x},t)}{2}\right)\right). \end{aligned} \tag{A.13}$$

Let us consider the magnetic field as an example for the treatment of the viscous term, the calculation is the same for the velocity field. We get

$$\begin{aligned}
& \lambda \mathbf{h}(\mathbf{x},t)\cdot\nabla^2\mathbf{h}(\mathbf{x},t) \\
= & \frac{1}{2}\lambda\nabla^2\mathbf{h}^2(\mathbf{x},t) - \lambda\sum_{i,j}\left(\frac{\partial h_i(\mathbf{x},t)}{\partial x_j}\right)^2, \\
= & \frac{1}{2}\lambda\nabla^2\mathbf{h}^2(\mathbf{x},t) - \frac{1}{2}\lambda\sum_{i,j}\left(\frac{\partial h_i(\mathbf{x},t)}{\partial x_j}+\frac{\partial h_j(\mathbf{x},t)}{\partial x_i}\right)^2 + \lambda\sum_{i,j}\left(\frac{\partial h_i(\mathbf{x},t)}{\partial x_j}\frac{\partial h_j(\mathbf{x},t)}{\partial x_i}\right), \\
= & \frac{1}{2}\lambda\nabla^2\mathbf{h}^2(\mathbf{x},t) - \varepsilon^\lambda(\mathbf{x},t) + \lambda\nabla\cdot[\mathbf{h}(\mathbf{x},t)\cdot\nabla\mathbf{h}(\mathbf{x},t)]. \qquad \text{(A.14)}
\end{aligned}$$

A.2.2 The balance equation for the cross helicity

We multiply (2.13) by $\mathbf{h}(\mathbf{x},t)$ and (2.14) by $\mathbf{u}(\mathbf{x},t)$ and add the resulting equations together. This reads

$$\begin{aligned}
\frac{\partial}{\partial t}\mathbf{u}(\mathbf{x},t)\cdot\mathbf{h}(\mathbf{x},t) &+ \mathbf{h}(\mathbf{x},t)\cdot[\mathbf{u}(\mathbf{x},t)\cdot\nabla\mathbf{u}(\mathbf{x},t)] - \mathbf{h}(\mathbf{x},t)\cdot[\mathbf{h}(\mathbf{x},t)\cdot\nabla\mathbf{h}(\mathbf{x},t)] \\
&+ \mathbf{u}(\mathbf{x},t)\cdot[\mathbf{u}(\mathbf{x},t)\cdot\nabla\mathbf{h}(\mathbf{x},t)] - \mathbf{u}(\mathbf{x},t)\cdot[\mathbf{h}(\mathbf{x},t)\cdot\nabla\mathbf{u}(\mathbf{x},t)] \\
= &- \frac{1}{\rho}\mathbf{h}(\mathbf{x},t)\cdot\nabla\left(p(\mathbf{x},t)+\rho\frac{\mathbf{h}^2(\mathbf{x},t)}{2}\right) \\
&+ \nu\mathbf{h}(\mathbf{x},t)\cdot\nabla^2\mathbf{u}(\mathbf{x},t) + \lambda\mathbf{u}(\mathbf{x},t)\cdot\nabla^2\mathbf{h}(\mathbf{x},t), \qquad \text{(A.15)}
\end{aligned}$$

and make use of the relation

$$\begin{aligned}
\nabla\cdot[\mathbf{u}(\mathbf{x},t)\times(\mathbf{u}(\mathbf{x},t)\times\mathbf{h}(\mathbf{x},t))] = & \ \mathbf{h}(\mathbf{x},t)\cdot(\mathbf{u}(\mathbf{x},t)\cdot\nabla\mathbf{u}(\mathbf{x},t)) + \mathbf{u}(\mathbf{x},t)\cdot(\mathbf{u}(\mathbf{x},t)\cdot\nabla\mathbf{h}(\mathbf{x},t)), \\
- & \ 2\mathbf{u}(\mathbf{x},t)\cdot(\mathbf{h}(\mathbf{x},t)\cdot\nabla\mathbf{u}(\mathbf{x},t)).
\end{aligned}$$

Inserting this relation for the term $\mathbf{u}(\mathbf{x},t)\cdot(\mathbf{h}(\mathbf{x},t)\cdot\nabla\mathbf{u}(\mathbf{x},t))$ yields

$$\begin{aligned}
\frac{\partial}{\partial t}\mathbf{u}(\mathbf{x},t)\cdot\mathbf{h}(\mathbf{x},t) &+ \frac{1}{2}\mathbf{h}(\mathbf{x},t)\cdot(\mathbf{u}(\mathbf{x},t)\cdot\nabla\mathbf{u}(\mathbf{x},t)) - \mathbf{h}(\mathbf{x},t)\cdot(\mathbf{h}(\mathbf{x},t)\cdot\nabla\mathbf{h}(\mathbf{x},t)), \\
&+ \frac{1}{2}\mathbf{u}(\mathbf{x},t)\cdot(\mathbf{u}(\mathbf{x},t)\cdot\nabla\mathbf{h}(\mathbf{x},t)) + \frac{1}{2}\nabla\cdot[\mathbf{u}(\mathbf{x},t)\times(\mathbf{u}(\mathbf{x},t)\times\mathbf{h}(\mathbf{x},t))], \\
= &- \mathbf{h}(\mathbf{x},t)\cdot\nabla\left(p(\mathbf{x},t)+\frac{\mathbf{h}^2(\mathbf{x},t)}{2}\right) \\
&+ \nu\mathbf{h}(\mathbf{x},t)\cdot\nabla^2\mathbf{u}(\mathbf{x},t) + \lambda\mathbf{u}(\mathbf{x},t)\cdot\nabla^2\mathbf{h}(\mathbf{x},t). \qquad \text{(A.16)}
\end{aligned}$$

The same procedure as in (A.13) can now be performed, and we get for the current density

$$\begin{aligned}
\mathbf{J}^{cross}(\mathbf{x},t) = & \ \mathbf{u}(\mathbf{x},t)\frac{\mathbf{u}(\mathbf{x},t)\cdot\mathbf{h}(\mathbf{x},t)}{2} - \mathbf{h}(\mathbf{x},t)\frac{\mathbf{h}^2(\mathbf{x},t)}{2} + \frac{1}{2}\mathbf{u}(\mathbf{x},t)\times(\mathbf{u}(\mathbf{x},t)\times\mathbf{h}(\mathbf{x},t)) \\
& \ \mathbf{h}(\mathbf{x},t)\left(p(\mathbf{x},t)+\frac{\mathbf{h}^2(\mathbf{x},t)}{2}\right) - \nu A^\dagger\mathbf{h}(\mathbf{x},t) - \lambda B^\dagger\mathbf{u}(\mathbf{x},t). \qquad \text{(A.17)}
\end{aligned}$$

The vector product can be rewritten and we get

$$\begin{aligned}
\mathbf{J}^{cross}(\mathbf{x},t) = & \ [\mathbf{u}(\mathbf{x},t)+\mathbf{h}(\mathbf{x},t)]\frac{\mathbf{u}(\mathbf{x},t)\cdot\mathbf{h}(\mathbf{x},t)}{2} \\
- & \ \mathbf{h}(\mathbf{x},t)\left[\frac{\mathbf{u}^2(\mathbf{x},t)}{2} - p(\mathbf{x},t)\right] - \nu A^\dagger\mathbf{h}(\mathbf{x},t) - \lambda B^\dagger\mathbf{u}(\mathbf{x},t) \\
q(\mathbf{x},t) = & \ \mathbf{f}(\mathbf{x},t)\cdot\mathbf{h}(\mathbf{x},t) + \mathbf{g}(\mathbf{x},t)\cdot\mathbf{u}(\mathbf{x},t) + (\nu+\lambda)\mathbf{j}(\mathbf{x},t)\cdot\boldsymbol{\omega}(\mathbf{x},t) \qquad \text{(A.18)}
\end{aligned}$$

For the viscous terms we have introduced the velocity and magnetic gradient tensors

$$A_{ij} = \frac{\partial u_i}{\partial x_j} \tag{A.19}$$

$$B_{ij} = \frac{\partial h_i}{\partial x_j} \, . \tag{A.20}$$

Appendix B

B.1 The calculus of isotropic tensors

A good overview over the calculus of isotropic tensors can be found in [Arg07], [Mon71] and [Hil01]. In the following we focus on homogeneous and isotropic tensor fields. Considering the second order tensor

$$A_{ij}(\mathbf{x}, \mathbf{x}+\mathbf{r}, t) = \langle u_i(\mathbf{x}, t) u_j(\mathbf{x}+\mathbf{r}, t)\rangle, \tag{B.1}$$

we conclude that under the assumption of homogeneity, $A_{ij}(\mathbf{x}, \mathbf{x}+\mathbf{r}, t)$ depends solely on the relative distance $\mathbf{r}$ between the velocity field $u_i(\mathbf{x}, t)$ and $u_j(\mathbf{x}+\mathbf{r}, t)$, and not on $\mathbf{x}$

$$A_{ij}(\mathbf{x}, \mathbf{x}+\mathbf{r}, t) = A_{ij}(\mathbf{r}, t). \tag{B.2}$$

Turning next to the symmetry of isotropy, the basic idea is that the tensor $A_{ij}(\mathbf{r}, t)$ has to be invariant under rotations. This means that it has to be invariant under a coordinate transformation of the form

$$\mathbf{r}' = U\mathbf{r} \qquad U^t U = \mathbf{1}, \tag{B.3}$$

where $U \in \mathrm{SO}(3)$.
This implies that

$$A'_{ij}(\mathbf{r}', t) = \sum_{kl} U_{ik} U_{jl} A_{ij}(\mathbf{r}', t), \tag{B.4}$$

has to be equal to $A_{ij}(\mathbf{r}, t)$, which yields

$$A_{ij}(\mathbf{r}, t) = \sum_{kl} U_{ik} U_{jl} A_{ij}(U^t\mathbf{r}, t). \tag{B.5}$$

One can show that for this tensor of second order only the basic tensors $r_i r_j / r^2$, δ_{ij} and $\varepsilon_{ijk} r_k / r$ fulfill the invariance condition. A general tensor of second order can thus be written as

$$A_{ij}(\mathbf{r}, t) = A_1(r, t)\frac{r_i r_j}{r^2} + A_2(r, t)\delta_{ij} + A_3(r, t)\varepsilon_{ijk}\frac{r_k}{r}, \tag{B.6}$$

For the next order we can write

$$A_{ijk}(\mathbf{r}, t) = A_1(r, t)\frac{r_i r_j r_k}{r^3} + A_2(r, t)\delta_{ij}\frac{r_k}{r} + A_3(r, t)\delta_{ik}\frac{r_j}{r} + A_4(r, t)\delta_{jk}\frac{r_i}{r} + A_5(r, t)\varepsilon_{ijk}. \tag{B.7}$$

The last basic tensor involving the ε-tensor can be omitted if the tensor $A_{ijk}(\mathbf{r}, t)$ is invariant under reflections, which is certainly the case for the tensor considered in (B.7), since both u_i and u_j change their sign under a reflection. Nevertheless, tensors like $B_{ijk}(\mathbf{r}, t) = \langle \omega_i(\mathbf{x} + \mathbf{r}, t) u_j(\mathbf{x}, t) u_k(\mathbf{x}, t)\rangle$ are not invariant under these reflections since only u_j and u_k change their sign, whereas ω_i is an axial vector that keeps its orientation [Rob40]. The resulting tensor is skew and can be written as

$$B_{ijk}(\mathbf{r}, t) = B_1(r, t)\varepsilon_{ijk}\ . \tag{B.8}$$

Since $\mathbf{H}$ can be seen in analogy with the vorticity $\boldsymbol{\omega}$ one has to respect that in the corresponding normal forms of the tensors of MHD turbulence.

B.1.1 Longitudinal and transverse correlation functions

If we consider the velocity fields $\mathbf{u}(\mathbf{x}+\mathbf{r},t)$ at point $\mathbf{x}+\mathbf{r}$ and $\mathbf{u}(\mathbf{x},t)$ at point $\mathbf{x}$, we can divide the vector $\mathbf{u}$ into a part $\mathbf{u}^l$ parallel to $\mathbf{r}$, and a transverse part $\mathbf{u}^t$. These parts are thereby given as

$$\begin{aligned} \mathbf{u}^l &= \frac{\mathbf{r}}{r}\left(\frac{\mathbf{r}}{r}\cdot\mathbf{u}\right), \\ \mathbf{u}^t &= -\left(\frac{\mathbf{r}}{r}\times\left(\frac{\mathbf{r}}{r}\times\mathbf{u}\right)\right). \end{aligned} \tag{B.9}$$

The longitudinal correlation function

$$C_{rr}^{\mathbf{uu}}(r,t) = \langle \mathbf{u}^l(\mathbf{x},t)\cdot\mathbf{u}^l(\mathbf{x}+\mathbf{r},t)\rangle, \tag{B.10}$$

can be calculated in multiplying the two-point correlation tensor $C_{ij}^{\mathbf{uu}}(\mathbf{r},t) = \langle u_i(\mathbf{x},t)u_j(\mathbf{x}+\mathbf{r},t)\rangle$ by r_i and r_j.
Assuming that $C_{ij}^{\mathbf{uu}}(\mathbf{r},t) = \langle u_i(\mathbf{x},t)u_j(\mathbf{x}+\mathbf{r},t)\rangle$ is isotropic and mirror symmetric, it follows from (B.6) that its general form is given as

$$C_{ij}^{\mathbf{uu}}(\mathbf{r},t) = \left(C_{rr}^{\mathbf{uu}}(r,t) - C_{tt}^{\mathbf{hh}}(r,t)\right)\frac{r_i r_j}{r^2} + C_{tt}^{\mathbf{uu}}(r,t)\delta_{ij}, \tag{B.11}$$

where $C_{tt}(r,t)$ is the transverse correlation function

$$C_{tt}^{\mathbf{uu}}(r,t) = \langle \mathbf{u}^t(\mathbf{x}+\mathbf{r},t)\cdot\mathbf{u}^t(\mathbf{x},t)\rangle. \tag{B.12}$$

Turning next to the third order correlation function

$$C_{ij,k}^{\mathbf{uuu}}(\mathbf{r},t) = \langle u_i(\mathbf{x},t)u_j(\mathbf{x},t)u_k(\mathbf{x}+\mathbf{r},t)\rangle \tag{B.13}$$

we conclude that it is symmetric in i and j, therefore $A_3(r,t) = A_4(r,t)$in (3.53). From (B.7) we get the longitudinal correlation function in multiplying by r_i r_j and r_k

$$C_{rrr}^{\mathbf{uuu}}(r,t) = A_1(r,t) + A_2(r,t) + 2A_3(r,t). \tag{B.14}$$

B.1.2 The correlation functions for incompressible, isotropic and homogeneous fields

B.1.2.1 The correlation functions of second order

Due to the incompressibility condition it is possible to reduce the tensorial form of $C_{ij}^{\mathbf{uu}}(\mathbf{r},t)$, to a dependence of the longitudinal structure function $C_{rr}^{\mathbf{uu}}(r,t)$ only. The incompressibility condition is therefore used according to

$$\frac{\partial}{\partial r_i}C_{ij}^{\mathbf{uu}}(\mathbf{r},t) = \left\langle \frac{\partial u_i(\mathbf{x}+\mathbf{r},t)}{\partial r_i}u_j(\mathbf{x},t), \right\rangle = 0. \tag{B.15}$$

where the summation over equal indices is implied.
By making use of the relation $\frac{\partial}{\partial r_i} = \frac{r_i}{r}\frac{\partial}{\partial r}$, one gets

$$\frac{\partial}{\partial r_i}D_{ij}^{\mathbf{vv}}(\mathbf{r},t) = \frac{\partial}{\partial r}(D_{rr}^{\mathbf{vv}}(r,t) - D_{tt}^{\mathbf{vv}}(r,t))\frac{r_j}{r} + \frac{2}{r}(D_{rr}^{\mathbf{vv}}(r,t) - D_{tt}^{\mathbf{vv}}(r,t))\frac{r_j}{r} + \frac{\partial}{\partial r}D_{tt}^{\mathbf{vv}}(r,t)\frac{r_j}{r} = 0, \tag{B.16}$$

which yields

$$C_{tt}^{\mathbf{uu}}(r,t) = \frac{1}{2r}\frac{\partial}{\partial r}\left(r^2 C_{rr}^{\mathbf{uu}}(r,t)\right). \tag{B.17}$$

The correlation function $C^{\mathbf{uu}}_{i\,j}(\mathbf{r},t)$ can therefore be described solely in terms of the longitudinal correlation function $C^{\mathbf{uu}}_{r\,r}(r,t)$.
In summing $C^{\mathbf{uu}}_{i\,j}(\mathbf{r},t)$ over equal indices i,j, we get $Q_{kin}(r,t)$, introduced in the section of the Kármán-Howarth equation 3.4.4 in equation (3.61)

$$Q_{kin}(r,t) = \sum_{i=j} C^{\mathbf{uu}}_{i\,j}(\mathbf{r},t) = C^{\mathbf{uu}}_{r\,r}(r,t) + 2C^{\mathbf{uu}}_{t\,t}(r,t) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^3 C^{\mathbf{uu}}_{r\,r}(r,t)\right). \tag{B.18}$$

The same relations hold for the two-point magnetic correlation function $C^{\mathbf{uu}}_{i\,j}(\mathbf{r},t)$. This gives

$$C^{\mathbf{hh}}_{t\,t}(r,t) = \frac{1}{2r}\frac{\partial}{\partial r}\left(r^2 C^{\mathbf{hh}}_{r\,r}(r,t)\right), \tag{B.19}$$

and

$$Q_{mag}(r,t) = \sum_{i=j} C^{\mathbf{hh}}_{i\,j}(\mathbf{r},t) = C^{\mathbf{hh}}_{r\,r}(r,t) + 2C^{\mathbf{hh}}_{t\,t}(r,t) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^3 C^{\mathbf{hh}}_{r\,r}(r,t)\right). \tag{B.20}$$

However, no relation similar to (B.19) can be derived for the cross helicity correlation function

$$C^{\mathbf{uh}}_{i\,j}(\mathbf{r},t) = \langle u_i(\mathbf{x},t) h_j(\mathbf{x}+\mathbf{r},t)\rangle = C^{\mathbf{uh}}(r,t)\epsilon_{ijl}\frac{r_l}{r}, \tag{B.21}$$

since the incompressibility condition

$$\frac{\partial}{\partial r_i} C^{\mathbf{uh}}_{i\,j}(\mathbf{r},t) = 0, \tag{B.22}$$

is fulfilled without imposing any restrictions on the defining scalar $C^{\mathbf{uh}}(r,t)$.

B.1.2.2 The correlation functions of third order

The incompressibility condition for the third order correlation function

$$\frac{\partial}{\partial r_k} C^{\mathbf{uuu}}_{i\,j,k}(\mathbf{r},t) = 0, \tag{B.23}$$

has the following implications. Applying (B.23) to (B.14), yields

$$\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 A_1(r,t)\right) + 2r\frac{\partial}{\partial r}\frac{A_3(r,t)}{r}\right)\frac{r_i r_j}{r^2} + \left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 A_2(r,t)\right) + 2\frac{A_3(r,t)}{r}\right)\delta_{ij} = 0. \tag{B.24}$$

Since both brackets in (B.24) have to vanish identically in order to satisfy the equation, we get two equations along with (B.14) for the three prefactors $A_1(r,t)$, $A_2(r,t)$ and $A_3(r,t)$. This equation system is solved by

$$\begin{aligned} A_1(r,t) &= -\frac{r^2}{2}\frac{\partial}{\partial r}\left(\frac{C^{\mathbf{uuu}}_{r\,r\,r}(r,t)}{r}\right), \\ A_2(r,t) &= -\frac{C^{\mathbf{uuu}}_{r\,r\,r}(r,t)}{2}, \\ A_3(r,t) &= \frac{1}{4r}\frac{\partial}{\partial r}\left(r^2 C^{\mathbf{uuu}}_{r\,r\,r}(r,t)\right). \end{aligned} \tag{B.25}$$

Therefore the third order correlation function can be written in terms of $C^{\mathbf{uuu}}_{r\,r\,r}(r,t)$ only

$$\begin{aligned} C^{\mathbf{uuu}}_{i\,j,k}(\mathbf{r},t) = &- \frac{r^2}{2}\frac{\partial}{\partial r}\left(\frac{C^{\mathbf{uuu}}_{r\,r\,r}(r,t)}{r}\right)\frac{r_i r_j r_k}{r^3} \\ &+ \frac{1}{4r}\frac{\partial}{\partial r}\left(r^2 C^{\mathbf{uuu}}_{r\,r\,r}(r,t)\right)\left(\frac{r_i}{r}\delta_{jk} + \frac{r_j}{r}\delta_{ik}\right) - \frac{C^{\mathbf{uuu}}_{r\,r\,r}(r,t)}{2}\frac{r_k}{r}\delta_{ij}. \end{aligned} \tag{B.26}$$

For the derivation of the Kármán-Howarth equation in MHD turbulence we need

$$J_k^{kin}(\mathbf{r},t) = \sum_{i=j}[C_{k\,i\,j}^{\mathbf{hhu}}(\mathbf{r},t) - C_{k\,i\,j}^{\mathbf{uuu}}(\mathbf{r},t)] \tag{B.27}$$

By making use of the relation $C_{r\,r\,r}^{\mathbf{hhu}}(r,t) = -2C_{t\,t\,r}^{\mathbf{hhu}}(r,t)$, we finally arrive at

$$J_k^{kin}(\mathbf{r},t) = \sum_{i=j}[C_{k\,i\,j}^{\mathbf{hhu}}(\mathbf{r},t) - C_{k\,i\,j}^{\mathbf{uuu}}(\mathbf{r},t)] = \frac{1}{2r^3}\frac{\partial}{\partial r}\left(r^4[-2C_{t\,t\,r}^{\mathbf{hhu}}(r,t) - C_{r\,r\,r}^{\mathbf{uuu}}(r,t)]\right)\frac{r_k}{r}. \tag{B.28}$$

The evaluation of the incompressibility condition of the skew antisymmetric third order tensor

$$C_{i\,j,k}^{\mathbf{uhu}}(\mathbf{r},t) = \langle u_k(h_i'u_j' - u_i'h_j')\rangle, \tag{B.29}$$

is needed, for the evolution equation of the cross helicity correlation function (3.79). Since it is skew and antisymmetric, its general form is given as

$$C_{i\,j,k}^{\mathbf{uhu}}(\mathbf{r},t) = C_1(r,t)\varepsilon_{ijk} + C_2(r,t)\left(\frac{r_i}{r}\varepsilon_{jkl}\frac{r_l}{r} - \frac{r_j}{r}\varepsilon_{ikl}\frac{r_l}{r}\right). \tag{B.30}$$

The incompressibility condition yields

$$\frac{\partial}{\partial r_k}C_{i\,j,k}^{\mathbf{uhu}}(\mathbf{r},t) = \left(\frac{\partial}{\partial r}C_1(r,t) - 2\frac{C_2(r,t)}{r}\right)\varepsilon_{jil}\frac{r_l}{r} = 0, \tag{B.31}$$

which can be fulfilled according to

$$\begin{aligned} C_1(r,t) &= \frac{1}{2}C^{\mathbf{uhu}}(r,t), \\ C_2(r,t) &= r\frac{\partial}{\partial r}C^{\mathbf{uhu}}(r,t). \end{aligned} \tag{B.32}$$

This gives

$$C_{i\,j,k}^{\mathbf{uhu}}(\mathbf{r},t) = C^{\mathbf{uhu}}(r,t)\varepsilon_{ijk} + r\frac{\partial}{\partial r}C^{\mathbf{uhu}}(r,t)\left(\frac{r_i}{r}\varepsilon_{jkl}\frac{r_l}{r} - \frac{r_j}{r}\varepsilon_{ikl}\frac{r_l}{r}\right). \tag{B.33}$$

B.2 Tensor calculus for the hierarchy of structure functions in MHD turbulence

This section contains the calculation used in the chapter 4 of the hierarchy of structure functions in MHD turbulence.

B.2.1 Structure functions of second order

The structure functions in chapter 4 were introduced as moments of the magnetic and velocity increments at the points $\mathbf{x}$ and $\mathbf{x}' = \mathbf{x} + \mathbf{r}$. The velocity structure function of second order thus reads

$$\begin{aligned} D_{i\,j}^{\mathbf{vv}}(\mathbf{x},\mathbf{x}',t) &= \langle (u_i(\mathbf{x},t) - u_i(\mathbf{x}',t))(u_j(\mathbf{x},t) - u_j(\mathbf{x}',t))\rangle \\ &= \langle u_i(\mathbf{x},t)u_j(\mathbf{x},t)\rangle + \langle u_i(\mathbf{x}',t)u_j(\mathbf{x}',t)\rangle - \langle u_i(\mathbf{x},t)u_j(\mathbf{x}',t)\rangle - \langle u_i(\mathbf{x}',t)u_j(\mathbf{x},t),\rangle \\ &= 2\langle u_i(\mathbf{x},t)u_j(\mathbf{x},t)\rangle - 2\langle u_i(\mathbf{x},t)u_j(\mathbf{x}',t)\rangle, \end{aligned} \tag{B.34}$$

where homogeneity and isotropy were used in the last step, so that

$$\begin{aligned} \langle u_i(\mathbf{x}',t)u_j(\mathbf{x}',t)\rangle &= \langle u_i(\mathbf{x},t)u_j(\mathbf{x},t)\rangle = C^{\mathbf{uu}}_{ij}(0,t), \\ \langle u_i(\mathbf{x},t)u_j(\mathbf{x}',t)\rangle &= \langle u_i(\mathbf{x},t)u_j(\mathbf{x}',t)\rangle = C^{\mathbf{uu}}_{ij}(\mathbf{r},t). \end{aligned} \tag{B.35}$$

Therefore, the structure function $D^{\mathbf{vv}}_{ij}(\mathbf{r},t)$ can be written in term of the correlation function $C^{\mathbf{uu}}_{ij}(\mathbf{r},t)$ according to

$$D^{\mathbf{vv}}_{ij}(\mathbf{r},t) = 2C^{\mathbf{uu}}_{ij}(0,t) - C^{\mathbf{uu}}_{ij}(\mathbf{r},t). \tag{B.36}$$

The same relations hold for the magnetic structure function of second order

$$D^{\mathbf{bb}}_{ij}(\mathbf{r},t) = 2C^{\mathbf{hh}}_{ij}(0,t) - 2C^{\mathbf{hh}}_{ij}(\mathbf{r},t), \tag{B.37}$$

whereas the cross helicity structure function behaves in another way due to the lack of mirror symmetry

$$\begin{aligned} D^{\mathbf{vb}}_{ij}(\mathbf{x},\mathbf{x}',t) &= \langle (u_i(\mathbf{x},t) - u_i(\mathbf{x}',t))(h_j(\mathbf{x},t) - h_j(\mathbf{x}',t))\rangle \\ &= \langle u_i(\mathbf{x},t)h_j(\mathbf{x},t)\rangle + \langle u_i(\mathbf{x}',t)h_j(\mathbf{x}',t)\rangle - \langle u_i(\mathbf{x},t)h_j(\mathbf{x}',t)\rangle - \langle u_i(\mathbf{x}',t)h_j(\mathbf{x},t),\rangle \\ &= 2\langle u_i(\mathbf{x},t)h_j(\mathbf{x},t)\rangle. \end{aligned} \tag{B.38}$$

In the last step we made use of the homogeneity assumption

$$\langle u_i(\mathbf{x},t)h_j(\mathbf{x},t)\rangle = \langle u_i(\mathbf{x}',t)h_j(\mathbf{x}',t)\rangle, \tag{B.39}$$

and

$$\langle u_i(\mathbf{x},t)h_j(\mathbf{x}',t)\rangle = -\langle u_i(\mathbf{x}',t)h_j(\mathbf{x},t)\rangle. \tag{B.40}$$

This last relation is responsible for the different decomposition behavior of the structure function tensor of the cross helicity, which is used in section 4.2.3. The structure function reads

$$D^{\mathbf{vb}}_{ij}(\mathbf{r},t) = 2C^{\mathbf{uh}}_{ij}(0,t), \tag{B.41}$$

so that the structure function is independent of $\mathbf{r}$.

B.2.1.1 The Kármán-Howarth relation

In general we are interested in relations between longitudinal and transversal structure functions. If we consider the general tensor form mentioned above, we can write the tensor of the velocity structure function of second order

$$D^{\mathbf{vv}}_{ij}(\mathbf{r},t) = (D^{\mathbf{vv}}_{rr}(r,t) - D^{\mathbf{vv}}_{tt}(r,t))\frac{r_i r_j}{r^2} + D^{\mathbf{vv}}_{tt}(r,t)\delta_{ij}. \tag{B.42}$$

The subscripts denote the longitudinal rr and the transverse tt structure functions.
It follows from equation (B.36) and (B.15) that

$$\frac{\partial}{\partial r_i} D^{\mathbf{vv}}_{ij}(\mathbf{r},t) = 0. \tag{B.43}$$

From the incompressibility condition we can therefore write a first relation between the longitudinal and the transverse structure functions.

$$\frac{\partial}{\partial r_i} D^{\mathbf{vv}}_{ij}(\mathbf{r},t) = \frac{\partial}{\partial r}(D^{\mathbf{vv}}_{rr}(r,t) - D^{\mathbf{vv}}_{tt}(r,t))\frac{r_j}{r} + \frac{2}{r}(D^{\mathbf{vv}}_{rr}(r,t) - D^{\mathbf{vv}}_{tt}(r,t))\frac{r_j}{r} + \frac{\partial}{\partial r} D^{\mathbf{vv}}_{tt}(r,t)\frac{r_j}{r} = 0, \tag{B.44}$$

where we have made use of $\frac{\partial}{\partial r_i} = \frac{r_i}{r}\frac{\partial}{\partial r}$.
This yields

$$D^{\mathbf{vv}}_{t\,t}(r,t) = \frac{1}{2r}\frac{\partial}{\partial r}\left(r^2 D^{\mathbf{vv}}_{r\,r}(r,t)\right), \tag{B.45}$$

which is known as the Kármán-Howarth relation. The same relation holds for the magnetic structure function of second order

$$D^{\mathbf{bb}}_{t\,t}(r,t) = \frac{1}{2r}\frac{\partial}{\partial r}\left(r^2 D^{\mathbf{bb}}_{r\,r}(r,t)\right). \tag{B.46}$$

Again, such a relation does not exist for the tensor $D^{\mathbf{vb}}_{i\,j}(\mathbf{r},t)$ due to the lack of mirror symmetry and

$$\frac{\partial}{\partial r_i} D^{\mathbf{vb}}_{i\,j}(\mathbf{r},t) = 2\frac{\partial}{\partial r_i} C^{\mathbf{uh}}_{i\,j}(0,t) = 0, \tag{B.47}$$

is fulfilled in any way.
Summing over equal i and j in (B.42) and the relation (B.45) yields

$$\langle v^2(r,t)\rangle = D^{\mathbf{vv}}_{r\,r}(r,t) + 2D^{\mathbf{vv}}_{t\,t}(r,t) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^3 D^{\mathbf{vv}}_{r\,r}(r,t)\right), \tag{B.48}$$

and similarly

$$\langle b^2(r,t)\rangle = D^{\mathbf{bb}}_{r\,r}(r,t) + 2D^{\mathbf{bb}}_{t\,t}(r,t) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^3 D^{\mathbf{bb}}_{r\,r}(r,t)\right), \tag{B.49}$$

which is needed fort the equation of energy balance in MHD turbulence (4.20).

B.2.2 Structure functions of third order

The third order velocity structure function reads

$$D^{\mathbf{vvv}}_{i\,j\,n}(\mathbf{x},\mathbf{x}',t) = \langle (u_i(\mathbf{x},t) - u_i(\mathbf{x}',t))(u_j(\mathbf{x},t) - u_j(\mathbf{x}',t))(u_n(\mathbf{x},t) - u_n(\mathbf{x}',t))\rangle \tag{B.50}$$

Since $\langle u_i(\mathbf{x},t)u_j(\mathbf{x},t)u_n(\mathbf{x},t) = 0\rangle$, the structure function decomposes under the assumption of homogeneity and isotropy according to

$$D^{\mathbf{vvv}}_{i\,j\,n}(\mathbf{r},t) = -2(C^{\mathbf{uuu}}_{i\,j,k}(\mathbf{r},t) + C^{\mathbf{uuu}}_{j\,k,i}(\mathbf{r},t) + 2C^{\mathbf{uuu}}_{k\,i,j}(\mathbf{r},t)). \tag{B.51}$$

Inserting (B.98) for $C^{\mathbf{uuu}}_{i\,j,k}(\mathbf{r},t)$ gives

$$D^{\mathbf{vvv}}_{i\,j\,n}(\mathbf{r},t) = 3r^2\frac{\partial}{\partial r}\left(\frac{C^{\mathbf{uuu}}_{r\,r\,r}(r,t)}{r}\right)\frac{r_ir_jr_n}{r^3} - \frac{\partial}{\partial r}(rC^{\mathbf{uuu}}_{r\,r\,r}(r,t))\left(\frac{r_n}{r}\delta_{ij} + \frac{r_i}{r}\delta_{jn} + \frac{r_j}{r}\delta_{in}\right). \tag{B.52}$$

Contracting this tensor to a tensor of second order yields

$$D^{\mathbf{vvv}}_{r\,j\,n}(\mathbf{r},t) = \frac{r_i}{r}D^{\mathbf{vvv}}_{i\,j\,n}(\mathbf{r},t) = \left(r\frac{\partial}{\partial r}C^{\mathbf{uuu}}_{r\,r\,r}(r,t) - 5C^{\mathbf{uuu}}_{r\,r,r}(r,t)\right)\frac{r_jr_n}{r^2} - \frac{\partial}{\partial r}(rC^{\mathbf{uuu}}_{r\,r\,r}(r,t))\delta_{jn}. \tag{B.53}$$

Comparing this relation to the general form of a tensor of second order (B.42) gives

$$\begin{aligned} D^{\mathbf{vvv}}_{r\,t\,t}(r,t) &= -\frac{\partial}{\partial r}(rC^{\mathbf{uuu}}_{r\,r\,r}(r,t)), \\ D^{\mathbf{vvv}}_{r\,r\,r}(r,t) &= \left(r\frac{\partial}{\partial r}C^{\mathbf{uuu}}_{r\,r\,r}(r,t) - 5C^{\mathbf{uuu}}_{r\,r,r}(r,t)\right) + 2D^{\mathbf{vvv}}_{r\,t\,t}(r,t) = -6C^{\mathbf{uuu}}_{r\,r\,r}(r,t). \end{aligned} \tag{B.54}$$

Therefore, a relation between the mixed and longitudinal structure function of can be derived as

$$D^{\mathbf{vvv}}_{r\,t\,t}(r,t) = \frac{1}{6}\frac{\partial}{\partial r}(rD^{\mathbf{vvv}}_{r\,r\,r}(r,t)). \tag{B.55}$$

Summing (B.53) over equal indices j and n gives

$$D^{\mathbf{vvv}}(r,t) = \langle v_r(r,t)\mathbf{v}(r,t)^2\rangle = D^{\mathbf{vvv}}_{r\,r\,r}(r,t) + 2D^{\mathbf{vvv}}_{r\,t\,t}(r,t), \tag{B.56}$$

which can be rewritten with the relation (B.55) as

$$D^{\mathbf{vvv}}(r,t) = \frac{1}{3r^3}\frac{\partial}{\partial r}\left(r^4 D^{\mathbf{vvb}}_{r\,r\,r}(r,t)\right). \tag{B.57}$$

This is the quantity which is needed for the equation of energy balance in MHD turbulence (4.20).
The mixed third order tensor $D^{\mathbf{bbv}}_{i\,j\,n}(r,t) - D^{\mathbf{vbb}}_{i\,j\,n}(r,t)$ from equation (4.25) that enters in the equation of energy balance (4.20)

$$\begin{aligned} D^{\mathbf{bbv}}_{i\,j\,n}(\mathbf{r},t) - D^{\mathbf{vbb}}_{i\,j\,n}(\mathbf{r},t) = &+ 2(\langle h_j h_n u_i'\rangle + \langle h_i h_n u_j'\rangle + \langle h_i h_j u_n'\rangle) \\ &- 2(\langle (u_n h_j - u_j h_n)h_i'\rangle + \langle (u_n h_i - u_i h_n)h_j'\rangle - \langle (u_j h_i + u_i h_j)h_n'\rangle), \end{aligned} \tag{B.58}$$

can be divided into

$$U_{ijn}(r,t) = -2(-\langle h_j h_n u_i'\rangle - \langle h_i h_n u_j'\rangle + \langle h_i h_j u_n'\rangle), \tag{B.59}$$

and

$$H_{ijn}(r,t) = -2(\langle (u_n h_j - u_j h_n)h_i'\rangle + \langle (u_n h_i - u_i h_n)h_j'\rangle - \langle (u_j h_i + u_i h_j)h_n'\rangle)\ . \tag{B.60}$$

Turning first to $U_{ijn}(r,t)$, we have to evaluate correlation functions like $C^{\mathbf{hhu}}_{i\,j,n}(\mathbf{r},t) = \langle h_i h_j u_n'\rangle$, whose prefactors are again determined by the incompressibility condition

$$\frac{\partial}{\partial r_n} C^{\mathbf{hhu}}_{i\,j,n}(\mathbf{r},t) = 0\ . \tag{B.61}$$

We get

$$\begin{aligned} C^{\mathbf{hhu}}_{i\,j\,n}(\mathbf{r},t) = & -\frac{r^2}{2}\frac{\partial}{\partial r}\left(\frac{\partial C^{\mathbf{hhu}}_{r\,r\,r}(r,t)}{r}\right)\frac{r_i r_j r_n}{r^3} \\ &+ \frac{1}{4r}\frac{\partial}{\partial r}\left(r^2 C^{\mathbf{hhu}}_{r\,r\,r}(r,t)\right)\left(\frac{r_i}{r}\delta_{jn} + \frac{r_j}{r}\delta_{in}\right) - \frac{C^{\mathbf{hhu}}_{r\,r\,r}(r,t)}{2}\frac{r_n}{r}\delta_{ij}\ . \end{aligned} \tag{B.62}$$

Inserting this in (B.59) gives

$$\begin{aligned} U_{ijn}(\mathbf{r},t) = & \left(C^{\mathbf{hhu}}_{r\,r\,r}(r,t) - r\frac{\partial}{\partial r}C^{\mathbf{hhu}}_{r\,r\,r}(r,t)\right)\frac{r_i r_j r_n}{r^3} \\ &- C^{\mathbf{hhu}}_{r\,r\,r}(r,t)\left(\frac{r_i}{r}\delta_{jn} + \frac{r_j}{r}\delta_{in}\right) + \left(3C^{\mathbf{hhu}}_{r\,r\,r}(r,t) + r\frac{\partial}{\partial r}C^{\mathbf{hhu}}_{r\,r\,r}(r,t)\right)\frac{r_n}{r}\delta_{ij}. \end{aligned} \tag{B.63}$$

This yields the following relations between the structure function $U_{ijn}(r,t)$ and the correlation function $C^{\mathbf{hhu}}_{r\,r\,r}(r,t)$

$$U_{rrr}(r,t) = 2C^{\mathbf{hhu}}_{r\,r\,r}(r,t) \tag{B.64}$$

$$U_{rtt}(r,t) = -C^{\mathbf{hhu}}_{r\,r\,r}(r,t) \tag{B.65}$$

$$U_{ttr}(r,t) = 3C^{\mathbf{hhu}}_{r\,r\,r}(r,t) + r\frac{\partial}{\partial r}C^{\mathbf{hhu}}_{r\,r\,r}(r,t). \tag{B.66}$$

For $H_{ijn}(r,t)$ we need the antisymmetric tensor

$$\langle (h_j u_n - u_j h_n) h'_i \rangle = C^{\mathbf{uhh}}(r,t) \left(\frac{r_j}{r}\delta_{in} - \frac{r_n}{r}\delta_{ij} \right), \tag{B.67}$$

and its symmetric counterpart

$$\langle (h_j u_n + u_j h_n) h'_i \rangle = C^{\mathbf{uhh}}_{j\,n,i}(r,t), \tag{B.68}$$

which fulfills the same equation as $C^{\mathbf{hhu}}_{i\,j,n}(\mathbf{r},t)$, namely (B.62)
One arrives at

$$\begin{aligned} H_{ijn}(\mathbf{r},t) = & \left(C^{\mathbf{uhh}}_{r\,r\,r}(r,t) - r\frac{\partial}{\partial r} C^{\mathbf{uhh}}_{r\,r\,r}(r,t) \right) \frac{r_i r_j r_n}{r^3}, \\ + & \left(\frac{1}{2r}\frac{\partial}{\partial r}(r^2 C^{\mathbf{uhh}}_{r\,r\,r}(r,t)) + 2C^{\mathbf{uhh}}(r,t) \right) \left(\frac{r_i}{r}\delta_{jn} + \frac{r_j}{r}\delta_{in} \right), \\ + & (C^{\mathbf{uhh}}_{r\,r\,r}(r,t) - 4C^{\mathbf{uhh}}(r,t)) \frac{r_n}{r}\delta_{ij}. \end{aligned} \tag{B.69}$$

This yields the following relations

$$H_{rrr}(r,t) = 2C^{\mathbf{uhh}}_{r\,r\,r}(r,t), \tag{B.70}$$

$$H_{rtt}(r,t) = \frac{1}{4r}\frac{\partial}{\partial r}(r^2 H_{rrr}(r,t)) + 2C^{\mathbf{uhh}}(r,t) \tag{B.71}$$

$$H_{ttr}(r,t) = -\frac{1}{2}H_{rrr}(r,t) - 4C^{\mathbf{uhh}}(r,t). \tag{B.72}$$

We need the function $D^{\mathbf{bbv}}(r,t) - D^{\mathbf{vbb}}(r,t)$ for the equation of energy balance in spherical coordinates.
By making use of

$$C^{\mathbf{uhh}}_{r\,r\,r}(r,t) = -2C^{\mathbf{uhh}}_{t\,t\,r}(r,t)\ , \tag{B.73}$$

we get

$$\begin{aligned} D^{\mathbf{bbv}}(r,t) - D^{\mathbf{vbb}}(r,t) &= \langle v_r(r,t) b^2(r,t) \rangle - 2\langle b_r(r,t)\mathbf{v}(r,t)\cdot\mathbf{b}(r,t) \rangle \\ &= U_{rrr}(r,t) + 2U_{ttr}(r,t) + H_{rrr}(r,t) + 2H_{ttr}(r,t) \\ &= -\frac{4}{r^3}\frac{\partial}{\partial r}(r^4 C^{\mathbf{hhu}}_{t\,t\,r}(r,t)) - 8C^{\mathbf{uhh}}(r,t). \end{aligned} \tag{B.74}$$

Astonishingly, the contributions from the symmetric correlation tensor $\langle (u_j h_i + u_i h_j) h'_n \rangle)$ vanish from this expression.

B.2.3 The viscous term

As an example of the treatment of the viscous terms in (4.17) we focus on the viscous velocity contributions. The calculation is the same for the viscous magnetic field contributions. The viscous terms in (4.17) read

$$v_j(\nabla^2_{\mathbf{x}} + \nabla^2_{\mathbf{x}'})v_i + v_i(\nabla^2_{\mathbf{x}} + \nabla^2_{\mathbf{x}'})v_j. \tag{B.75}$$

We rewrite the Laplace operators in $\mathbf{x}$- and $\mathbf{x}'$-space according to

$$\nabla^2_{\mathbf{x}} = \frac{\partial}{\partial x_n}\frac{\partial}{\partial x_n} \qquad \text{and} \qquad \nabla^2_{\mathbf{x}'} = \frac{\partial}{\partial x'_n}\frac{\partial}{\partial x'_n}, \tag{B.76}$$

and make use of the identity

$$\frac{\partial}{\partial x_n}\frac{\partial}{\partial x_n}(fg) = f\frac{\partial}{\partial x_n}\frac{\partial}{\partial x_n}g + 2\left(\frac{\partial f}{\partial x}\right)\left(\frac{\partial g}{\partial x_n}\right) + g\frac{\partial}{\partial x_n}\frac{\partial}{\partial x_n}f. \tag{B.77}$$

Therefore, we can rewrite (B.75) as

$$\begin{aligned} & v_j\left(\frac{\partial}{\partial x_n}\frac{\partial}{\partial x_n} + \frac{\partial}{\partial x'_n}\frac{\partial}{\partial x'_n}\right)v_i + v_i\left(\frac{\partial}{\partial x_n}\frac{\partial}{\partial x_n} + \frac{\partial}{\partial x'_n}\frac{\partial}{\partial x'_n}\right)v_j \\ = & \left(\frac{\partial}{\partial x_n}\frac{\partial}{\partial x_n} + \frac{\partial}{\partial x'_n}\frac{\partial}{\partial x'_n}\right)v_iv_j - 2\left[\left(\frac{\partial v_i}{\partial x_n}\right)\left(\frac{\partial v_j}{\partial x_n}\right) + \left(\frac{\partial v_i}{\partial x'_n}\right)\left(\frac{\partial v_j}{\partial x'_n}\right)\right]. \end{aligned} \tag{B.78}$$

Note that $\left(\frac{\partial v_i}{\partial x_n}\right)\left(\frac{\partial v_j}{\partial x_n}\right) = \left(\frac{\partial u_i}{\partial x_n}\right)\left(\frac{\partial u_j}{\partial x_n}\right)$ and $\left(\frac{\partial v_i}{\partial x'_n}\right)\left(\frac{\partial v_j}{\partial x'_n}\right) = \left(\frac{\partial u'_i}{\partial x'_n}\right)\left(\frac{\partial u'_j}{\partial x'_n}\right)$ and that

$$\left(\frac{\partial}{\partial x_n}\frac{\partial}{\partial x_n} + \frac{\partial}{\partial x'_n}\frac{\partial}{\partial x'_n}\right) = 2\left(\frac{\partial}{\partial r_n}\frac{\partial}{\partial r_n} + \frac{1}{4}\frac{\partial}{\partial X_n}\frac{\partial}{\partial X_n}\right), \tag{B.79}$$

can be rewritten by making use of

$$\frac{\partial}{\partial x_i} = \frac{\partial}{\partial r_i} + \frac{1}{2}\frac{\partial}{\partial X_i} \qquad \text{and} \qquad \frac{\partial}{\partial x'_i} = -\frac{\partial}{\partial r_i} + \frac{1}{2}\frac{\partial}{\partial X_i}. \tag{B.80}$$

If we rewrite the equations again with the Laplacian operators in **r**- and **X**-space, we get

$$v_j(\nabla^2_{\mathbf{x}} + \nabla^2_{\mathbf{x}'})v_i + v_i(\nabla^2_{\mathbf{x}} + \nabla^2_{\mathbf{x}'})v_j = 2\left[\left(\nabla^2_{\mathbf{r}} + \frac{1}{4}\nabla^2_{\mathbf{X}}\right)v_iv_j - \varepsilon^{\mathbf{uu}}_{ij}\right], \tag{B.81}$$

with

$$\varepsilon^{\mathbf{uu}}_{ij} = \left(\frac{\partial u_i}{\partial x_l}\right)\left(\frac{\partial u_j}{\partial x_l}\right) + \left(\frac{\partial u'_i}{\partial x'_n}\right)\left(\frac{\partial u'_j}{\partial x'_n}\right). \tag{B.82}$$

B.2.4 Tensors for the fourth order structure functions

The tensor of fourth order, symmetric in all four indices is given by [Mon71] in Vol. II by formula (13.82). It has the form

$$\begin{aligned} & D_{ijkn}(\mathbf{r},t) = (D_{rrrr}(r,t) - 6D_{rrtt}(r,t) + D_{tttt}(r,t))\frac{r_ir_jr_kr_n}{r^4} \\ + & (D_{rrtt}(r,t) - \frac{1}{3}D_{tttt}(r,t))\left[\frac{r_ir_j}{r^2}\delta_{kn} + \frac{r_ir_k}{r^2}\delta_{jn} + \frac{r_ir_n}{r^2}\delta_{jk} + \frac{r_jr_k}{r^2}\delta_{in} + \frac{r_jr_n}{r^2}\delta_{ik} + \frac{r_kr_n}{r^2}\delta_{ij}\right] \\ + & \frac{1}{3}D_{rrtt}(r,t)[\delta_{ij}\delta_{kn} + \delta_{ik}\delta_{jn} + \delta_{in}\delta_{jk}]. \end{aligned} \tag{B.83}$$

If we calculate its divergence we get a tensor of third order, whose tensorial form is given in general by [Mon71] in Vol. II by formula (13.80), therefore we obtain

$$\begin{aligned} \frac{\partial}{\partial r_n}D_{ijkn}(\mathbf{r},t) &= \left(\frac{\partial}{\partial r_n}D_{rrrn}(\mathbf{r},t) - 3\frac{\partial}{\partial r_n}D_{rttn}(\mathbf{r},t)\right)\frac{r_ir_jr_n}{r^3} \\ &+ \frac{\partial}{\partial r_n}D_{rttn}(\mathbf{r},t)\left[\frac{r_i}{r}\delta_{jk} + \frac{r_j}{r}\delta_{ik} + \frac{r_k}{r}\delta_{ij}\right]. \end{aligned} \tag{B.84}$$

This tensorial form can be compared with the original calculations. We get

$$\begin{aligned} \frac{\partial}{\partial r_n}D_{rrrn}(\mathbf{r},t) - 3\frac{\partial}{\partial r_n}D_{rttn}(\mathbf{r},t) &= \left(\frac{\partial}{\partial r} + \frac{2}{r}\right)(D_{rrrr}(r,t) - 6D_{rrtt}(r,t) + D_{tttt}(r,t)) \\ &+ \left(3\frac{\partial}{\partial r} - \frac{6}{r}\right)\left(D_{rrtt}(r,t) - \frac{1}{3}D_{tttt}(r,t)\right), \end{aligned} \tag{B.85}$$

and

$$\frac{\partial}{\partial r_n} D_{rttn}(\mathbf{r},t) = \left(\frac{\partial}{\partial r} + \frac{4}{r}\right) D_{rrtt}(r,t) - \frac{4}{3r} D_{tttt}(r,t). \tag{B.86}$$

The last equation has to be inserted into (B.85) in order to get the equation for the longitudinal structure function $\frac{\partial}{\partial r_n} D_{rrrn}(\mathbf{r},t)$.

For the antisymmetric tensor

$$D^{\mathbf{vvbb}}_{ij,kn}(\mathbf{r},t) = \langle v_i v_j b_k b_n - b_i b_j v_k v_n \rangle = D^{\mathbf{vvbb}}_{rr,tt}(r,t) \left(\frac{r_i r_j}{r^2}\delta_{kn} - \frac{r_k r_n}{r^2}\delta_{ij}\right), \tag{B.87}$$

we calculate the divergence as

$$\begin{aligned} \frac{\partial}{\partial r_n} D^{\mathbf{vvbb}}_{ij,kn}(\mathbf{r},t) &= \frac{\partial}{\partial r} D^{\mathbf{vvbb}}_{rr,tt}(r,t) \left[\frac{r_i r_j r_k}{r^3} - \frac{r_k}{r}\delta_{ij}\right] \\ &+ \frac{D^{\mathbf{vvbb}}_{rr,tt}(r,t)}{r} \left[\frac{r_i}{r}\delta_{jk} + \frac{r_j}{r}\delta_{ik} - 2\frac{r_i r_j r_k}{r^3} - 2\frac{r_k}{r}\delta_{ij}\right]. \end{aligned} \tag{B.88}$$

Let us define $A_{ijk,n}(\mathbf{r},t)$ as

$$A_{ijk,n}(\mathbf{r},t) = D^{\mathbf{vvbb}}_{ij,kn}(\mathbf{r},t) + D^{\mathbf{vvbb}}_{kj,in}(\mathbf{r},t) + D^{\mathbf{vvbb}}_{ik,jn}(\mathbf{r},t). \tag{B.89}$$

We obtain

$$\begin{aligned} \frac{\partial}{\partial r_n} A_{ijk,n}(\mathbf{r},t) &= \left(3\frac{\partial}{\partial r} D^{\mathbf{vvbb}}_{rr,tt}(r,t) - \frac{6}{r} D^{\mathbf{vvbb}}_{rr,tt}(r,t)\right) \frac{r_i r_j r_k}{r^3} \\ &- \frac{\partial}{\partial r} D^{\mathbf{vvbb}}_{rr,tt}(r,t) \left[\frac{r_i}{r}\delta_{jk} + \frac{r_j}{r}\delta_{ik} + \frac{r_k}{r}\delta_{ij}\right]. \end{aligned} \tag{B.90}$$

Therefore, the coefficients can be read of analogous to (B.85) and (B.112) as

$$\frac{\partial}{\partial r_n} A_{rrrn}(\mathbf{r},t) - 3\frac{\partial}{\partial r_n} A_{rttn}(\mathbf{r},t) = \left(3\frac{\partial}{\partial r} - \frac{6}{r}\right) D^{\mathbf{vvbb}}_{rr,tt}(r,t), \tag{B.91}$$

and

$$\frac{\partial}{\partial r_n} A_{rttn}(\mathbf{r},t) = -\frac{\partial}{\partial r} D^{\mathbf{vvbb}}_{rr,tt}(r,t). \tag{B.92}$$

B.2.5 The equations for 2D MHD

In the following the tensor calculus is applied to the case of 2D MHD turbulence.

B.2.5.1 Kármán-Howarth relation in 2D

The only difference for the tensor calculus that arises from a change of 3D turbulence to 2D turbulence is the calculation of the divergence of r_n, namely

$$\frac{\partial}{\partial r_n} r_n = 2. \tag{B.93}$$

Therefore the calculation of the incompressibility condition for the second order velocity structure function from equation (B.42) yields

$$\frac{\partial}{\partial r_i} D^{\mathbf{uu}}_{ij}(\mathbf{r},t) = \frac{\partial}{\partial r}(D^{\mathbf{uu}}_{rr}(r,t) - D^{\mathbf{uu}}_{tt}(r,t))\frac{r_j}{r} + \frac{1}{r}(D^{\mathbf{uu}}_{rr}(r,t) - D^{\mathbf{uu}}_{tt}(r,t))\frac{r_j}{r} + \frac{\partial}{\partial r} D^{\mathbf{uu}}_{tt}(r,t)\frac{r_j}{r} = 0, \tag{B.94}$$

where we have made use of $\frac{\partial}{\partial r_i} = \frac{r_i}{r}\frac{\partial}{\partial r}$.
This yields

$$D^{\mathbf{uu}}_{t\,t}(r,t) = \frac{\partial}{\partial r}\left(r D^{\mathbf{uu}}_{r\,r}(r,t)\right). \tag{B.95}$$

The kinetic and magnetic energy thus read

$$\begin{aligned} \langle v(r,t)^2\rangle &= D^{\mathbf{uu}}_{r\,r}(r,t) + D^{\mathbf{uu}}_{t\,t}(r,t) = \frac{1}{r}\frac{\partial}{\partial r}(r^2 D^{\mathbf{uu}}_{r\,r}(r,t)), &(\text{B.96})\\ \langle b(r,t)^2\rangle &= D^{\mathbf{bb}}_{r\,r}(r,t) + D^{\mathbf{bb}}_{t\,t}(r,t) = \frac{1}{r}\frac{\partial}{\partial r}(r^2 D^{\mathbf{bb}}_{r\,r}(r,t)). &(\text{B.97})\end{aligned}$$

B.2.5.2 Structure functions of third order in 2D

The third order correlation function in 2D is again determined by the incompressibility condition. This gives

$$\begin{aligned} C^{\mathbf{uuu}}_{i\,j\,n}(\mathbf{r},t) = &- r^2\frac{\partial}{\partial r}\left(\frac{\partial C^{\mathbf{uuu}}_{r\,r\,r}(r,t)}{r}\right)\frac{r_i r_j r_n}{r^3} \\ &+ \frac{1}{2r}\frac{\partial}{\partial r}\left(r C^{\mathbf{uuu}}_{r\,r\,r}(r,t)\right)\left(\frac{r_i}{r}\delta_{jn} + \frac{r_j}{r}\delta_{in}\right) - \frac{C^{\mathbf{uuu}}_{r\,r\,r}(r,t)}{2}\frac{r_n}{r}\delta_{ij}. \end{aligned} \tag{B.98}$$

The third order velocity structure function is calculated as

$$D^{\mathbf{vvv}}_{i\,j\,n}(\mathbf{r},t) = -2(C^{\mathbf{uuu}}_{i\,j,n}(\mathbf{r},t) + C^{\mathbf{uuu}}_{j\,n,i}(\mathbf{r},t) + 2C^{\mathbf{uuu}}_{n\,i,j}(\mathbf{r},t)), \tag{B.99}$$

which yields

$$D^{\mathbf{vvv}}_{i\,j\,n}(\mathbf{r},t) = 6r^2\frac{\partial}{\partial r}\left(\frac{C^{\mathbf{uuu}}_{r\,r\,r}(r,t)}{r}\right)\frac{r_i r_j r_n}{r^3} - 2r\frac{\partial}{\partial r}C^{\mathbf{uuu}}_{r\,r\,r}(r,t))\left(\delta_{ij}\frac{r_n}{r} + \delta_{jn}\frac{r_i}{r} + \frac{r_j}{r}\delta_{in}\right). \tag{B.100}$$

The defining scalars can be read off according to

$$\begin{aligned} D^{\mathbf{vvv}}_{r\,t\,t}(r,t) &= -2\frac{\partial}{\partial r}C^{\mathbf{uuu}}_{r\,r\,r}(r,t), \\ D^{\mathbf{vvv}}_{r\,r\,r}(r,t) &= -6C^{\mathbf{uuu}}_{r\,r\,r}(r,t). \end{aligned} \tag{B.101}$$

Therefore, the relation between the mixed and longitudinal structure function reads

$$D^{\mathbf{vvv}}_{r\,t\,t}(r,t) = \frac{1}{3}r\frac{\partial}{\partial r}D^{\mathbf{vvv}}_{r\,r\,r}(r,t). \tag{B.102}$$

The quantity which enters the equation of energy balance is

$$D^{\mathbf{vvv}}(r,t) = \langle v_r(r,t)\mathbf{v}(r,t)^2\rangle = D^{\mathbf{vvv}}_{r\,r\,r}(r,t) + D^{\mathbf{vvv}}_{r\,t\,t}(r,t), \tag{B.103}$$

which can be rewritten with the relation (B.102) as

$$D^{\mathbf{vvv}}(r,t) = \frac{1}{3r^2}\frac{\partial}{\partial r}\left(r^3 D^{\mathbf{vvb}}_{r\,r\,r}(r,t)\right). \tag{B.104}$$

The same procedure has to be done for the mixed third order tensor $D^{\mathbf{bbv}}_{i\,j\,n}(r,t) - D^{\mathbf{vbb}}_{i\,j\,n}(r,t)$.

B.2.5.3 Equation of energy balance in 2D MHD turbulence and the 3/2 law

The equation of energy balance for stationary 2D MHD turbulence in polar coordinates reads

$$\begin{aligned} & \frac{1}{2r}\frac{\partial}{\partial r}\left(rD^{\mathbf{vvv}}(r,t)+r(D^{\mathbf{bbv}}(r,t)-D^{\mathbf{vbb}}(r,t))\right) \\ = \; & \nu\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}D^{\mathbf{vv}}(r,t)\right)+\lambda\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}D^{\mathbf{bb}}(r,t)\right)-2\left\langle\varepsilon^{\mathbf{v}}\right\rangle-2\left\langle\varepsilon^{\mathbf{b}}\right\rangle+Q(r,t), \end{aligned}$$

where we have assumed homogeneity and isotropy.
The following tensors have now to be inserted

$$D^{\mathbf{vv}}(r,t) = \langle\mathbf{v}(r,t)^2\rangle = \frac{1}{r}\frac{\partial}{\partial r}\left(r^2 D^{\mathbf{vv}}_{rr}(r,t)\right) \tag{B.105}$$

$$\frac{\partial}{\partial r}D^{\mathbf{vv}}(r,t) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^3\frac{\partial}{\partial r}D^{\mathbf{vv}}_{rr}(r,t)\right) \tag{B.106}$$

$$\begin{aligned} D^{\mathbf{vvv}}(r,t) &= \langle v_r(r,t)\mathbf{v}(r,t)^2\rangle \\ &= \frac{1}{3r^2}\frac{\partial}{\partial r}\left(r^3 D^{\mathbf{vvb}}_{rrr}(r,t)\right) \end{aligned} \tag{B.107}$$

$$\begin{aligned} D^{\mathbf{bbv}}(r,t)-D^{\mathbf{vbb}}(r,t) &= \langle v_r(r,t)\mathbf{b}(r,t)^2\rangle-2\langle b_r(r,t)\mathbf{v}(r,t)\cdot\mathbf{b}(r,t)\rangle \\ &= -\frac{4}{r^2}\frac{\partial}{\partial r}\left(r^3 C^{\mathbf{hhu}}_{ttr}(r,t)\right)-8C^{\mathbf{uhh}}(r,t). \end{aligned} \tag{B.108}$$

We arrive at

$$\begin{aligned} D^{\mathbf{vvv}}_{rrr}(r)-12C^{\mathbf{hhu}}_{ttr}(r) = & -\frac{3}{2}\left\langle\varepsilon^{\mathbf{v}}+\varepsilon^{\mathbf{b}}\right\rangle r+\frac{24}{r^3}\int_0^r \mathrm{d}r'\, r'^2 C^{\mathbf{uhh}}(r') \\ & + 6\nu\frac{\partial}{\partial r}D^{\mathbf{vv}}_{rr}(r)+6\lambda\frac{\partial}{\partial r}D^{\mathbf{bb}}_{rr}(r)+q(r). \end{aligned} \tag{B.109}$$

where the source $q(r)$ is given by

$$q(r)=\frac{6}{r^3}\int_0^r \mathrm{d}r' r'\int_0^{r'}\mathrm{d}r'' r'' Q(r''). \tag{B.110}$$

B.2.5.4 Tensors for the fourth order structure functions in 2D

In order to derive the next order equation for 2D MHD turbulence, we have to calculate the divergences of the corresponding tensors. In calculating the divergence of the symmetric fourth order tensor defined by equation (B.83), one arrives at

$$\begin{aligned} \frac{\partial}{\partial r_n}D_{rrrn}(\mathbf{r},t)-3\frac{\partial}{\partial r_n}D_{rttn}(\mathbf{r},t) = & \left(\frac{\partial}{\partial r}+\frac{1}{r}\right)\left(D_{rrrr}(r,t)-6D_{rrtt}(r,t)+D_{tttt}(r,t)\right) \\ & + \left(3\frac{\partial}{\partial r}-\frac{6}{r}\right)\left(D_{rrtt}(r,t)-\frac{1}{3}D_{tttt}(r,t)\right), \end{aligned} \tag{B.111}$$

and

$$\frac{\partial}{\partial r_n}D_{rttn}(\mathbf{r},t)=\left(\frac{\partial}{\partial r}+\frac{3}{r}\right)D_{rrtt}(r,t)-\frac{1}{r}D_{tttt}(r,t). \tag{B.112}$$

The last equation has to be inserted into (B.111) in order to get the equation for the longitudinal structure function $\frac{\partial}{\partial r_n}D_{rrrn}(\mathbf{r},t)$.

The equations for 2D turbulence now read

$$\frac{1}{r}\frac{\partial}{\partial r}\left[rD^{\mathbf{vvvv}}_{rrrr}(r)\right] - \frac{3}{r}D^{\mathbf{vvvv}}_{rrtt}(r) = -T_{rrr}(r), \tag{B.113}$$

$$\frac{1}{r^3}\frac{\partial}{\partial r}\left[r^3D^{\mathbf{vvvv}}_{rrtt}(r)\right] - \frac{1}{r}D^{\mathbf{vvvv}}_{tttt}(r) = -T_{rtt}(r). \tag{B.114}$$

For the antisymmetric tensor

$$D^{\mathbf{vvbb}}_{ij,kn}(\mathbf{r},t) = \langle v_iv_jb_kb_n - b_ib_jv_kv_n\rangle = D^{\mathbf{vvbb}}_{rr,tt}(r,t)\left(\frac{r_ir_j}{r^2}\delta_{kn} - \frac{r_kr_n}{r^2}\delta_{ij}\right), \tag{B.115}$$

we calculate the divergence as

$$\begin{aligned}\frac{\partial}{\partial r_n}D^{\mathbf{vvbb}}_{ij,kn}(\mathbf{r},t) &= \frac{\partial}{\partial r}D^{\mathbf{vvbb}}_{rr,tt}(r,t)\left[\frac{r_ir_jr_k}{r^3} - \frac{r_k}{r}\delta_{ij}\right] \\ &+ \frac{D^{\mathbf{vvbb}}_{rr,tt}(r,t)}{r}\left[\frac{r_i}{r}\delta_{jk} + \frac{r_j}{r}\delta_{ik} - \frac{r_ir_jr_k}{r^3} - \frac{r_k}{r}\delta_{ij}\right].\end{aligned} \tag{B.116}$$

Let us define $A_{ijk,n}(\mathbf{r},t)$ as

$$A_{ijk,n}(\mathbf{r},t) = D^{\mathbf{vvbb}}_{ij,kn}(\mathbf{r},t) + D^{\mathbf{vvbb}}_{kj,in}(\mathbf{r},t) + D^{\mathbf{vvbb}}_{ik,jn}(\mathbf{r},t). \tag{B.117}$$

We obtain

$$\begin{aligned}\frac{\partial}{\partial r_n}A_{ijk,n}(\mathbf{r},t) &= \left(3\frac{\partial}{\partial r}D^{\mathbf{vvbb}}_{rr,tt}(r,t) - \frac{3}{r}D^{\mathbf{vvbb}}_{rr,tt}(r,t)\right)\frac{r_ir_jr_k}{r^3} \\ &- \frac{\partial}{\partial r}D^{\mathbf{vvbb}}_{rr,tt}(r,t)\left[\frac{r_i}{r}\delta_{jk} + \frac{r_j}{r}\delta_{ik} + \frac{r_k}{r}\delta_{ij}\right].\end{aligned} \tag{B.118}$$

Therefore, the coefficients can be read of analogous to (B.85) and (B.112) as

$$\frac{\partial}{\partial r_n}A_{rrrn}(\mathbf{r},t) - 3\frac{\partial}{\partial r_n}A_{rttn}(\mathbf{r},t) = \left(3\frac{\partial}{\partial r} - \frac{3}{r}\right)D^{\mathbf{vvbb}}_{rr,tt}(r,t), \tag{B.119}$$

and

$$\frac{\partial}{\partial r_n}A_{rttn}(\mathbf{r},t) = -\frac{\partial}{\partial r}D^{\mathbf{vvbb}}_{rr,tt}(r,t). \tag{B.120}$$

Appendix C

C.1 Additional figures

C.1.1 The third order longitudinal magnetic structure function

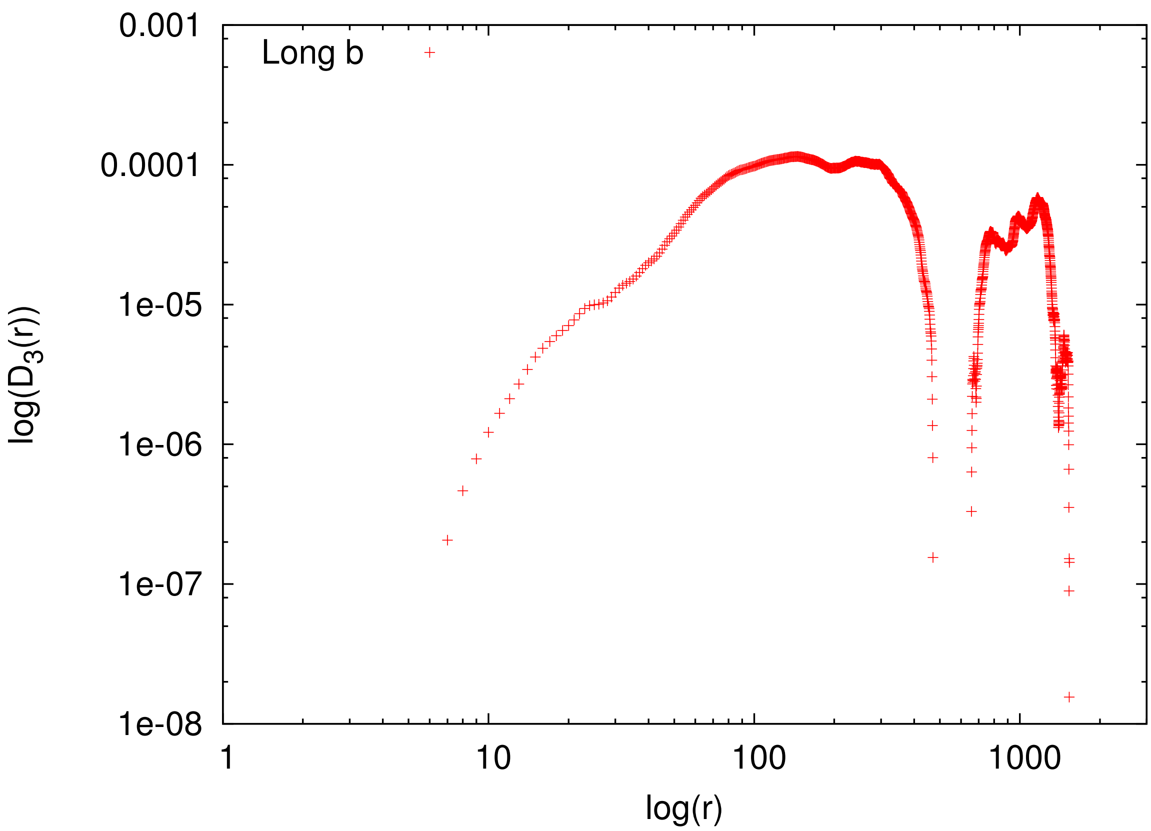


Figure C.1: The third order longitudinal magnetic structure function from the simulations in table 5.1.

List of Figures

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Erklärung zur Master-Arbeit

Hiermit versichere ich, diese Arbeit selbständig angefertigt und außer den angegebenen keine weiteren Hilfsmittel verwendet zu haben.

Münster, im November 2012