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New Templates for Inflationary Gravitational Waves

by

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Conventions

In this thesis the following conventions are used:

- Natural units: $\hbar = c = k_B = 1$
- Reduced Planck mass: $M_{\text{Pl}} = 1/\sqrt{8\pi G}$
- “Mostly plus” convention: $\eta_{\mu\nu} = \text{diag}(-1, +1, +1, +1)$

1 Introduction

“I never understood how anyone could feel small compared with the Universe. After all, man knows how overwhelmingly large it is, and a few others things besides, and that means he is not small. The fact that man has discovered all this precisely proves his greatness.”

— Harry Mulisch, *The Discovery of Heaven*

Since the first direct detection of gravitational waves (GW) in 2015 by the LIGO and VIRGO collaboration [1], *gravitational-wave astronomy* has entered an exciting new era, when multiple pulsar timing array (PTA) collaborations—including CPTA [2], EPTA/InPTA [3], MPTA [4], NANOGrav [5], and PPTA [6]—reported evidence for the detection of a stochastic gravitational-wave background (SGWB). These GWs contain information about some of the most extreme events in the Universe like black holes merging or even new physics beyond the Standard Model (SM). While the leading candidates for the SGWB are supermassive black-hole binaries (SMBHB) as an astrophysical explanation, the SGWB could also originate from cosmological processes in the early universe. Since the GWs evolve almost freely, they can be used to probe the early universe for even earlier times than the CMB (see fig. 1.1).

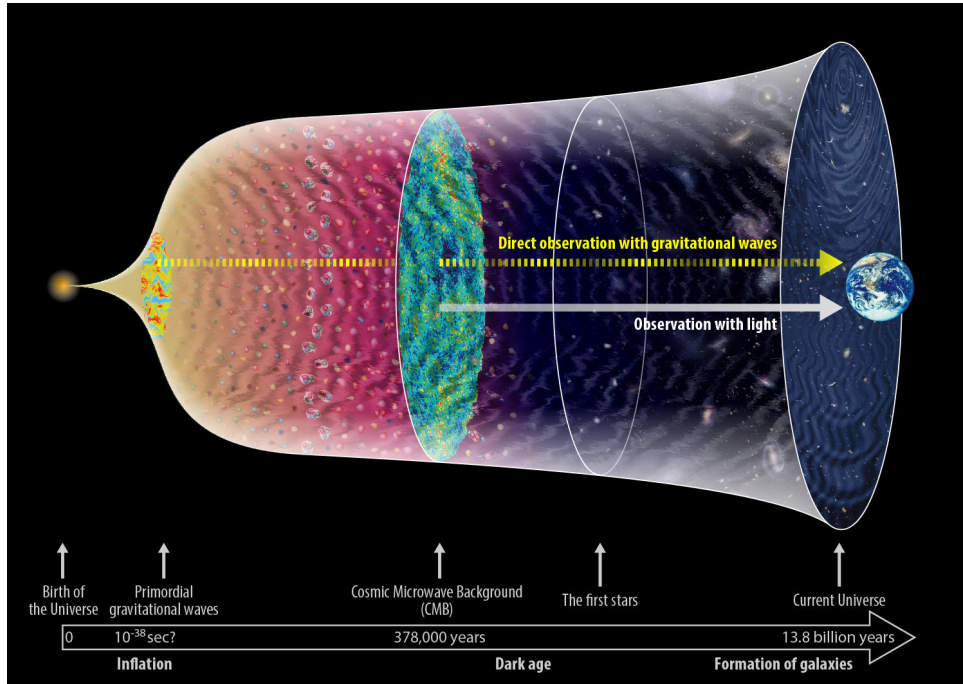


Figure 1.1: Illustration of the origin of GWs from the very early universe compared to the CMB. It can be seen that gravitational waves, if they are primordial, can be used as a messenger from the early universe much earlier than the CMB. [7]

Among the various cosmological mechanisms proposed, one of the most well-motivated scenarios is the generation of a SGWB during the *inflationary phase* of the Universe [8]. The perturbations produced during this phase form GWs that can be used to probe inflation. The resulting spectrum of the inflationary GWB can be conveniently decomposed

into two distinct contributions. The first is the *primordial tensor power spectrum*, which is generated during inflation itself and encodes information about the inflationary dynamics. The second is the *transfer function*, which describes the subsequent propagation of the GWs through the various cosmological epochs up to the present day [9–11]. While the primordial tensor power spectrum is directly sensitive to the underlying inflationary model, the transfer function is largely model-independent and instead reflects the thermal and expansion history of the Universe. One of these epochs in the expansion history is the so-called *reheating era*, which marks the transition from inflation to the Hot Big Bang. Although it is clear that this era has existed, not much is known about this phase. The goal of this thesis is thereby to construct an agnostic model of the transfer function for inflationary gravitational waves, including the effects of reheating, and to confront this model with PTA data from NANOGrav under the assumption that reheating occurred at an energy scale such that the corresponding feature in the GWB spectrum falls precisely within the NANOGrav frequency band. Similar analyses have been performed in [12, 13], but have assumed a EOS parameter w_{rh} , describing the ratio of pressure and energy density, during reheating of 0 and a constant decay rate of the inflaton field. New in this thesis is the treatment of w_{rh} as a free parameter and temperature-dependent decay rates. This approach allows the reheating-induced signature to be directly probed using current PTA observations, without committing to a specific model of reheating or inflation. Besides inflation and outside the scope of this thesis, cosmological SGWB sources such as phase transitions, cosmic strings, or domain walls have been proposed [14], typically predicting spectral shapes distinct from the tilted, reheating-modulated spectrum expected from inflation.

The remainder of this thesis is organized as follows: first, a short introduction to PTAs is given, then the theoretical background needed for creating new templates for inflation is explained. This includes a short introduction into inflation, primordial perturbations and reheating. Then, the principles of the PTA analysis pipeline is shown to fit the new templates to PTA data. In this thesis, the focus is set on the NANOGrav 15 yr (NG15) data set. This choice allows for a direct comparison with the reheating analysis performed in [12, 13], and the data set was readily accessible for this analysis. Lastly, the results for the new templates are shown and a conclusion as well as an outlook is made.

1.1 Pulsar timing arrays

PTAs like CPTA, EPTA/InPTA, MPTA, NANOGrav, and PPTA detect GWs by observing an array of millisecond pulsars. Millisecond pulsars are highly magnetized neutron stars with a rotational period in the low millisecond regime that emit radiation. Because the rotation of the pulsars is extremely stable, they can be used as astronomical clocks [15]. A gravitational wave passing between a pulsar and the Earth slightly perturbs the pulsar’s radio signal on its way to the telescope, causing its arrival time to shift slightly compared to what a regular, unperturbed pulsar clock would predict. The accumulated redshift can be written as a line integral of the perturbation along the photon path. This path integral reduces to two boundary terms: one at the pulsar, evaluated at emission

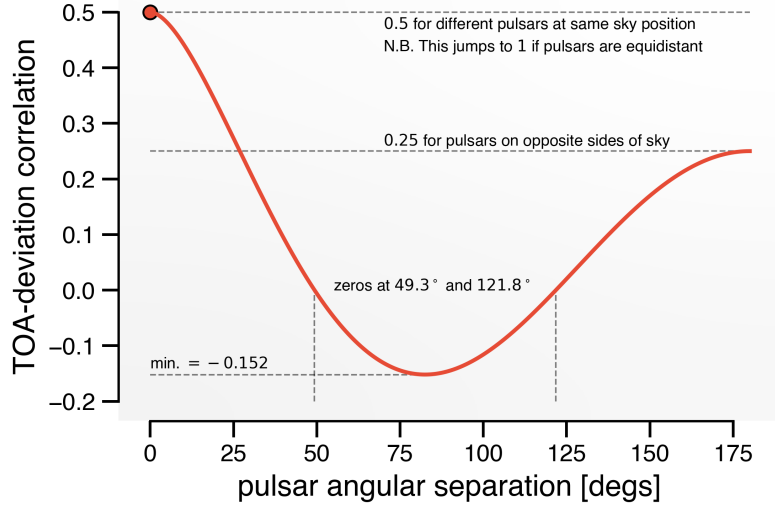


Figure 1.2: The Hellings–Downs curve, showing the expected correlation of pulsar timing residuals as a function of angular separation for an isotropic SGWB. [15]

(the pulsar term), and one at Earth, evaluated at reception (the Earth term). The pulsar term is uncorrelated between the pulsars and essentially behaves like extra noise, while the Earth term is shared between all pulsars observed at the same time. It is this shared Earth term that can be used to find correlations between pulsar pairs. A SGWB specifically induces a characteristic cross-correlation pattern in the timing residuals of the individual pulsars in the PTA. This pattern, the so-called Hellings–Downs curve [16] (see fig. 1.2), is the hallmark signature of a GWB signal and is now seen in the latest PTA data sets since summer 2023.

PTAs are sensitive to GWs in the nanohertz frequency range, roughly spanning $f \sim 1\text{--}100\text{ nHz}$. This range is set by the observational properties of the pulsar timing technique itself: the lower bound of the sensitivity band is determined by the total observation time span T_{obs} , typically corresponding to timescales of years to decades, of the pulsar timing campaign, with $f_{\text{min}} \sim 1/T_{\text{obs}}$. The upper bound, in turn, is set by the cadence of the observations, i.e., the typical time interval between successive measurements, $f_{\text{max}} \sim 1/(2\Delta t)$, which for current PTA campaigns corresponds to observation intervals on the order of weeks [17]. This nanohertz window makes PTAs uniquely complementary to ground-based interferometers such as LIGO/Virgo/KAGRA, which probe the GW spectrum at much higher frequencies ($\sim 10\text{--}10^4\text{ Hz}$) [18], as well as to future space-based detectors like LISA, which will target the millihertz band [19]. The quantity measured by the detectors is the (h^2 scaled) gravitational-wave energy density spectrum:

$$h^2\Omega_{\text{GW}}(f) = \frac{h^2}{\rho_c} \frac{d\rho_{\text{GW}}}{d\ln f} \quad (1.1)$$

with the GW energy density ρ_{GW} and the critical energy density $\rho_c = 3M_{\text{Pl}}^2 H_0^2$, conventionally expressed with the h^2 prefactor to remove the dependence on the present-day Hubble rate H_0 . The reason is the high uncertainty of H_0 due to the so-called Hubble tension [20]. h is defined with $H_0 = 100h \frac{\text{km}}{\text{sMpc}}$.

2 Theoretical background and models

In this chapter, the theoretical background required to introduce the analysis of this thesis is reviewed. After introducing the inflationary paradigm, the primordial perturbations generated during this stage are analyzed and concluded with a discussion about reheating.

2.1 Inflation

This section is primarily based on [21] as well as [22]. The old-school model of the Big Bang cosmology follows the cosmological principle: on cosmological scales, the Universe is homogeneous and isotropic. However, this model has some shortcomings. The most important are the so-called flatness and the horizon problem. In the following section, the two problems are briefly explained and inflation as a solution is introduced. The simplest model of inflation and its principles are explained.

2.1.1 The flatness problem

This problem comes from CMB observations indicating that the Universe is spatially flat. This means that the curvature parameter in the Friedmann–Lemaître–Robertson–Walker (FLRW) metric corresponds to $K = 0$. The general FLRW metric is given by:

$$ds^2 = -dt^2 + a^2(t) \left[\frac{dr^2}{1 - Kr^2} + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2) \right] \quad (2.1)$$

The FLRW metric describes an expanding isotropic and homogeneous universe, where $a(t)$ is the scale factor, t describes the cosmological time and r, θ, φ describe the comoving coordinates. The first Friedmann equation can be written as

$$H^2 = \frac{\rho + \rho_K}{3M_{\text{Pl}}^2} \quad (2.2)$$

Where $H = \dot{a}/a$ is the Hubble rate, ρ is the energy density and $\rho_K = -3M_{\text{Pl}}^2 K/a^2$ describes the energy density of the curvature. Matter and radiation typically decrease with the scale factor as $1/a^3$ and $1/a^4$. Comparing this to the curvature, one would expect that even with extremely small values of the curvature energy density at early times, the curvature should dominate today, which observationally is not the case. This means that either the curvature is extremely fine-tuned, or that at early times another process thinned out the curvature strongly.

2.1.2 The horizon problem

In the CMB, the temperature anisotropies are at the level $\delta T/T \sim 10^{-5}$, even for large angular scales of the order of the whole sky. This means that in the past, the large scale regions must have been in causal contact to form a thermodynamic equilibrium. With the old-school Hot Big Bang, one would expect that regions in the sky, which are separated by more than 2° should never have been in causal contact and the temperature differences should be of the order $\delta T/T \sim O(1)$.

These two problems can be solved by introducing a period of rapid expansion in the early Universe. This period is called *inflation*. With a long enough period of accelerated expansion, the curvature can be thinned out and regions, which were once in causal contact and formed an equilibrium can be separated. In principle, the accelerated expansion of the Universe leads to the fact that the Hubble radius, which describes the distance at which particles appear to move faster than light because of the expansion, starts to shrink.

2.1.3 Single field slow-roll inflation

The exact mechanisms for inflation are unknown and can be realized in many different ways. The easiest way to introduce accelerated expansion is the model of a single field slow-roll inflation. As it is the simplest way to form an inflation model, it will be used here to cover the basics of slow-roll inflation. In general, there is no need for inflation to just consist of a single scalar field. Famous examples for inflation models beyond single field slow-roll inflation are hybrid inflation, where there are two scalar fields, or axion inflation, where a pseudo-scalar field is used.

Consider the flat FLRW metric ($K = 0$):

$$ds^2 = -dt^2 + a^2 d\mathbf{x}^2 \quad (2.3)$$

In the single field slow-roll inflation, one introduces a single scalar field ϕ named the inflaton field. The scalar field moves in a potential $V(\phi)$ in curved space. As such, the action of the scalar field is

$$S_\phi = \int d^4x \sqrt{-g} \left[-\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right]. \quad (2.4)$$

From the action of the inflaton field, the energy momentum tensor can then be determined as

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta S_\phi}{\delta g^{\mu\nu}} = \partial_\mu \phi \partial_\nu \phi - g_{\mu\nu} \left[\frac{1}{2} g^{\rho\sigma} \partial_\rho \phi \partial_\sigma \phi + V(\phi) \right]. \quad (2.5)$$

This allows for a connection of the scalar field with Einstein's field equations. From this, the energy density ρ_ϕ and pressure p_ϕ with $p_\phi \delta_{ij} = T_{ij}$ can be determined as

$$\rho_\phi = T_{00} = \frac{1}{2} \dot{\phi}^2 + V(\phi), \quad (2.6)$$

$$p_\phi = \frac{1}{2} \dot{\phi}^2 - V(\phi). \quad (2.7)$$

The equation of motion (EOM) of a scalar field in a potential $V(\phi)$ here is the Klein-Gordon equation in curved space. Assuming that the inflaton field is spatially homogeneous, the gradients of the inflaton field can be neglected and the EOM reads

$$\ddot{\phi} + 3H\dot{\phi} + \frac{dV}{d\phi} = 0 \quad (2.8)$$

An important quantity of the inflaton is the equation of state (EOS) parameter, which describes the relation of the pressure and the energy. It is crucial to know the EOS

parameter to calculate the expansion rate and evolution of the energy density, as the energy density for a field with EOS parameter w is proportional to

$$\rho(a) \propto a^{-3(1+w)}. \quad (2.9)$$

For the inflaton field, the EOS parameter reads

$$w_\phi = \frac{p_\phi}{\rho_\phi} = \frac{\frac{1}{2}\dot{\phi}^2 - V(\phi)}{\frac{1}{2}\dot{\phi}^2 + V(\phi)}. \quad (2.10)$$

For comparison, cold matter as a pressure-less gas follows an EOS parameter of $w = 0$, meaning that with the expansion of the Universe it drops with $\rho_M \propto a^{-3}$, while radiation follows an EOS parameter of $w = 1/3$, such that the energy density is dropping as a^{-4} . This can naturally be understood as matter being diluted in expanding space, while radiation additionally loses energy by the wavelengths getting stretched. Inflation needs a phase of accelerated expansion. The second Friedmann equation for the inflaton in the flat FLRW metric reads

$$\frac{\ddot{a}}{a} = -\frac{\rho_\phi + 3p_\phi}{6M_{\text{Pl}}^2} = -\frac{(1 + 3w_\phi)\rho_\phi}{6M_{\text{Pl}}^2}. \quad (2.11)$$

Then, an accelerated expansion with $\ddot{a} > 0$ can be obtained by the condition $w_\phi < -1/3$. This can be realized in eq. 2.10 by taking the limit

$$\frac{1}{2}\dot{\phi}^2 \ll V(\phi) \quad \rightarrow \quad w_\phi = -1 \quad (2.12)$$

Because this means a low kinetic energy compared to the potential energy, this is called the *first slow-roll condition*. If $w_\phi = -1$ exactly, the metric is called a de Sitter space, where H is constant and the scale factor will be growing exponentially. In order to solve the flatness and the horizon problem, inflation needs to be sustained over an extended period of time. For the standard instantaneous reheating scenario, where the energy of the inflaton field is directly converted into SM particles, the slow-roll condition needs to be satisfied for around $\Delta N \sim 60$ e -folds. An e -fold is the time needed to grow the scale factor a by the factor e ($a \rightarrow e \cdot a$). It should be noted that this estimate assumes an instantaneous reheating scenario, where all the energy from the inflaton field gets transferred into SM particles instantly. This is not what is done in this thesis and will be discussed in detail in sec. 2.3. Either way, inflation needs to last relatively long and from this one can form the *second slow-roll condition*,

$$|\ddot{\phi}| \ll \left| \frac{dV}{d\phi} \right| \quad (2.13)$$

which describes that the kinetic energy should not grow fast, such that inflation is sustained. One could imagine single field slow-roll inflation as a sticky ball slowly rolling down the potential, creating an accelerated expansion because of its high potential energy. Using the Friedmann equations and the EOM of the inflaton, the slow-roll conditions can be described as conditions on the potential, from which the slow-roll parameters can be

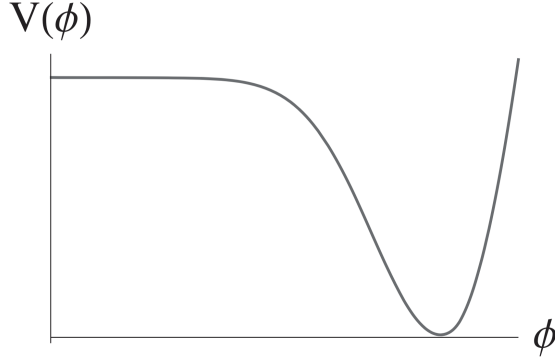


Figure 2.1: Example of a hill-top potential. At the beginning of inflation, the inflaton field sits at the top of the potential, slowly rolling down towards the minimum. When it reaches the minimum, it starts to oscillate, ending the inflation period. [21]

defined:

$$\epsilon_V = \frac{M_{\text{Pl}}^2}{2} \left(\frac{dV/d\phi}{V} \right)^2 \quad (2.14)$$

$$\eta_V = M_{\text{Pl}}^2 \frac{d^2 V/d\phi^2}{V} \quad (2.15)$$

In general, there is a great variety of models of inflation, ranging from so-called large-field models with super-Planckian field values $\phi \gtrsim M_{\text{Pl}}$ to small-field models with $\phi \ll M_{\text{Pl}}$. A common example for a small-field inflation is the so-called hill-top potential (see Fig. 2.1). The slow-roll parameters here allow for approximations independent of the choice of the specific model to calculate effects of inflation, leaving predictions agnostic of the exact mechanisms.

2.2 Primordial perturbations

The model presented in sec. 2.1 is purely classical. But of course, one would expect the inflaton field to also be a quantum field with quantum fluctuations. Due to the accelerated expansion, these fluctuations can be stretched to cosmological sizes, which could in principle be detected today to learn about the early Universe beyond the CMB. In general, the perturbations can be decomposed into scalar, vector and tensor perturbations, with the vector perturbations being negligible. This is because in linear perturbations theory of the field equations, the vector perturbations don't have a source in a perfect fluid, typically used to describe the Universe, and every other initial vector perturbations are damped strongly by the expansion of the universe [23]. In this chapter, the creation and evolution of these perturbations are shown in the scalar and the tensor case in the example of the single field slow-roll inflation. For details about the tensor derivation see sec. A.

2.2.1 Mukhanov-Sasaki equation

Going back to eq. 2.4, one sees that due to the curved space, there exists a non-trivial coupling between the metric and the inflaton field fluctuations. Depending on the chosen

gauge, the mixing changes drastically. In comoving gauge, the inflaton fluctuations vanish and all of the fluctuations go into the metric fluctuations. It is convenient however to work in the spatially flat gauge to connect the fluctuations to the inflaton field. Including scalar fluctuations in the metric of eq. 2.3 and introducing the conformal time η with $d\eta = dt/a$ gives

$$ds^2 = a^2 \left[-(1 + 2A) d\eta^2 + 2\partial_i B dx^i d\eta + \delta_{ij} dx^i dx^j \right] \quad (2.16)$$

where A and B describe the metric fluctuations. The inflaton field will be split in two parts, $\phi = \bar{\phi} + \delta\phi$, where $\bar{\phi}$ describes the unperturbed field and $\delta\phi$ describes the fluctuations. Varying the action of the inflaton field in this metric and staying at first order of fluctuations leads to the following EOM for the perturbations:

$$\delta\phi'' + 2\mathcal{H}\delta\phi' - \nabla^2\delta\phi = (A' + \nabla^2 B) \bar{\phi}' - 2a^2 \frac{dV}{d\phi} A - a^2 \frac{d^2 V}{d\phi^2} \delta\phi \quad , \quad (2.17)$$

where $f' = \frac{df}{d\eta}$ and $\mathcal{H} = \frac{a'}{a}$ describes the conformal Hubble rate. To simplify the expression, the perturbed Einstein equations can be used, which describe the evolution of the perturbations apart from the homogeneous background,

$$\delta G_{\mu\nu} = \frac{1}{M_{\text{Pl}}^2} \delta T_{\mu\nu} . \quad (2.18)$$

With this, A and B can be connected with $\delta\phi$, removing two unknowns from the equation:

$$A = \frac{\phi'}{2M_{\text{Pl}}^2 \mathcal{H}} \delta\phi, \quad (2.19)$$

$$\nabla^2 B = -\frac{\phi'}{2M_{\text{Pl}}^2 \mathcal{H}} \left[\delta\phi' + \left(\mathcal{H} - \frac{\phi''}{\phi'} - \frac{(\phi')^2}{2M_{\text{Pl}}^2 \mathcal{H}} \right) \delta\phi \right]. \quad (2.20)$$

Inserting this into eq. 2.17, performing a Fourier transform and defining the variable $f = a\delta\phi$ as a rescale of the inflaton field gives the *Mukhanov-Sasaki equation*:

$$f'' + \left(k^2 - \frac{z''}{z} \right) f = 0, \quad \text{where} \quad z \equiv \frac{a\bar{\phi}'}{\mathcal{H}}, \quad (2.21)$$

with the comoving wave number k . The equation describes a harmonic oscillator with a time-dependent frequency:

$$\omega^2(\eta, k) = k^2 - \frac{z''(\eta)}{z(\eta)}. \quad (2.22)$$

During slow-roll inflation, $\bar{\phi}'/\mathcal{H} = \dot{\phi}/H$ is approximately constant, due to the acceleration of the inflaton field being negligible and the Hubble rate staying constant, such that $z''/z = a''/a$. By using $\mathcal{H}' = \mathcal{H}^2 \left(1 + \frac{\dot{H}}{H^2} \right)$ with $\dot{H} \approx 0$ and $\frac{a''}{a} = \mathcal{H}' + \mathcal{H}^2$, the quantity z''/z can then be written as

$$\frac{z''}{z} = 2\mathcal{H}^2. \quad (2.23)$$

The comoving Hubble radius can be described by $\mathcal{H}^{-1}$, which allows one to describe the behavior of the perturbations. In early times in inflation, $k \gg \mathcal{H}$, such that the mode

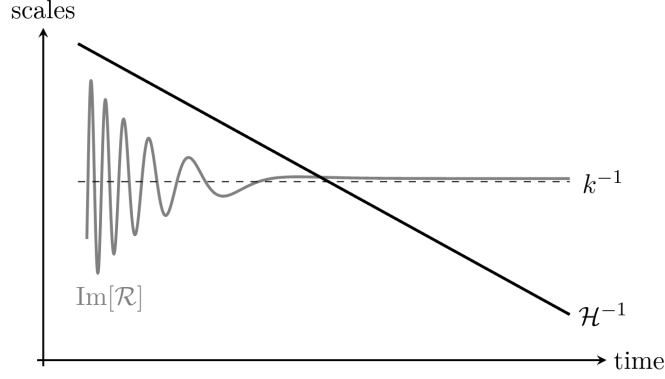


Figure 2.2: Evolution of a mode during slow-roll inflation. The subhorizon mode oscillates during a shrinking comoving Hubble radius until it becomes superhorizon and freezes to a constant value. Here, $\mathcal{R} = f/z$ describes the comoving curvature perturbation. [22]

will be inside the comoving Hubble radius (subhorizon) and oscillates freely while being damped by the expansion of the Universe:

$$f'' + k^2 f = 0 \quad \rightarrow \quad \delta\phi = f/a \sim \frac{\cos(k\eta)}{a(\eta)}. \quad (2.24)$$

For $k \ll \mathcal{H}$, such that the mode is outside of the Hubble radius (superhorizon), the equation becomes:

$$f'' - \frac{z''}{z} f = 0. \quad (2.25)$$

The solution $f \sim z$ shows that in this situation, the fluctuations can freeze and become a constant value $\delta\phi = f/a = \text{const.}$ This shows that for an accelerated Universe with shrinking comoving Hubble radius $\mathcal{H}^{-1}$, fluctuations produced during inflation will eventually leave the horizon and stay at their value (see fig. 2.2), which is called freezing, until the comoving Hubble radius starts to grow again during the Hot Big Bang, meaning after the end of inflation. Due to the expansion of the universe, this will be seen as perturbations getting enlarged to cosmological scales, serving as a preservation of the processes during inflation.

2.2.2 Quantum fields in inflation

Due to the quantum nature of the inflaton field, the modes need to be quantized. This quantum nature is also the natural reason for perturbations of the inflaton field, forming the sources of it. Taking the slow-roll limit and the de Sitter space, eq. 2.21 becomes:

$$f'' + \left(k^2 - \frac{2}{\eta^2}\right) f = 0 \quad (2.26)$$

The field f_k is promoted to a quantum operator $\hat{f}(\eta, \mathbf{x})$ with its conjugate momentum $\hat{\pi}(\eta, \mathbf{x})$. Performing the first quantization:

$$\left[\hat{f}(\eta, \mathbf{x}), \hat{\pi}(\eta, \mathbf{x}')\right] = i\delta(\mathbf{x} - \mathbf{x}') \quad (2.27)$$

and switching to Fourier space gives:

$$[\hat{f}_{\mathbf{k}}(\eta), \hat{\pi}_{\mathbf{k}'}(\eta)] = i(2\pi)^3 \delta(\mathbf{k} + \mathbf{k}') \quad (2.28)$$

It is convenient to write $\hat{f}_{\mathbf{k}}$ as

$$\hat{f}_{\mathbf{k}} = f_k(\eta) \hat{a}_{\mathbf{k}} + f_k^*(\eta) \hat{a}_{-\mathbf{k}}^\dagger \quad (2.29)$$

with $f_k(\eta)$ satisfying the classical EOM. $\hat{a}_{\mathbf{k}}$ and $\hat{a}_{\mathbf{k}}^\dagger$ function as creation and annihilation operators of the mode $\mathbf{k}$. Choosing the canonical normalization

$$f_k f_k'^* - f_k' f_k^* = i \quad (2.30)$$

leads to $[\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{k}'}^\dagger] = (2\pi)^3 \delta(\mathbf{k} - \mathbf{k}')$. Defining the vacuum as $\hat{a}_{\mathbf{k}}(\eta_i) |0\rangle = 0$ at the initial time η_i defines the initial conditions of the mode. However, since the Hamiltonian is time dependent, the vacuum is dependent on the specific η_i chosen. Choosing the ground state at the beginning of inflation means choosing the limit $\eta_i \rightarrow -\infty$ or $|k\eta_i| \gg 1$. Since at this time the mode is deep inside the horizon, the initial condition of the modes become:

$$\lim_{k\eta \rightarrow -\infty} f_k(\eta) = \frac{1}{\sqrt{2k}} e^{-ik\eta} \quad (2.31)$$

which sets the solution of the Mukhanov-Sasaki equation to the *Bunch-Davies mode function*:

$$f_k(\eta) = \frac{1}{\sqrt{2k}} \left(1 - \frac{i}{k\eta} \right) e^{-ik\eta} \quad (2.32)$$

The Bunch-Davies mode function thus can be used to describe the evolution of the quantum operator $\hat{f}_{\mathbf{k}}$ and can be used to compute the amount of fluctuations in the inflaton field.

2.2.3 Scalar perturbations

To get the statistics of the fluctuations, the expectation value and variance need to be determined:

$$\langle \hat{f} \rangle = \langle 0 | \hat{f} | 0 \rangle = 0 \quad (2.33)$$

$$\langle |\hat{f}|^2 \rangle = \langle 0 | \hat{f}(\eta, \mathbf{0}) \hat{f}(\eta, \mathbf{0}) | 0 \rangle = \int \frac{dk}{k} \frac{k^3}{2\pi^2} |f_k(\eta)|^2. \quad (2.34)$$

As expected by $f = a\delta\phi$, the expectation value vanishes for the inflaton field and the fluctuations are described by the variance. A useful quantity to describe the variance is the power spectrum. The power spectrum $P_f(k)$ for f can be defined from the two-point correlator,

$$\langle \hat{f}_{\mathbf{k}} \hat{f}_{\mathbf{k}'}^* \rangle = \frac{1}{V} (2\pi)^3 \delta(\mathbf{k} - \mathbf{k}') P_f(k) \quad , \quad (2.35)$$

where V denotes the normalization volume of the Fourier transform, such that the power spectrum can be determined as $P_f(k, \eta) = |f_k(\eta)|^2$. The “dimensionless” power spectrum

is defined as $\mathcal{P}_f(k) = \frac{k^3}{2\pi^2} P_f(k)$, such that

$$\mathcal{P}_f(\eta, k) = \frac{k^3}{2\pi^2} |f_k(\eta)|^2 \quad . \quad (2.36)$$

It is called the “dimensionless” power spectrum as the Fourier integral can then be computed on a logarithmic scale, removing k dependencies in the integral,

$$\langle |\hat{f}|^2 \rangle = \int d \ln k \mathcal{P}_f(k) \quad . \quad (2.37)$$

With eq. 2.32 the “dimensionless” power spectrum of the inflaton fluctuations becomes

$$\mathcal{P}_{\delta\phi}(k, \eta) = \frac{\mathcal{P}_f(k, \eta)}{a^2} = \left(\frac{H}{2\pi} \right)^2 [1 + (k\eta)^2] \quad (2.38)$$

Note that in the superhorizon limit $k\eta \rightarrow 0$, the power spectrum is a constant, due to H being constant in the slow-roll limit. It is convenient to describe the scalar fluctuations in form of the comoving curvature perturbation $\mathcal{R} = \frac{\mathcal{H}}{\phi'} \delta\phi$, leading to the power spectrum

$$\mathcal{P}_{\mathcal{R}}(k) = \left(\frac{H^2}{2\pi \dot{\phi}^2} \right)^2 \quad . \quad (2.39)$$

Unlike $\delta\phi$, which is gauge-dependent, the comoving curvature perturbation $\mathcal{R}$ is invariant under gauge transformation at linear order, making it the physically meaningful quantity to connect inflationary predictions to observations. However, it is important to include the time dependence of H , because inflation would never end if H stays constant. Taking the first order of slow-roll expansion leads to a spectral tilt of the comoving curvature perturbation power spectrum $\mathcal{P}_{\mathcal{R}}(k)$ in the form of:

$$\mathcal{P}_{\mathcal{R}}(k) = A_s \left(\frac{k}{k_*} \right)^{n_s - 1} \quad (2.40)$$

where A_s is the amplitude at the pivot scale k_* and $n_s - 1$ the spectral tilt with

$$n_s - 1 = 2\eta_V - 6\epsilon_V \quad . \quad (2.41)$$

Measuring the spectral tilt of the spectrum for scalar perturbations is thus crucial to learn about inflation with the slow-roll parameters (see eq. 2.14 and 2.15). Since in slow-roll inflation $\epsilon_V \ll 1$ and $\eta_V \ll 1$, n_s is close to 1, which is consistent with the measurement $n_s = 0.9649 \pm 0.0042$ [24] obtained from CMB measurements.

2.2.4 Tensor perturbations

The quantum fluctuations of the inflaton can not only produce scalar perturbations, but also tensor perturbations in the form of GWs. Similar to the Mukhanov-Sasaki equation, to compute the spectrum of GWs sourced by the inflaton field, the EOM of these tensor modes is needed. To get the evolution of the tensor perturbations, they have to be included

in the metric. The transverse-traceless part of the perturbed FLRW metric, describing the GW part, is

$$ds^2 = a^2 [-d\eta^2 + (\delta_{ij} + h_{ij}^{\text{TT}})dx^i dx^j] \quad , \quad (2.42)$$

where h_{ij}^{TT} is a transverse-traceless tensor. This time for the GWs, one looks at the transverse-traceless part of the eq. 2.18. Staying at first order of h_{ij}^{TT} , one finds

$$(h_j^{i\text{TT}})'' + 2\mathcal{H}(h_j^{i\text{TT}})' - \nabla^2 h_j^{i\text{TT}} = 16\pi G a^2 \sigma_j^{i\text{TT}} \quad , \quad (2.43)$$

where σ_{ij}^{TT} is the transverse-traceless part of the perturbed part of the energy-momentum tensor δT_{ij} describing the anisotropic stress. The tensor perturbations and the anisotropic stress can be described in the polarization tensor base in momentum space,

$$h_j^{i\text{TT}}(\eta, \mathbf{k}) = \sum_{A=+, \times} e_j^{iA}(\mathbf{k}) \cdot h_A(\eta, \mathbf{k}) \quad (2.44)$$

$$\sigma_j^{i\text{TT}}(\eta, \mathbf{k}) = \sum_{A=+, \times} e_j^{iA}(\mathbf{k}) \cdot \sigma_A(\eta, \mathbf{k}) \quad , \quad (2.45)$$

with the polarization tensors e_{ij}^A fulfilling $e_{ij}^A(\mathbf{k}) \cdot e_{ij}^{A'}(\mathbf{k}) = 2\delta^{AA'}$. A describes the two possible polarization modes $+, \times$. Choosing this base and performing a Fourier transform leads to the mode equation of the tensor perturbations,

$$h_A'' + 2\mathcal{H}h_A' + k^2 h_A = 16\pi G a^2 \sigma_A \quad (2.46)$$

In most cases, the anisotropic stress is negligibly small, leading to $\sigma_{ij} = 0$, which will be used in the following analysis. The mode equation is similar to the scalar perturbation case. Here, the tensor modes can also freeze during inflation, stretching them to cosmological scales. The quantization process will be very much the same. However, h_A is not a canonically normalized field. To get the power spectrum, one thus needs to compute the second order of perturbation to perform normalization. Going to the second order, the action can be written as

$$S_2[h] = \frac{M_{\text{Pl}}^2}{8} \int d^3x d\eta a^2 [\partial_\eta h_{ij}^{\text{TT}} \partial_\eta h_{ij}^{\text{TT}} - \partial_k h_{ij}^{\text{TT}} \partial_k h_{ij}^{\text{TT}}] \quad (2.47)$$

Using the polarization tensor base then gives

$$S_2[h] = -\frac{1}{2} \sum_A \int d^4x \sqrt{-\bar{g}} \bar{g}^{\mu\nu} \partial_\mu \varphi_A \partial_\nu \varphi_A \quad , \quad (2.48)$$

with the background metric $\bar{g}^{\mu\nu}$ and $\varphi_A = \frac{M_{\text{Pl}}}{\sqrt{2}} h_A$. With this choice of φ_A , the action now describes two canonically normalized scalar fields in curved space. For a homogeneous and isotropic source, the tensor power spectrum can again then be described through the variance:

$$\langle h_A(\eta, \mathbf{k}) \cdot h_{A'}^*(\eta, \mathbf{k}') \rangle = \frac{1}{4} \frac{(2\pi)^3}{V} \frac{2\pi^2}{k^3} \delta(\mathbf{k} - \mathbf{k}') \delta_{AA'} P_t(k) \quad . \quad (2.49)$$

Using the description of the canonically normalized scalar field φ_A and with $\mathcal{P}_t(k) = \frac{2}{M_{\text{Pl}}^2} \mathcal{P}_\varphi(k)$ and the dimensionless power spectrum $\mathcal{P}_t(k) = \frac{k^3}{2\pi^2} P_t(k)$, the computations follow the scalar perturbation case. In lowest order of slow-roll expansion, the dimensionless power spectrum then in the superhorizon limit gives

$$\mathcal{P}_t(k) = \frac{2}{\pi^2} \left(\frac{H}{M_{\text{Pl}}} \right)^2. \quad (2.50)$$

In the lowest order, similar to the scalar case, the power spectrum is again a constant. In first order of slow roll needed for inflation to end, the spectrum gets a spectral tilt n_t , which depends on the time dependence of H , leading to

$$n_t = \left(\frac{d \log V}{d \log k} \right)_{k_*} = -2\epsilon_V \quad (2.51)$$

at the pivot scale k_* . In single field slow-roll inflation, one thus expects a slightly red-tilted spectrum. With the spectral tilt, the tensor power spectrum then can be written as

$$\mathcal{P}_t(k) = A_t \left(\frac{k}{k_*} \right)^{n_t}. \quad (2.52)$$

An important ratio to measure to describe perturbations in inflation is the tensor-to-scalar ratio

$$r = (\mathcal{P}_t/\mathcal{P}_\mathcal{R})_{k_*} = A_t/A_s. \quad (2.53)$$

In first order of slow-roll inflation, the spectral tilt takes the value of a slightly red-tilted spectra $n_t = -2\epsilon_V < 0$ and the tensor-to-scalar ratio is $r = 16\epsilon_V = -8n_t$. Since ϵ_V does not appear explicitly anymore in this relation, the measure is independent of the choice of the inflationary potential, making the experimental confirmation of this relation very convincing evidence for single field slow-roll inflation. However, the relation between n_t and r is not always as trivial when models beyond the single field slow-roll approximation are taken into account. While the values of n_s are constrained by experiments, n_t is not. Blue-tilted spectra with $n_t > 0$ can be generated in many non-minimal inflation scenarios like axion-vector dynamics during inflation (see for example [25, 26]) or other models (see for example [27, 28]). To not choose a specific inflation model, the tensor power spectrum for this thesis will be chosen in a way such that for blue-tilted spectra the maximum of $\mathcal{P}_t$ is at the end of inflation and then will be cut off for $k > k_{\text{end}}$,

$$\boxed{\mathcal{P}_t(k) = A_t^{\text{max}} \left(\frac{k}{k_{\text{end}}} \right)^{n_t} \Theta(k_{\text{end}} - k) = A_t^{\text{max}} \left(\frac{k}{\mathcal{H}_{\text{end}}} \right)^{n_t} \Theta(\mathcal{H}_{\text{end}} - k)}. \quad (2.54)$$

As this thesis focuses on blue-tilted spectra, the quantity A_t^{max} thus describes the maximum amplitude of the power spectrum. The restriction is motivated by the observation that the SGWB measured by PTAs is blue-tilted. The cutoff originates from the fact that modes cannot exit the horizon after the end of inflation, since the comoving Hubble horizon is starting to grow in the Hot Big Bang, and thus are not stretched to super-horizon scales such that no primordial tensor modes can be generated for $k > k_{\text{end}}$. However, even

for blue-tilted spectra, r is still useful to measure for constraining models, which will be used in sec. 2.4.

2.3 Reheating

This section is based on [29]. To connect inflation to the rest of the Hot Big Bang, the energy of the inflaton field needs to be transferred into SM particles. When the inflaton field decays into SM particles, the energy density of the SM particles rises, reheating the Universe. This phase to connect inflation and Hot Big Bang is thus called the reheating era. This phase of the Universe is highly interesting, because it is responsible for forming the known Universe and its structures, although much is not known about it. Here, the tensor perturbations of inflation can lead to significant discoveries. As soon as inflation ends, the comoving Hubble radius starts to grow again and the tensor perturbations unfreeze and start to evolve through the universe. The first modes that become subhorizon enter during the reheating era and thus can carry information how this phase could have looked. This chapter focuses on the general dynamics of the inflaton field and radiation during reheating, while staying agnostic about the specific form of the potential of the inflaton field. In the next steps, this model of reheating is then used to determine the GW spectrum created by the inflaton field.

2.3.1 Boltzmann equations

To describe reheating, one takes a look at the Boltzmann equations of the inflaton field. Using eq. 2.6 and 2.7 in eq. 2.8 leads to the Boltzmann equation for the inflaton field in curved space,

$$\dot{\rho}_\phi + 3H(1 + w_\phi)\rho_\phi = 0. \quad (2.55)$$

At the end of inflation, the inflaton field starts to oscillate in a minimum and decays into SM particles. This can be modeled effectively with a decay rate Γ in the EOM:

$$\ddot{\phi} + (3H + \Gamma)\dot{\phi} + \frac{dV}{d\phi} = 0, \quad (2.56)$$

or in the Boltzmann equation:

$$\dot{\rho}_\phi + (3H + \Gamma)(1 + w_\phi)\rho_\phi = 0. \quad (2.57)$$

The Hubble rate term is called Hubble friction, where the expansion of the Universe causes the energy density to fall off. If reheating should sustain, the Hubble rate needs to be chosen such that $\Gamma \ll H$ during reheating, otherwise decay would immediately take over and reheating would end. Depending on the form of the potential around that minimum, the inflaton field behaves differently. Looking at 2.10, one can average over the oscillations of the field,

$$w_{\text{rh}} = \langle w_\phi \rangle = \frac{\langle p_\phi \rangle}{\langle \rho_\phi \rangle} \quad (2.58)$$

To compute the average, one assumes that ρ_ϕ stays constant during an oscillation. This is well-motivated by the fact that in an oscillation, the velocity at the maximum amplitudes is 0 and maximal in the minimum, meaning that the particle spends most of the time of the oscillation close to the maximum amplitudes. However in the most extreme cases, where the potential is extremely steep, this approximation can fail and the oscillations of the field need to be taken fully into account. With $w_\phi = \frac{\dot{\phi}^2}{\rho_\phi} - 1$, the average then becomes

$$w_{\text{rh}} = \frac{\langle \dot{\phi}^2 \rangle}{\rho_\phi} - 1. \quad (2.59)$$

With $dt = d\phi/\dot{\phi}$ and assuming a symmetric potential $V(\phi) = V(-\phi)$, averaging over one oscillation can be written as

$$\langle \dot{\phi}^2 \rangle = \frac{4}{T_{\text{osc}}} \int_{\phi_{\text{min}}}^{\phi_{\text{max}}} \dot{\phi} d\phi \quad T_{\text{osc}} = \oint dt = 4 \int_{\phi_{\text{min}}}^{\phi_{\text{max}}} \frac{d\phi}{\dot{\phi}}, \quad (2.60)$$

where $\phi_{\text{min/max}}$ is the minimum/maximum amplitude during one oscillation. The factor 4 comes from the fact that the inflaton needs to go two times from minimum to maximum and two times from maximum to minimum for one oscillation. With eq. 2.6, the averaged EOS parameter thus becomes

$$w_{\text{rh}} = 2 \frac{\int_{\phi_{\text{min}}}^{\phi_{\text{max}}} (1 - V/\rho_\phi)^{1/2} d\phi}{\int_{\phi_{\text{min}}}^{\phi_{\text{max}}} (1 - V/\rho_\phi)^{-1/2} d\phi} - 1. \quad (2.61)$$

To stay agnostic about the exact form of the potential, the following potential around the minimum will be taken:

$$V(\phi) = \Lambda^4 \left| \frac{\phi}{M_{\text{Pl}}} \right|^p. \quad (2.62)$$

This potential can then approximate any form near the minimum and reproduce different inflation models with the parameters Λ and p . The parameter Λ depends on the Hubble rate at the end of inflation H_{end} . The first Friedmann equation reads

$$H^2 = \frac{\rho_{\text{tot}}}{3M_{\text{Pl}}^2}. \quad (2.63)$$

Assuming that at the end of inflation there is only the inflaton field such that $\rho_{\text{tot}} = \rho_\phi$, one can use the fact that $\ddot{a} = 0$ and $\ddot{\phi} = 0$ and the second Friedmann equation 2.11 to connect Λ and H_{end} :

$$\Lambda^4 = 2M_{\text{Pl}}^2 H_{\text{end}}^2 \left(\frac{\phi_{\text{end}}}{M_{\text{Pl}}} \right)^p, \quad (2.64)$$

where ϕ_{end} is the value of ϕ at the end of inflation. Using again the Friedmann equations and the properties of end of inflation, one can show that

$$\frac{\phi_{\text{end}}}{M_{\text{Pl}}} = \frac{p}{\sqrt{2}}, \quad (2.65)$$

leading to a direct connection between H_{end} and Λ ,

$$\Lambda^4 = 2M_{\text{Pl}}^2 H_{\text{end}}^2 \left(\frac{p}{\sqrt{2}} \right)^{-p}. \quad (2.66)$$

The averaged EOS parameter w_{rh} can then be calculated by setting $\phi_{\text{min}} = 0$ and using $V(\phi_{\text{max}}) = \rho_\phi$ to determine the maximum amplitude $\phi_{\text{max}} = M_{\text{Pl}} \left(\frac{\rho_\phi}{\Lambda^4} \right)^{\frac{1}{p}}$. Using the substitution $t = \frac{\Lambda^4}{\rho_\phi} \left(\frac{\phi}{M_{\text{Pl}}} \right)^p$ leads to

$$w_{\text{rh}} = 2 \frac{B\left(\frac{1}{p}, \frac{3}{2}\right)}{B\left(\frac{1}{p}, \frac{1}{2}\right)} - 1 = \frac{p-2}{p+2}, \quad (2.67)$$

where $B(x, y)$ is the beta function,

$$B(x, y) = \int_0^1 dt \cdot t^{x-1} (1-t)^{y-1} = \frac{\Gamma(x) \cdot \Gamma(y)}{\Gamma(x+y)}. \quad (2.68)$$

The form of the potential p can thus influence how the averaged EOS parameter during reheating looks like. In existing literature, the standard case that is taken for inflation is the case $w_{\text{rh}} = 0$ leading to $p = 2$ (see for example [12, 13, 30]). Here however, p can take any value $p > 1$. It can for example take the form of a quartic potential $p = 4$, leading to a radiation-like behavior with $w_{\text{rh}} = 1/3$. Another example is a box-shaped potential $p \rightarrow \infty$, corresponding to $w_{\text{rh}} \rightarrow 1$, which is called a stiff equation of state. This makes p a good choice to stay agnostic about inflation models while studying different scenarios for reheating. However, one needs to be careful again about steep potentials, where the approximation for the EOS parameter can fail. After the end of inflation, the inflaton field starts to decay with the decay rate Γ into radiation. When the inflaton energy density ρ_ϕ drops below the radiation energy density ρ_R , reheating is over and the Hot Big Bang starts with radiation domination (RD). A good measure to describe reheating in a more effective way is to use the reheating temperature T_{rh} . The reheating temperature describes the temperature of the Universe at which reheating is ending $\rho_\phi = \rho_R$. This happens, when decay becomes effective, meaning $\Gamma \sim H$. With the Friedmann equation, this leads to a connection between the radiation energy density and the decay rate,

$$\Gamma \sim \sqrt{\frac{2\rho_R}{3M_{\text{Pl}}^2}}. \quad (2.69)$$

The temperature and ρ_R can be connected via the relativistic degrees of freedom (DOFs),

$$\rho_R = g_*(T) \frac{\pi^2}{30} \cdot T^4. \quad (2.70)$$

It originates from looking at the effective number of relativistic particle species, where the energy density of bosons/fermions is determined by the energy distribution of the

particles,

$$\rho = g \int \frac{d^3 k_{\text{ph}}}{(2\pi)^3} \frac{E_k}{e^{(E_k - \mu)/T} \mp 1}. \quad (2.71)$$

g is the number of spin or helicity states, μ the chemical potential and $E_k = \sqrt{m^2 + k_{\text{ph}}^2}$. In the relativistic limit $T \gg m$ and $T \gg \mu$, this leads for bosons to

$$\rho_B = g \frac{\pi^2}{30} T^4 \quad (2.72)$$

and for fermions to

$$\rho_F = \frac{7}{8} \times g \frac{\pi^2}{30} T^4 \quad (2.73)$$

To account for bosons and fermions in the total ρ_R and also including that some of the particles have decoupled from the relativistic cosmological fluid, the energy density of radiation and the temperature can be connected via the effective number of relativistic degrees of freedom $g_*(T)$:

$$\rho_R = \sum_{i=\text{bosons}} g_i \frac{\pi^2}{30} T^4 + \frac{7}{8} \times \sum_{i=\text{fermions}} g_i \frac{\pi^2}{30} T^4 = g_*(T) \frac{\pi^2}{30} T^4 \quad (2.74)$$

This leads to a description of Γ with T_{rh} ,

$$T_{\text{rh}} \sim \left(\frac{45}{\pi^2 g_*^{\text{rh}}} \right)^{1/4} \sqrt{\Gamma M_{\text{Pl}}}. \quad (2.75)$$

In general, the decay rate depends on the specific interaction of the inflaton with the SM particles [31]. For example, in a scalar trilinear interaction $\phi \leftrightarrow \chi + \chi$, where the inflaton decays into two scalar fields, the interaction Lagrangian is

$$\mathcal{L}_{\text{int}} = -A m_\phi \phi |\chi|^2, \quad (2.76)$$

and the decay rate can become temperature-dependent $\Gamma \sim T^{-1}$. Or in a Yukawa interaction $\phi \leftrightarrow \psi + \bar{\psi}$,

$$\mathcal{L}_{\text{int}} = -y \phi \psi \bar{\psi}, \quad (2.77)$$

where the inflaton decays into a fermion and an antifermion, the decay rate can become $\Gamma \sim T$. Further details are given in [31]. The decay rate generally also depends on the amplitude of the inflaton and the thermalization rate, describing at which rate the radiation reaches thermal equilibrium. To model generally that different scenarios can affect reheating, the decay rate will be temperature-dependent,

$$\Gamma(T) = \Gamma_0 \left(\frac{T}{T_{\text{rh}}} \right)^q. \quad (2.78)$$

In this model for example, $q = -1$ corresponds to the trilinear interaction and $q = 1$ to the Yukawa interaction. This is only an approximation, but can be used to show the effects of temperature dependence of the decay rate on reheating, while staying agnostic to the exact decay mechanisms of the inflaton field. Adding the decay rate and averaging over

inflaton oscillations leads to the averaged evolution of the inflaton field during reheating,

$$\dot{\rho}_\phi + 3H(1 + w_{\text{rh}})\rho_\phi = -\Gamma(T)(1 + w_{\text{rh}})\rho_\phi. \quad (2.79)$$

The transfer of energy into radiation can be determined by using the energy conservation in curved space $\nabla_\mu T_0^\mu = 0$, leading to the continuity equation

$$\dot{\rho}_{\text{tot}} + 3H(\rho_{\text{tot}} + p_{\text{tot}}) = 0, \quad (2.80)$$

with $\rho_{\text{tot}} = \rho_\phi + \rho_R$ and $p_{\text{tot}} = p_\phi + p_R$. Using eq. 2.79 leads then to the Boltzmann equation of radiation,

$$\dot{\rho}_R + 3H(1 + w_R)\rho_R = \Gamma(T)(1 + w_{\text{rh}})\rho_\phi. \quad (2.81)$$

with the radiation EOS parameter $w_R = \frac{p_R}{\rho_R}$. The first law of thermodynamics reads

$$s = \frac{\rho_R + p_R}{T} = \frac{\rho_R}{T}(1 + w_R), \quad (2.82)$$

which can be used together with the relativistic DOFs of entropy density s ,

$$s = \frac{2\pi^2}{45} g_*^s(T) T^3, \quad (2.83)$$

to compute the radiation EOS parameter

$$w_R = \frac{4}{3} \frac{g_*^s}{g_*} - 1. \quad (2.84)$$

The relativistic DOFs of the entropy density again come from the fact that the entropy density of a single relativistic boson is $\frac{2\pi^2}{45} T^3$ for each spin state, while for a relativistic fermion it is multiplied by 7/8. Adding all relativistic particles, this leads to the effective number of relativistic degrees of freedom of the entropy density:

$$g_{*,s}(T) = \sum_{i=\text{bosons}} g_i \left(\frac{T_i}{T} \right)^3 + \frac{7}{8} \sum_{i=\text{fermions}} g_i \left(\frac{T_i}{T} \right)^3. \quad (2.85)$$

Generally, with falling temperatures the values g_* and $g_{*,s}$ also fall and if there are no phase transitions at that temperature, $\frac{g_*^s}{g_*} = 1$ and the radiation EOS parameter is the standard value $w_R = 1/3$. During phase transitions however, they can start to differ and the equation of state parameter for radiation can change. The values for g_* and $g_{*,s}$ used for this thesis are based on [32] and shown in fig. 2.3.

In general, the Boltzmann equation for radiation reads

$$\dot{\rho}_R + 4H \frac{g_*^s(T)}{g_*(T)} \rho_R = \Gamma(T)(1 + w_{\text{rh}})\rho_\phi. \quad (2.86)$$

Together, eq. 2.79 and 2.86 form the evolution of the reheating era. During inflation, every SM energy density gets thinned out and can be neglected, leading to $\rho_R(t_{\text{end}}) = 0$. With

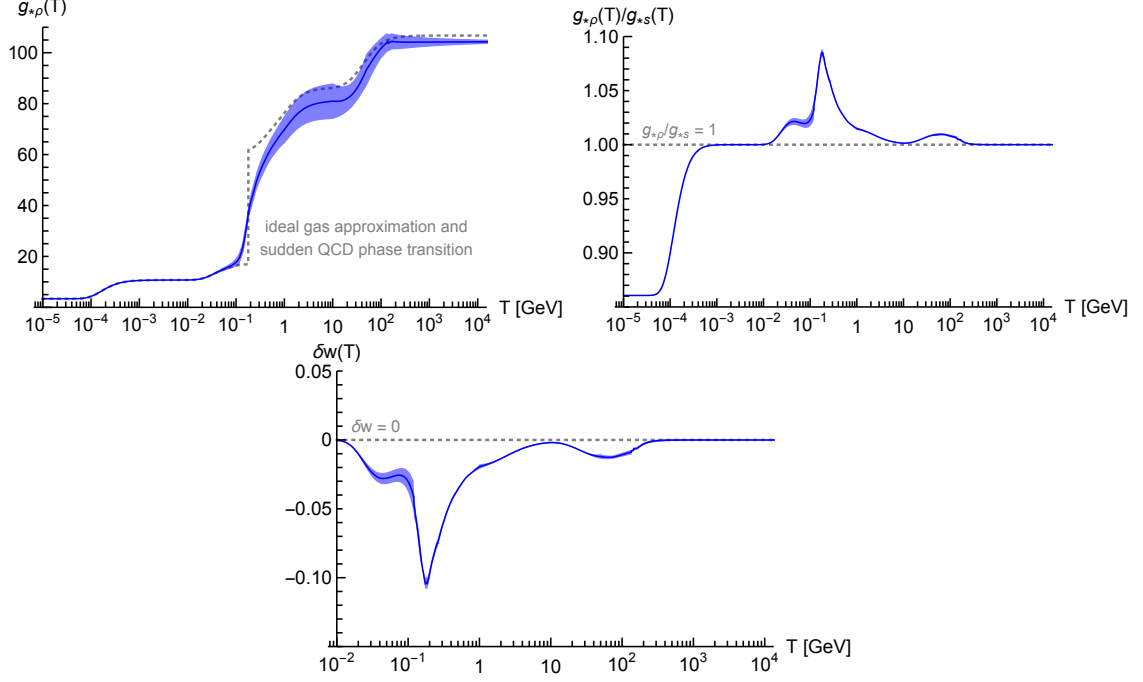


Figure 2.3: Temperature dependence of $g_*(T)$ (top left), the ratio of $g_*(T)$ and $g_{*,s}(T)$ (top right) and the deviation δw from the standard value of $w_R = 1/3$ (bottom). [32].

the Friedmann equation one can calculate the energy density of the inflaton field at the end of inflation,

$$\rho_{\text{end}} = 3M_{\text{Pl}}^2 H_{\text{end}}^2. \quad (2.87)$$

With this, it can be seen that reheating can be modeled here by the Hubble rate at the end of inflation H_{end} , the reheating temperature T_{rh} , the averaged inflaton EOS parameter w_{rh} , the decay rate temperature proportionality q and the two Boltzmann equations. A comparison for different reheating scenarios is shown in fig. 2.4, including the pseudo equation of state parameter $w_{\text{pseudo}} = \frac{2}{3Ht} - 1$. It is equal to the total EOS parameter if it is constant and is a convenient way to describe the average EOS parameter. One sees that the temperature dependence q of the decay rate only changes the form of the transition, but not the general trend of reheating.

The general shapes of the lines can be understood analytically in the $q = 0$ case. For the reheating era, where $\rho_\phi \gg \rho_R$, the decay rate is not yet effective, which leads to $\Gamma \ll H$. In this approximation, the EOS parameter of radiation can also be set to a constant value $w_R = 1/3$. Using this approximation for eq. 2.79 gives the Boltzmann equation for the inflaton field without decay,

$$\dot{\rho}_\phi + 3H(1 + w_{\text{rh}})\rho_\phi = 0 \quad (2.88)$$

which can be solved by

$$\rho_\phi(t) = \rho_\phi(t_0) \cdot e^{-3(1+w_{\text{rh}}) \int_{t_0}^t H(t') dt'} \quad (2.89)$$

Using the definition of an e -fold $N = \ln a$ and $Hdt = \frac{da}{a} = d \ln a = dN$ leads to the

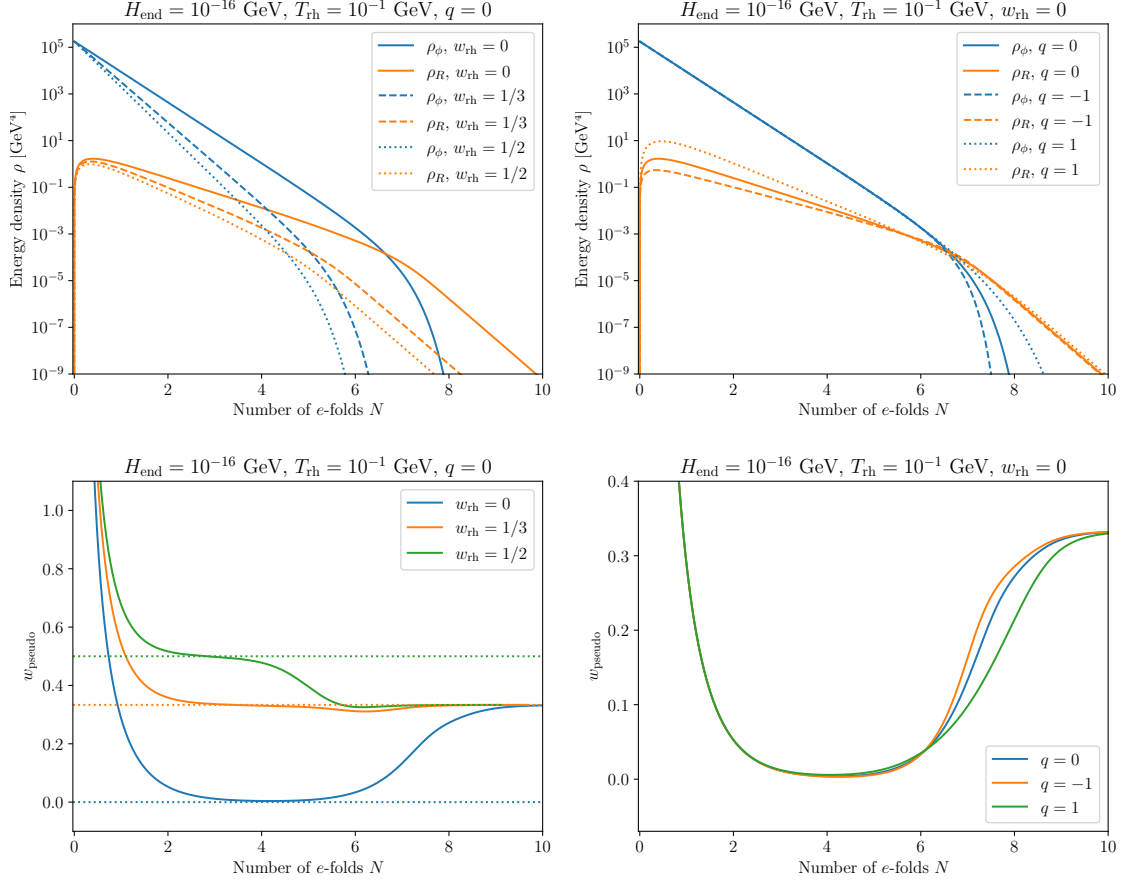


Figure 2.4: Different scenarios of reheating for different values of the average EOS of the inflaton (left) and for different temperature dependencies of the decay rate. The pseudo EOS parameter $w_{\text{pseudo}} = \frac{2}{3Ht} - 1$ is a measure to describe the averaged EOS parameter, because $w = w_{\text{pseudo}}$ if $w = \text{const}$. The blue, orange and green dotted lines in the lower left plot correspond to $w_{\text{pseudo}} = 0$, $w_{\text{pseudo}} = 1/3$, $w_{\text{pseudo}} = 1/2$.

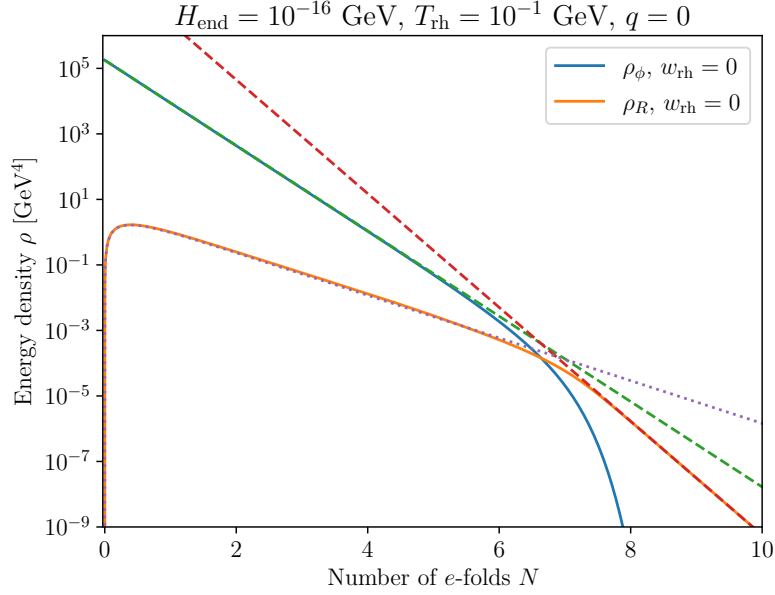


Figure 2.5: Example reheating case with analytical approximations. The green dashed line corresponds to eq. 2.90, the red dashed line corresponds to eq. 2.91 and the purple dotted line corresponds to eq. 2.94.

solution

$$\rho_\phi(N) = \rho_\phi(N_0)e^{-3(1+w_{\text{rh}})(N-N_0)} \quad \rho_\phi \gg \rho_R. \quad (2.90)$$

The same can be done for RD, when $\rho_\phi \ll \rho_R$, leading to

$$\rho_R(N) = \rho_R(N_0)e^{-4(N-N_0)} \quad \rho_\phi \ll \rho_R. \quad (2.91)$$

In the reheating era, the energy density is mostly determined by the decay of the inflaton field, as the Hubble friction is still dominant and causes already produced radiation to fall off faster than new radiation is produced. Using again $Hdt = dN$ in eq. 2.86 leads to

$$\frac{d\rho_R}{dN} + 4\rho_R = \frac{\Gamma}{H}(1+w_{\text{rh}})\rho_\phi(N) \quad (2.92)$$

Because ρ_R is subdominant, the Hubble rate becomes $H = \sqrt{\frac{\rho_\phi}{3M_{\text{Pl}}^2}}$ which leads with eq. 2.90 to a linear differential equation with a source term,

$$\frac{d\rho_R}{dN} + 4\rho_R = S(N) = \Gamma(1+w_{\text{rh}})\sqrt{3\rho_\phi(N_0)}M_{\text{Pl}}e^{-\frac{3}{2}(1+w_{\text{rh}})(N-N_0)}. \quad (2.93)$$

The energy density of radiation ρ_R is determined by the source term and the solution is then the inhomogeneous part of the differential equation, leading to

$$\rho_R(N) = \frac{\Gamma(1+w_{\text{rh}})}{4-\frac{3}{2}(1+w_{\text{rh}})} \left(e^{-\frac{3}{2}(1+w_{\text{rh}})(N-N_0)} - e^{-4(N-N_0)} \right) \quad \rho_\phi \gg \rho_R. \quad (2.94)$$

An example for the approximations can be seen in fig. 2.5, explaining the behavior of the energy densities in the different regions.

2.3.2 Transfer function

After the end of inflation, the comoving Hubble radius $\mathcal{H}^{-1}$ starts to grow and the once-frozen tensor perturbations start to evolve through the Universe again as GWs with the mode equation,

$$\boxed{h_A'' + 2\mathcal{H}h_A' + k^2 h_A = 0}. \quad (2.95)$$

Since they are sourced as fluctuations from an isotropic and homogeneous inflation, they are isotropically and homogeneously distributed today and form the so-called inflationary gravitational-wave background (IGWB). More precisely, with the rescaling $\chi_A = h_A/a$ the mode equation reads

$$\chi_A'' + \left(k^2 - \frac{a''}{a}\right) \chi_A = 0. \quad (2.96)$$

In typical epochs like radiation domination, matter domination or other scenarios, $a''/a \sim \mathcal{H} \sim \eta^{-2}$. In the superhorizon limit $k \ll \mathcal{H} \Leftrightarrow k\eta \ll 1$, the equation becomes

$$\chi_A'' - \frac{a''}{a} \chi_A = 0, \quad (2.97)$$

which can be solved by $h_A = \text{const.}$, showing that the superhorizon modes are frozen. In the subhorizon limit $k \gg \mathcal{H} \Leftrightarrow k\eta \gg 1$, the equation becomes a harmonic oscillator,

$$\chi_A'' + k^2 \chi_A = 0, \quad (2.98)$$

and the modes start to oscillate while being damped by the expansion of the universe,

$$h_A \sim \frac{\cos(k\eta)}{a(\eta)}. \quad (2.99)$$

To find the spectrum of the IGWB today, the evolution of the modes from when they are created at η_k^{prim} to today at η_0 needs to be determined. This transition factor is called the transfer function $T(k)$:

$$h_A(\eta_0, k) = \mathcal{T}(k) \cdot h_A(\eta_k^{\text{prim}}, k). \quad (2.100)$$

Since the modes stay at a constant value until they cross the horizon $k = \mathcal{H}$, it is enough to connect the modes at the time, when they enter the horizon η_k^{in} and become subhorizon. By averaging over the oscillations of the modes, the averaged transfer function can be determined as

$$\bar{\mathcal{T}}(k) = \tau_k \frac{a_k^{\text{in}}}{a_0}, \quad (2.101)$$

where $a_k^{\text{in}} = a(\eta_k^{\text{in}})$ and $a_0 = a(\eta_0)$. In some cases, τ_k can be computed analytically. For example in RD, where the EOS parameter is $w = 1/3$, the conformal Hubble parameter becomes $\mathcal{H} = 1/\eta$ and the solution of the EOM is

$$h_A(\eta) = \frac{\sin(k\eta)}{k\eta} \quad (2.102)$$

Together with the horizon crossing condition $k = \mathcal{H}$ and $a \propto \eta$ in RD, the solution to the EOM deep in RD reads

$$h_A = \frac{a_k^{\text{in}}}{a} \sin(k\eta) \quad \Rightarrow \quad \tau_k^{\text{RD}} = 1. \quad (2.103)$$

For general or even time dependent w , there exists no analytic solution, but the prefactor τ_k can be determined numerically by

$$\tau_k = \left(2 \cdot \left\langle \left(\frac{a \cdot h_A(\eta, k)}{a_k^{\text{in}}} \right)^2 \right\rangle \right)^{1/2} \quad (2.104)$$

The transfer function depends on the period during which the GWs enter. This thesis is focused on the transition between end of inflation and radiation domination, so it is useful to split the transfer function into a reheating part and a radiation domination part,

$$\boxed{\mathcal{T}(k) = \tau_k \frac{a_k^{\text{in}}}{a_0} = \frac{k_0}{k} \cdot \frac{\tau_{k_*} a_{k_*}^{\text{in}} k_*}{a_0 k_0} \cdot \frac{\tau_k a_k^{\text{in}} k}{\tau_{k_*} a_{k_*}^{\text{in}} k_*} = \frac{k_0}{k} \cdot T_{\text{RD}} \cdot T_{\text{RH}}(k)}, \quad (2.105)$$

where k_* is a pivot mode that enters deep inside RD, determining the reheating transfer function as

$$T_{\text{RH}}(k) = \frac{\tau_k a_k^{\text{in}} k}{a_{k_*}^{\text{in}} k_*}, \quad (2.106)$$

with $\tau_{k_*} = \tau_{k_*}^{\text{RD}} = 1$. The evolution of the reheating era significantly influences the unfreezing of the modes. If instantaneous reheating is chosen, where one goes from inflation directly to radiation domination, a mode k can enter much later than in the standard scenario with $w_{\text{rh}} = 0$ (see fig. 2.6). In the most general case, the reheating transfer function cannot be solved for analytically. Especially at transition point for example from reheating to RD or in phase transitions in RD, the factors need to be solved for numerically. But special cases can give an insight how the transfer function could look like. For a Universe with a constant EOS parameter w , the conformal Hubble rate $\mathcal{H}$ and the scale factor a evolves as

$$\mathcal{H} = \frac{2}{(1+3w)\eta}, \quad a \propto \eta^{\frac{2}{1+3w}}. \quad (2.107)$$

The modes will cross the horizon when $\mathcal{H} = k$. In this case, the reheating transfer function will become

$$T_{\text{RH}}(k) \propto a_k^{\text{in}} k \propto k^{-\frac{1-3w}{1+3w}} \quad (2.108)$$

In RD with $w = 1/3$ for example, T_{RH} will then be a constant. If the transition point from reheating to RD happens at k_{rh} , the reheating transfer function will go to 1 for $k \ll k_{\text{rh}}$ in RD and to $T_{\text{RH}} \propto k^{-\frac{1-3w_{\text{rh}}}{1+3w_{\text{rh}}}}$ for $k \gg k_{\text{rh}}$.

The RD part of the transfer function T_{RD} can be determined by looking at the Friedmann

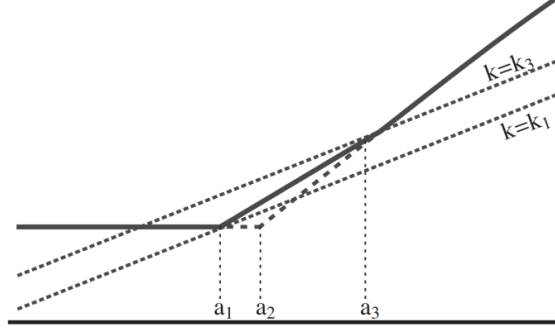


Figure 2.6: Freezing and unfreezing of two modes k_1 and k_3 for different reheating scenarios. The thick line and the thick dashed line correspond to the Hubble horizon H^{-1} . Both modes freeze as they exit the horizon at early time during inflation (left) and enter the horizon after inflation (right). The mode k_1 however enters the Hubble horizon much earlier in the instantaneous reheating scenario in RD (thick line), but with non-instantaneous reheating in the standard case $w_{\text{rh}} = 0$ (dashed line), the mode enters the horizon at a later time. [21]

equation during RD,

$$H^2 = H_0^2 \cdot \Omega_r = H_0^2 \cdot \Omega_r^0 \left(\frac{g_*(T)}{g_*(T_0)} \right) \left(\frac{g_*^s(T_0)}{g_*^s(T)} \right)^{4/3} \left(\frac{a_0}{a} \right)^4, \quad (2.109)$$

where Ω_r^0 is the radiation energy density parameter today, H_0 is the Hubble rate today and T_0 is the temperature of radiation today. The ratio $\frac{a_k^{\text{in}}}{a_0}$ for RD can then be determined as

$$\frac{a_k^{\text{in}}}{a_0} = \frac{k H_0}{k_0 H} = \left(\frac{k}{k_0} \right) \left(\frac{g_*^0}{g_*^k} \right)^{1/2} \left(\frac{g_{*,s}^k}{g_{*,s}^0} \right)^{2/3} \left(\frac{a_k^{\text{in}}}{a_0} \right)^2 \frac{1}{(\Omega_r^0)^{1/2}}. \quad (2.110)$$

This can then be rearranged to compute the radiation transfer function

$$T_{\text{RD}} = \left(\frac{g_{*,s}^k}{g_{*,s}^0} \right)^{1/2} \left(\frac{g_{*,s}^0}{g_{*,s}^k} \right)^{2/3} (\Omega_r^0)^{1/2}. \quad (2.111)$$

Combining $T_{\text{RH}}(k)$ and T_{RD} leads to the full transfer function and computes that the primordial spectrum will look today.

2.3.3 Gravitational-wave spectrum

As already introduced in sec. 1.1, the gravitational-wave spectrum is measured with the quantity

$$h^2 \Omega_{\text{GW}}(k) = \frac{h^2}{\rho_c} \frac{d \ln \rho_{\text{GW}}^0}{d \ln k}. \quad (2.112)$$

The GW energy density can be computed with

$$\rho_{\text{GW}} = \frac{M_{\text{Pl}}^2}{4a^2} \langle (h_{ij}^{\text{TT}})' (h^{\text{TT}ij})' \rangle. \quad (2.113)$$

For subhorizon modes, the GW becomes $h_A \sim \frac{\cos(k\eta)}{a(\eta)}$. When looking at a few oscillations of the GW, the scale factor $a(\eta)$ does not change significantly and can be treated as a constant, such that $h'_A \sim k \frac{\cos(k\eta)}{a}$. Using Fourier transformation, the polarization base $h(k, \eta)$ and the transfer function leads to

$$\rho_{\text{GW}} = \frac{M_{\text{Pl}}^2}{4\pi^2 a^2} \sum_{A=+, \times} \int dk k^4 |\bar{\mathcal{T}}(\eta, k)|^2 \langle |h(\eta_k^{\text{in}}, k)|^2 \rangle, \quad (2.114)$$

where $\bar{\mathcal{T}}(\eta, k)$ describes the averaged transfer function at conformal time η , leading to $\bar{\mathcal{T}}(\eta_0, k) = \bar{\mathcal{T}}(k)$. Because the mode will be frozen when it is superhorizon, $h(\eta_k^{\text{in}}, k) = h(\eta_k^{\text{prim}}, k)$. Then, using the definition of the power spectrum leads to the gravitational-wave spectrum:

$$h^2 \Omega_{\text{GW}}(k) = \frac{h^2}{\rho_c} \frac{d \ln \rho_{\text{GW}}^0}{d \ln k} = \frac{h^2}{\rho_c} \frac{M_{\text{Pl}}^2 k^2}{4a_0^2} |\bar{\mathcal{T}}(k)|^2 \mathcal{P}_t(k) = \frac{h^2}{12} \left(\frac{k}{k_0} \right)^2 |\bar{\mathcal{T}}(k)|^2 \mathcal{P}_t(k) \quad (2.115)$$

Typically, the frequency f of the gravitational wave is affected by redshift. The comoving wavenumber k and the frequency f can be connected via

$$f = \frac{1}{\lambda_{\text{phys}}} = \frac{k}{2\pi a_0}. \quad (2.116)$$

The inflationary gravitational-wave spectrum can then be computed by determining τ_k numerically to compute the transfer function and using the power spectrum. To get τ_k , the evolution of the reheating era needs to be simulated and with the power spectrum, six parameters are needed to describe the inflationary GW spectrum: H_{end} , T_{rh} , w_{rh} and q for the reheating itself and A_t^{max} and n_t for the power spectrum. Example reheating transfer functions computed numerically for different values of w_{rh} and q are shown in fig. 2.7.

2.3.4 Backreaction

In the discussion before, the Friedmann equations used to compute the evolution of the universe only include the energy density of the inflaton field and the radiation. In principle however, since the GWs themselves have an energy density, the GWs can have an effect on the evolution of the reheating era. The energy density of the inflaton field with EOS parameter w_{rh} will fall off with $\rho_\phi(a) \propto a^{-3(1+w_{\text{rh}})}$, while GWs that behave like radiation $w_{\text{GW}} = 1/3$ will fall off like $\rho_{\text{GW}}(a) \propto a^{-4}$. In cases, where the equation of state is stiff, meaning $w_{\text{rh}} > w_{\text{GW}} = 1/3$, the energy density of the inflaton field will drop faster than the energy density of GWs. If the power spectrum in this region is strong enough, the GWs can then have a significant impact on the evolution of reheating. This effect is called backreaction and leads especially for high w_{rh} to the fact that it might need to be included in the Friedmann equations,

$$H^2 = \frac{\rho_\phi + \rho_R + \rho_{\text{GW}}}{3M_{\text{Pl}}^2}. \quad (2.117)$$

The effects of backreaction on a stiff EOS with $w_{\text{rh}} = 1$ have been studied in [33], which shows that the effect in this case is not negligible. The GW energy density ρ_{GW} can be

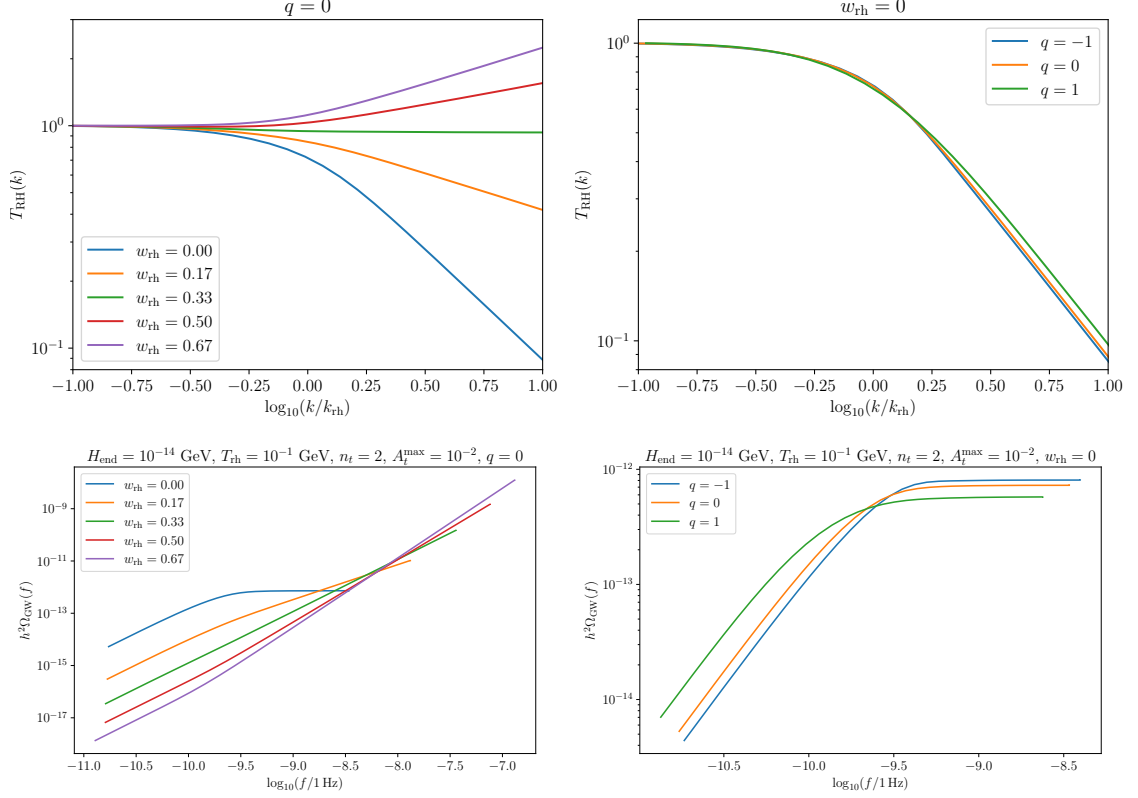


Figure 2.7: Example reheating transfer functions and spectra for different values of w_{rh} and q .

computed using eq. 2.114 for subhorizon modes with the time-dependent transfer function,

$$\mathcal{T}(\eta, k) = \tau_k \frac{a_k^{\text{in}}}{a(\eta)}. \quad (2.118)$$

Using the tensor power spectrum and including that only the subhorizon modes contribute to the energy density with $k > \mathcal{H}$ leads to

$$\rho_{\text{GW}}(t) = \frac{M_{\text{Pl}}^2}{8 \cdot a^4(t)} \int_{\mathcal{H}(t)}^{\mathcal{H}_{\text{end}}} \frac{dk}{k} (\tau_k a_k^{\text{in}} k)^2 P_t(k). \quad (2.119)$$

To compute the energy density of GWs however, one needs to know the numerical factor τ_k , which is not yet known for modes at the Hubble horizon as it is determined by averaging over a few oscillations of the subhorizon GW. Since this thesis deals with blue-tilted power spectra of GWs, this also means that the first modes that unfreeze after inflation give the highest contribution to the energy density. As such a way is needed to include the energy density for the first GWs to enter the horizon without knowing the numerical factor τ_k . The total EOS parameter of the reheating era is then also modified by the GW energy density

$$w_{\text{tot}} = \frac{p_\phi + p_R + p_{\text{GW}}}{\rho_\phi + \rho_R + \rho_{\text{GW}}} = \frac{w_{\text{rh}} \cdot \rho_\phi + w_R \cdot \rho_R + \frac{1}{3} \cdot \rho_{\text{GW}}}{\rho_\phi + \rho_R + \rho_{\text{GW}}}. \quad (2.120)$$

If the GW energy density is high enough compared to the inflaton field, it can push the total EOS parameter slightly towards $1/3$, which can lead to a slight modification of the reheating, the transfer function and the spectrum of GWs itself. The effect can be seen for an extreme example with high GW energy density in fig. 2.8 and 2.9, where the maximum portion of gravitational-wave energy density $\rho_{\text{GW}}/\rho_{\text{tot}}$ is approximately 44%.

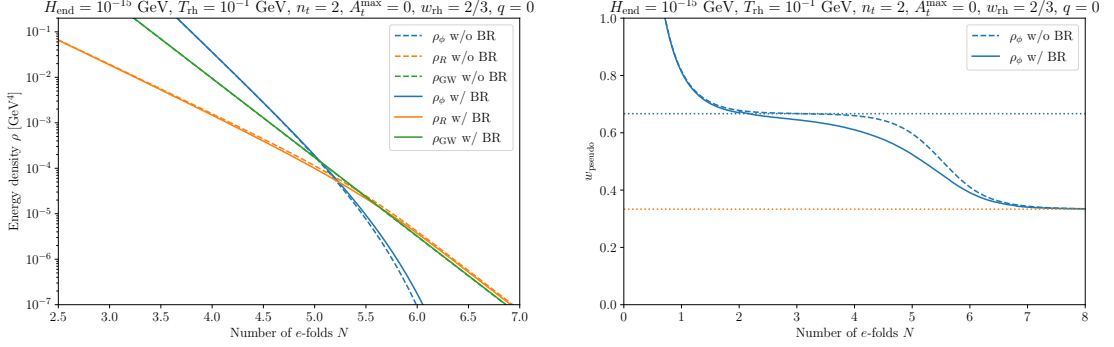


Figure 2.8: Example of the evolution of reheating with and without backreaction. In this case, the maximum portion of gravitational-wave energy density $\rho_{\text{GW}}/\rho_{\text{tot}}$ is approximately 44%. The blue and orange dotted line in the right plot correspond to $w_{\text{pseudo}} = 2/3$ and $w_{\text{pseudo}} = 1/3$

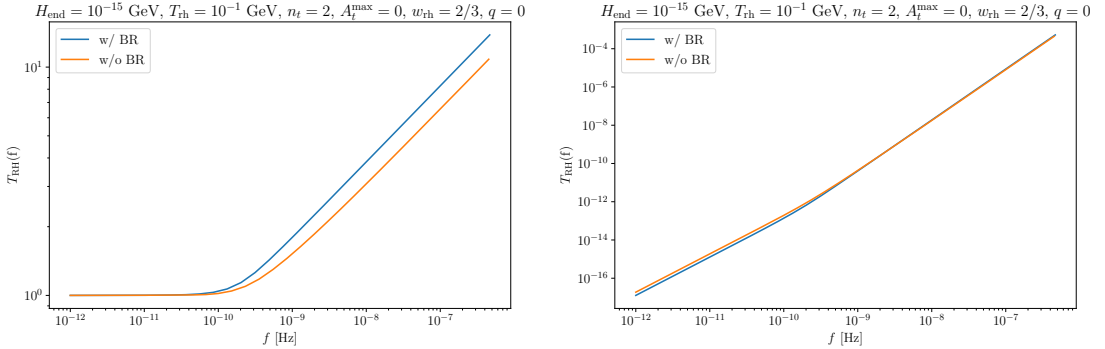


Figure 2.9: Example of the reheating transfer function and spectrum with and without backreaction. As in fig. 2.8, the maximum portion of gravitational-wave energy density $\rho_{\text{GW}}/\rho_{\text{tot}}$ is approximately 44%.

2.3.5 Analytical approximations

To better understand the mechanics of reheating and get a better feeling for how the parameters are connected to the measurements, analytical approximations can be made. The analytical approximations can then be used to move the spectrum such that it falls directly into the PTA band. The most important quantities for reheating are the number of e -folds for reheating N_{rh} , the reheating frequency f_{rh} and the cutoff frequency at the end of inflation f_{end} .

Number of e -folds for reheating N_{rh} : The number of e -folds for reheating is defined as $N_{\text{rh}} = \ln\left(\frac{a_{\text{rh}}}{a_{\text{end}}}\right)$, where a_{end} is the scale factor at the end of inflation and a_{rh} is the

scale factor at the time at which $\rho_\phi = \rho_R$. If one assumes that $w_{\text{tot}} = w_{\text{rh}}$ stays constant during the entire reheating era, the scale factor evolves as

$$a \propto \rho^{-\frac{1}{3(1+w_{\text{rh}})}} , \quad (2.121)$$

leading to

$$N_{\text{rh}} = \frac{1}{3(1+w_{\text{rh}})} \ln \left(\frac{\rho_{\text{end}}}{\rho_{\text{rh}}} \right) . \quad (2.122)$$

Using eq. 2.70 with $\rho_{\text{rh}} = \rho_R + \rho_\phi = 2\rho_R$ and the fact that at the end of inflation $\rho_{\text{end}} = 3M_{\text{Pl}}^2 H_{\text{end}}^2$, the number of e -folds of reheating can be determined as

$$N_{\text{rh}} = \frac{1}{3(1+w_{\text{rh}})} \ln \left(\frac{45M_{\text{Pl}}^2 H_{\text{end}}^2}{\pi^2 g_{*}^{\text{rh}} T_{\text{rh}}^4} \right) . \quad (2.123)$$

Reheating frequency f_{rh} : The reheating frequency f_{rh} marks the transition point between RD and reheating and can be calculated via

$$f_{\text{rh}} = \frac{k_{\text{rh}}}{2\pi a_0} = \frac{H_{\text{rh}}}{2\pi} \frac{a_{\text{rh}}}{a_0} . \quad (2.124)$$

After reheating, the entropy is conserved, leading to $g_{*,s}(T) T a^3 = \text{const}$. The Hubble rate at the end of reheating H_{rh} is described by the Friedmann equation and again $\rho_{\text{rh}} = 2\rho_R$. Connecting these equations leads to an analytical estimate of the reheating frequency,

$$f_{\text{rh}} = \frac{1}{6M_{\text{Pl}}} \sqrt{\frac{g_{*}(T_{\text{rh}})}{5}} \left(\frac{g_{*,s}(T_0)}{g_{*,s}(T_{\text{rh}})} \right)^{1/3} T_0 T_{\text{rh}} . \quad (2.125)$$

Cutoff frequency f_{end} : Another important quantity is the frequency at the end of inflation f_{end} , which also describes the cutoff point of the power spectrum. It can be computed via

$$f_{\text{end}} = \frac{k_{\text{end}}}{2\pi a_0} = \frac{H_{\text{end}}}{2\pi} \frac{a_{\text{end}}}{a_{\text{rh}}} \frac{a_{\text{rh}}}{a_0} . \quad (2.126)$$

Using the same types of approximations, the frequency at the end of inflation then reads

$$f_{\text{end}} = \frac{H_{\text{end}}}{2\pi} \cdot \left(\frac{g_{*,s}^0}{g_{*,s}^{\text{rh}}} \right)^{1/3} \cdot \left(\frac{\pi^2 g_{*}^{\text{rh}} T_{\text{rh}}^4}{45M_{\text{Pl}}^2 H_{\text{end}}^2} \right)^{\frac{1}{3(1+w_{\text{rh}})}} \cdot \frac{T_0}{T_{\text{rh}}} \quad (2.127)$$

Moving the spectrum to the PTA band: Ultimately, one is interested in whether reheating can be observed by PTAs like NANOGrav, which measures frequencies $f \sim 10^{-9} - 10^{-7}$ Hz. To have f_{rh} in this region, the reheating temperature thus needs to be at around

$$T_{\text{rh}} \sim 10^{-2} - 10^1 \text{ GeV} . \quad (2.128)$$

Additionally, the reheating era should not be the longest period of the Universe, as it is expected to be short. If one chooses the number of e -folds to be ≤ 10 , one can get an upper bound for H_{end} to be at $H_{\text{end}} \lesssim 10^{-12} \text{ GeV}$, depending on how high one goes with w_{rh} . Lastly, f_{end} should at least be in the region of the NANOGrav spectrum, such that the

cutoff does not happen before the spectrum. f_{end} should then fulfill roughly $f_{\text{end}} \gtrsim 28 \text{ nHz}$ [5]. This gives an approximate lower bound for the Hubble rate at the end of inflation to be at around $H_{\text{end}} \gtrsim 10^{-18} \text{ GeV}$. In total, the bounds for H_{end} are then

$$10^{-18} \text{ GeV} \leq H_{\text{end}} \leq 10^{-12} \text{ GeV} \quad (2.129)$$

Of course these are not hard bounds because there is no need for reheating to fall exactly in the NANOGrav spectrum or be only 10 e -folds long, but if one wants to analyze reheating with PTAs in a typical reheating scenario, the parameters should be in this region. As such, for this thesis, these bounds will be used approximately, leaving room for corrections from the equations.

2.4 Constraints on the gravitational-wave spectrum

Until now, no constraints on the parameters used to determine the spectrum have been made. There are several ways to exclude different regions of the parameters by looking at theoretical and phenomenological bounds.

2.4.1 Theoretical constraints

Perturbativity bound: The simplest constraint to apply is requiring perturbativity to remain intact. In the theoretical framework it has been assumed that the tensor perturbations are much smaller than the background metric,

$$||h_{ij}|| \ll 1. \quad (2.130)$$

For the stochastic GWs, the tensor perturbations can be connected to the tensor power spectrum in eq. 2.49. Using the definition of the dimensionless power spectrum leads to

$$\mathcal{P}_t \ll 1. \quad (2.131)$$

Because the power spectrum has been set to have a maximum at $k = k_{\text{end}}$ (see eq. 2.54), the constraint set by perturbativity is to ensure that

$$A_t^{\text{max}} < 1. \quad (2.132)$$

Reheating consistency constraint: To ensure that reheating occurs after inflation, the Hubble rate at the end of inflation must exceed the Hubble rate at reheating $H_{\text{end}} > H_{\text{rh}}$, which is equivalent to $\rho_{\text{end}} > \rho_{\text{rh}}$ and $f_{\text{end}} > f_{\text{rh}}$. This also ensures that $\Gamma < H_{\text{end}}$ and $N_{\text{rh}} > 0$. With eq. 2.70 this then leads to a constraint between H_{end} and T_{rh} :

$$H_{\text{end}} > \left(\frac{\pi^2}{45} g_*(T_{\text{rh}}) \right)^{1/2} \frac{T_{\text{rh}}^2}{M_{\text{Pl}}} \quad (2.133)$$

These two constraints are necessary to keep the mathematical structure of the model intact, as such they need to be used in every analysis.

2.4.2 Phenomenological constraints

Other constraints set by experiments are useful to further reduce the allowed parameter space, but are not necessary for the mathematical structure.

N_{eff} bound: One of these constraints is set by the so-called N_{eff} bound. The energy density of radiation consists of photons ρ_γ and neutrinos ρ_ν . With the 3 neutrino species in the SM, one can relate the energy density of the photons to the neutrinos as

$$\frac{\rho_\nu}{\rho_\gamma} = \frac{7}{8} \left(\frac{4}{11} \right)^{\frac{4}{3}} \cdot 3. \quad (2.134)$$

Here, the $\frac{7}{8}$ corresponds to the difference between the different relativistic degrees of freedom of fermions and bosons and the factor $\left(\frac{4}{11} \right)^{\frac{4}{3}}$ describes the cooling of the photons relative to the cooling of neutrinos after electron-positron annihilation. Before electron-positron annihilation, the SM predicts the degrees of freedom of neutrinos at a bit higher value at $N_{\text{eff}}^{\text{SM}} \approx 3.044$ [14]. If however additional components like GWs are accounted for in radiation, the number of relativistic degrees of freedom increases. In this scenario, the ratio of energy densities can be generally rewritten as

$$\frac{\rho_\nu}{\rho_\gamma} = \frac{7}{8} \left(\frac{4}{11} \right)^{\frac{4}{3}} \cdot N_{\text{eff}}. \quad (2.135)$$

The extra relativistic degrees of freedom ΔN_{eff} can be measured at times of Big Bang Nucleosynthesis (BBN), recombination and baryon acoustic oscillations (BAO), leading to an upper limit. Following the discussion of [14] leads to the following connection of the spectrum today and the extra degrees of freedom:

$$\int_{f_{\text{BBN}}}^{f_{\text{end}}} \frac{df}{f} h^2 \Omega_{\text{GW}} \lesssim 5.6 \times 10^{-6} \Delta N_{\text{eff}}, \quad (2.136)$$

where the measured value of the effective DOFs is $N_{\text{eff}} = 2.990 \pm 0.070$ [34]. The frequency $f_{\text{BBN}} \sim 10^{-12}$ Hz describes the frequency of the GW modes that enter the horizon during BBN. For definiteness, the frequency is set to $f_{\text{BBN}} = 10^{-12}$ Hz.

LVK bound: The measurement of the GWB has also been done by multiple experiments. Although an SGWB has not been detected yet by the ground-based experiments LIGO, Virgo and KAGRA (LVK), they have set an upper bound on the amplitude of the gravitational-wave spectrum [35],

$$\Omega_{\text{GW}}(f = 25 \text{ Hz}) \leq 2.8 \times 10^{-9}. \quad (2.137)$$

However, due to the parameter ranges explored in this thesis in this scenario (see sec. 2.3.5), f_{end} can only reach a maximum value of about $f_{\text{end}} \sim 10^{-4}$ Hz far below the LVK detection band, making the constraint in this scenario never binding.

Tensor-to-scalar ratio bound: And lastly, an important measure to describe inflation is the tensor-to-scalar ratio described in eq. 2.53. This quantity can be constrained by using CMB and baryon acoustic oscillation data (BAO). The tensor perturbations can lead to distortions, which can be constrained by the measurements. This leads to the constraint [36]

$$r < 0.034 \quad (2.138)$$

It is important to note however that the r constraint is for scales at CMB and BAO, which are at frequencies $f \sim 10^{-16}$ Hz ($k \sim 0.05 \text{ Mpc}^{-1}$ [36]), which are at much lower frequencies than the typical nHz frequencies for PTAs. To use this constraint, it needs then to be assumed that r does not change its value between the PTA frequencies and the CMB/BAO frequencies. If r stays constant, it can be evaluated by using eq. 2.53:

$$r = \frac{A_t^{\max}}{\mathcal{P}_s^{\text{CMB}}} \left(\frac{f_{\text{CMB}}}{f_{\text{end}}} \right)^{n_t} \quad (2.139)$$

with $\mathcal{P}_s^{\text{CMB}} = 2.10 \times 10^{-9}$ and $f_{\text{CMB}} = 7.73 \times 10^{-17}$ Hz [24].

3 PTA analysis

When performing an analysis with PTAs, one uses the timing residuals of the pulsars to determine the GW signal. These timing residuals are small and overlaid with noise from many pulsars, each of which has free noise parameters that influence the measured spectrum. Performing a simple best-fit model thus is not sufficient to explore the whole parameter space. An alternative is to use Bayesian statistics, where the parameters are treated as probability distributions. In the Bayesian framework, prior assumptions of what the true parameter might be are made (prior) and are then combined with the likelihood, which describes the probability distribution of a parameter given the data. Thus, Bayesian inference provides a natural framework for estimating model parameters in the presence of complex noise processes and significant parameter degeneracies.

3.1 Bayes' theorem and MCMC sampling

Bayesian statistics is based on Bayes' theorem:

$$P(\vec{\theta}|\vec{x}) = \frac{P(\vec{x}|\vec{\theta})P(\vec{\theta})}{P(\vec{x})}, \quad (3.1)$$

where $P(\vec{\theta}|\vec{x})$ is the probability of the model parameters $\vec{\theta}$ given the data $\vec{x}$ and thus the main quantity of interest called the posterior distribution. $P(\vec{\theta})$ is the prior of the parameters $\vec{\theta}$ and describes prior knowledge about the parameters before seeing any data, and $P(\vec{x}|\vec{\theta})$ is the likelihood and describes how the data is assumed to be distributed. The probability of finding data x can be written as

$$P(\vec{x}) = \int d^n\theta P(\vec{x}|\vec{\theta})P(\vec{\theta}) \quad (3.2)$$

and thus functions as the normalization. This integral is hard to compute, as in many cases there is no closed form for it. To get an approximation for the distributions, one can use Markov chain Monte Carlo (MCMC) sampling. Here, a Markov chain explores the parameter space by randomly moving through it and computing the likelihood for a randomly sampled parameter set. Over time, the chain spends more time in regions with high posterior probability and less time in unlikely regions. The collection of visited points therefore represents samples from the posterior distribution. The details on how to test if chains converged, see sec. C.2.

3.2 PTA likelihood

The likelihood used to fit the timing residuals is Gaussian, with a covariance matrix built from pulsar-specific white and red noise together with a common, spatially correlated process shared between all pulsars, the signal of interest, which for an isotropic SGWB follows the Hellings-Downs correlation pattern introduced in sec. 1.1. This thesis uses the standard NG15 noise model within `enterprise` [37], together with `PTArcade` [38] and its standard `PTMCMCSampler` [39] backend. `PTArcade` allows an arbitrary theoretical spectrum

$h^2\Omega_{\text{GW}}(f)$ to be used as the common-process model, making it possible to directly evaluate the reheating-modulated spectrum derived in sec. 2.3.3 against the NG15 data, without assuming a concurrent astrophysical (SMBHB) background.

The timing residuals used to compute the likelihood in **PTArcade** describe the assumed and measured pulsar pulse time of arrival and are constructed as

$$\vec{\delta t} = \vec{n} + \mathbf{F}\vec{a} + \mathbf{M}\vec{\epsilon}. \quad (3.3)$$

It consists of the white noise $\vec{n}$, red noise $\mathbf{F}\vec{a}$ and deviations from the initial best-fit values of the pulsar timing model $\mathbf{M}\vec{\epsilon}$. $\mathbf{F}$ is the Fourier design matrix containing the sine–cosine pairs evaluated at the different observation times for the frequency bins of the spectrum. The Fourier coefficients $\vec{a}$ contain the pulsar-intrinsic red noise and the GW-induced shift, assumed to be normally-distributed random variables with zero mean. The covariance matrix $\langle \vec{a}\vec{a}^T \rangle = \phi$ is given by

$$[\phi]_{(ai)(bj)} = \delta_{ij}(\Gamma_{ab}\Phi_i + \delta_{ab}\varphi_{a,i}), \quad (3.4)$$

with the pulsar indices a, b , the frequency harmonics i, j and the Hellings-Downs curve Γ_{ab} . Φ_i are the model-dependent parameters at the i -th frequency related to the SGWB spectrum Ω_{SGWB} ,

$$\Omega_{\text{SGWB}} = \frac{8\pi^4 f^5}{H_0^2} \Phi(f) T_{\text{obs}}, \quad (3.5)$$

and φ describes the pulsar-intrinsic red noise described by a power law. The likelihood can then be evaluated with

$$p(\vec{\delta t}|\vec{\theta}) = \frac{\exp\left(-\frac{1}{2}\vec{\delta t}^T \mathbf{C}^{-1} \vec{\delta t}\right)}{\sqrt{\det(2\phi\mathbf{C})}}, \quad (3.6)$$

with $\mathbf{C} = \langle \vec{n}\vec{n}^T \rangle + [\mathbf{M}, \mathbf{F}]\text{diag}(\infty, \phi)[\mathbf{M}, \mathbf{F}]^T$.

3.3 Priors

Because one has no knowledge about the parameters, the priors of the parameters will be uniformly distributed in the desired parameter ranges. The parameter ranges for T_{rh} and H_{end} are based on sec. 2.3.5. Starting from the standard case $w_{\text{rh}} = 0$, the prior range for w_{rh} is extended symmetrically around the RD case up to $w_{\text{rh}} = 2/3$ and the amplitude A_t^{max} is chosen close to the perturbativity bound. Since this thesis is focused on blue-tilted spectra and based on the earlier analysis [12], the prior range of n_t reaches from 0 to 6. The chosen parameter ranges are again shown in table 3.1.

The upper bound for w_{rh} is set to $2/3$ because the approximation made to connect w_{rh} to p can fail and numerical computations of reheating for higher values become extremely expensive and are prone to backreaction, increasing the computation cost even more.

Parameter	Prior range
$\log_{10}(T_{\text{rh}}/\text{GeV})$	$[-2.5, 2.5]$
$\log_{10}(H_{\text{end}}/\text{GeV})$	$[-18, -12]$
w_{rh}	$[0, 2/3]$
$\log_{10} A_t^{\text{max}}$	$[-6, 0]$
n_t	$[0, 6]$

Table 3.1: Priors for the model parameters used in the PTA analysis.

3.4 Fit function

Determining the spectrum for many different parameter points numerically is computationally expensive. As such, a fit function for the reheating transfer function will first be created to reduce computation time. To check the viability of the fit in the analysis, the fit function will be compared to the numeric case as well as to existing fits in the literature for the standard case $w_{\text{rh}} = q = 0$, and subsequently to partly and fully numerical PTA analyses. As mentioned in sec. 2.3.2, the reheating transfer function T_{RH} needs to fulfill $T_{\text{RH}} \rightarrow 1$ for $k \ll k_{\text{rh}}$ and $T_{\text{RH}} \sim k^{-\frac{1-3w_{\text{rh}}}{1+3w_{\text{rh}}}}$ with $w_{\text{rh}} = \frac{p-2}{p+2}$ while being smoothly connected to each other. To keep the model independent of other parameters that are sensitive to the specific temperature of each frequency or backreaction, the model will be created with the assumption $w_R = 1/3 \Leftrightarrow g_{*,s}/g_* = 1$ and negligible ρ_{GW} compared to the total energy density. The chosen fit function will be

$$T_{\text{RH}}(x) = (1 + a(p, q) \cdot x + c(p, q) \cdot x^{\alpha(p, q)})^{\frac{1}{2\alpha(p, q)} \frac{p-4}{p-1}}, \quad (3.7)$$

with $x = k/k_{\text{rh}} = f/f_{\text{rh}}$. The parameters α , a and c allow for a broad calibration of the transition between RD and reheating era. At first, the fit function will be calibrated to 100 equidistant values of p from 2 to 6, leading to $w_{\text{rh}} = 0 - 2/3$, for each $q = -1, 0, 1$ separately. Then, the optimal parameters for each p will be refitted in a smooth function to get an overall fit. In the following PTA analysis, it is also applied up to $w_{\text{rh}} = 2/3$, where its accuracy is additionally tested.

3.5 Backreaction

Including backreaction into the numerical framework by adding the energy density of the GWs as described in sec. 2.3.4 is not possible, because eq. 2.119 itself requires knowledge of the evolution of the GW modes. It needs the factor τ_k , which is determined by averaging over the oscillations of the GW. GWs, which are not yet deep enough inside the horizon and cannot oscillate freely, can thus not be included in the computation of ρ_{GW} . However, this is a crucial part as the first GW modes, which enter the horizon after the end of inflation, are the most energetic ones for blue-tilted spectra and can then have a great impact on the evolution of reheating. As an approximation, these modes will be included by using the fit function of the transfer function to model the energy density of the GWs. Although this is not the exact full solution, the approximation is justified since ρ_{GW} remains subdominant throughout reheating: an error in its precise value therefore only enters the background

evolution at second order and does not strongly affect the dynamics. The approximation thus captures the correct order of magnitude at low computational cost.

3.6 Analysis runs

The PTA analysis will be split into different parts. First in the no-backreaction case, the new model will be compared to the earlier model in the standard reheating case $w_{\text{rh}} = 0$ with NG15 data (see [12, 13]), and to an analysis with EPTA data [30]. In the new model, an agnostic w_{rh} and different q can be integrated, and as such in the next step, the effect of different w_{rh} will be included for different approximations: a completely analytical fit with analytical approximations for the transfer function, then an analytical fit with numerical parameters for the transfer function, then the effect of the relativistic DOFs on the spectrum, and lastly a fully numeric analysis. For the fully numeric case, the effect of the temperature dependence of the inflaton decay rate q will be evaluated for $q = -1$ and $q = 1$. In the next step, backreaction is included and compared to the fully numeric analysis without backreaction. Finally, the constraints from sec. 2.4 will be used to do a constrained analysis. In all of the cases except the fully analytical case, because the numerical computations are too slow to yield a fast enough sample rate for the MCMC sampling, the spectra for the different parameters will be computed beforehand on a regular grid and then linearly interpolated to still perform a PTA analysis. For the no-backreaction case, a grid with 30 points in each parameter direction is chosen, while for the backreaction case, the grid will be 25 points in each direction because of the higher computational cost, leading to grids with 30^5 or 25^5 points for each analysis.

The constrained analysis is performed in an alternative way using `ceffyl` [40], which is based on the free spectrum, a model-independent representation of the timing residual power spectrum in which the power in each frequency bin is treated as a free parameter, rather than being fixed to a specific functional form such as a power law. Because this free spectrum has already been computed in advance, `ceffyl` can evaluate the likelihood of a given theoretical model directly, without recomputing the full timing likelihood from the raw data. While this does not constitute a complete alternative for the `enterprise` analysis, it nonetheless allows for a number of meaningful conclusions to be drawn regarding the model parameters under consideration. A dedicated analysis using `enterprise` is left for future work.

4 Results

In the following, the results of the fit to the reheating transfer function are presented, followed by a discussion of the parameter regions in which reheating can have a non-negligible impact, and finally an analysis comparing the model spectrum to the NG15 data.

4.1 Reheating transfer function

The details for fitting the transfer function with eq. 3.7 are explained in sec. B. The relative error of the fit to the numerical result is less than 2% in all cases up to $w_{\text{rh}} = 2/3$ (see fig. B.3), resulting in a highly accurate model for the transfer function. Comparing the new fit to existing fits in the literature for the standard case $w_{\text{rh}} = q = 0$ also reveals much higher accuracy (see fig. 4.1). However, the precision of the fit function can be compromised by phase transitions in RD, which can change w_R to values different than $1/3$. The fit function is also dependent on k_{rh} or f_{rh} , because the k or f values are always connected with the ratio to the former. If the parameters of interest in the analysis differ from these, they need to be computed either in an analytic or numeric way. While the analytic way can lead to loss of precision, the numeric way again suffers from the much higher computational cost. Additionally, backreaction can also have a significant effect on w_{rh} and can thus also lead to loss of precision. These problems for the analytic fit function will be addressed later in the PTA analyses.

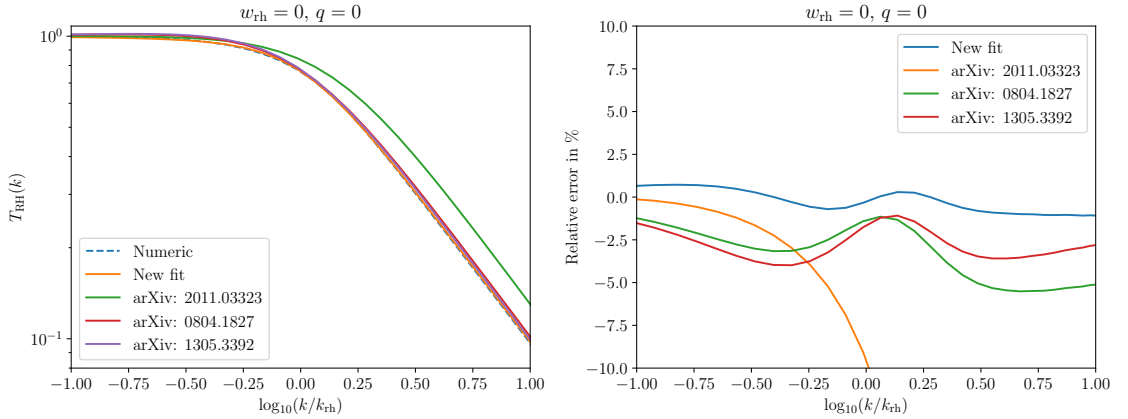


Figure 4.1: Comparison of the new fit and existing literature fits [9–11] to the numerical reheating transfer function for the standard case $w_{\text{rh}} = q = 0$ with the literature.

4.2 Influence of backreaction

The influence of backreaction is determined by the amount of energy density of the GWs in the total energy density. When the energy density of the GWs becomes significant, it can change the EOS parameter w_{tot} to push it towards a radiation-like EOS parameter $w_R = 1/3$, because the GWs behave as radiation itself. Especially in cases where the EOS parameter gets stiffer, meaning $w_{\text{rh}} > 1/3$, the energy of the inflaton field drops faster than the energy density of GWs, leading to a higher contribution of GWs in the total

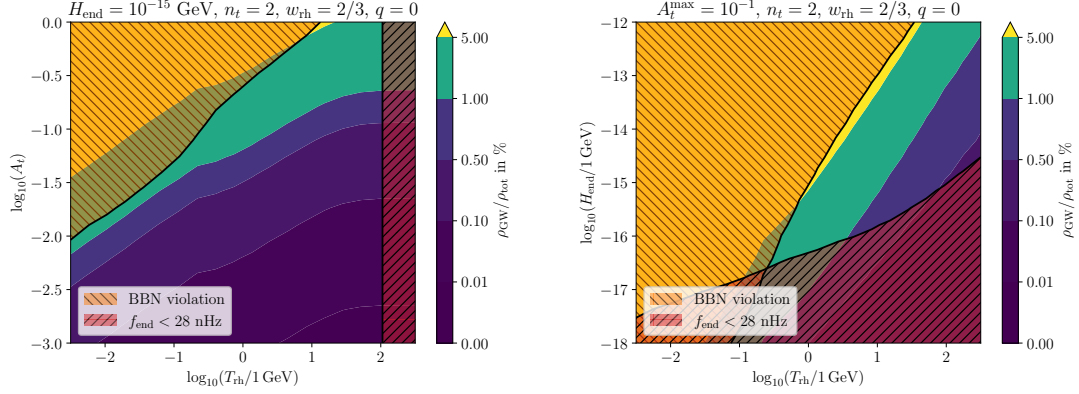


Figure 4.2: Examples of parameter spaces constrained by ΔN_{eff} and f_{end} , including the maximum portion of gravitational-wave energy density $\rho_{\text{GW}}/\rho_{\text{tot}}$. The parameter spaces also show regions where the BBN bound is violated and where f_{end} is inside the NANOGrav spectrum or lower.

energy density. However, because they behave as radiation (so-called dark radiation), they are also contributing to the effective number of relativistic degrees of freedom N_{eff} (see sec. 2.4). Constraining N_{eff} thus also constrains the amount of backreaction allowed in the model. Fig. 4.2 shows the maximum portion of GW energy density in the total energy density for a specific parameter space with high $w_{\text{rh}} = 2/3$. One can see that the most extreme regions of backreaction, where $\rho_{\text{GW}}/\rho_{\text{tot}} > 5\%$, are mostly ruled out by N_{eff} . However, large portions where $\rho_{\text{GW}}/\rho_{\text{tot}} > 1\%$ still remain. This means that backreaction cannot be neglected in all cases when high precision of the transfer function is needed. It is thus necessary to compare the no-backreaction to the backreaction case in order to assess how strong the effects on the spectrum analyses are.

4.3 NG15 analysis

In the following analysis with NG15 data, the new model will be compared first to the standard reheating model $w_{\text{rh}} = q = 0$ and the analyses performed in [12, 13, 30]. Then a general analysis is performed first in the no-backreaction case and then in the backreaction case. Lastly, constraints from other observations are added. The tables with details about the posterior results are shown in sec. C.1 and the convergence diagnostics of the chains are shown in sec. C.2. For all **enterprise** runs, the convergence diagnostics shows a marginal convergence, but not a full convergence. Single pulsar parameters might not have converged yet, but the model parameters and the vast majority of pulsar parameters have, except for the backreaction model, where the H_{end} parameter is not converged ($\hat{R} \approx 1.11$, the highest value among the model parameters in this run, though still at the boundary of the weak-convergence threshold). Since for the backreaction model it is the only unconverged parameter, it is expected not to differ strongly from the converged result. It is thus possible to interpret the data, but some small deviations from a full convergence are to be expected. The **ceffy1** runs on the other hand show a full convergence, but are not fully comparable with the full **enterprise** run.

The posterior distributions shown in the corner plots were generated using `GetDist`, with kernel density smoothing applied to the sampled posterior chains. For all runs a custom 2D smoothing scale of `smooth_scale_2D = 0.3` was chosen. For runs performed with `enterprise`, the default automatic `smooth_scale_1D` setting of `GetDist` was used. For runs performed with `ceffyl`, a custom smoothing scale factor of `smooth_scale_1D = 0.3` was applied instead, in order to reduce sampling noise and improve the visual clarity of the resulting posterior distributions. These values control the width of the Gaussian kernel used in the density estimation relative to the default bin width, with smaller values corresponding to less smoothing and larger values producing a smoother, more visually smoother posterior shape. The chosen scale factor provides a reasonable trade-off between suppressing stochastic noise arising from a finite number of samples and preserving the essential structure of the underlying posterior distribution.

4.3.1 Standard case $w_{\text{rh}} = 0$, $q = 0$

In the standard reheating case $w_{\text{rh}} = q = 0$, the numerical model leads to the posterior distributions in fig. 4.3 (and table C.1). The reheating temperature T_{rh} , the spectral tilt n_t and the Hubble rate at the end of inflation H_{end} are well constrained, with

$$T_{\text{rh}} = 10^{0.509 \pm 0.631} \text{ GeV} \quad n_t = 2.746 \pm 0.926 \quad H_{\text{end}} = 10^{-14.942 \pm 1.114} \text{ GeV}. \quad (4.1)$$

The amplitude A_t^{max} , by contrast, is essentially unconstrained by the data from above within the prior range, and consequently piles up close to the perturbativity bound $A_t^{\text{max}} < 1$ (see sec. 2.4). Only a lower limit around $10^{-1.331}$ can be quoted at 68% credibility.

A clear negative degeneracy appears between T_{rh} and n_t . This can be understood from the position of the reheating frequency f_{rh} relative to the NANOGrav band: for high reheating temperatures, f_{rh} lies above the NG15 frequencies, so the data probes the spectrum already in the RD-dominated part of the transfer function, where $T_{\text{RH}}(k) \rightarrow 1$ (see sec. 2.3.2). In this regime the spectral shape within the NG15 band is set purely by n_t , and the data prefers $n_t \simeq 2$. For low reheating temperatures, on the other hand, f_{rh} falls inside or below the NG15 band, so the observed frequencies probe the reheating part of the transfer function itself. Since $T_{\text{RH}}^2(k) \propto k^{-2 \cdot (1-3w_{\text{rh}})/(1+3w_{\text{rh}})}$ with $w_{\text{rh}} = 0$ contributes an additional tilt of k^{-2} on top of the primordial spectrum, a primordial tilt of $n_t \simeq 4$ is then needed to reproduce the same observed slope. This trade-off produces the two preferred branches visible in the $T_{\text{rh}}-n_t$ panel of fig. 4.3, together with an in-between transition region between the two cases.

A further degeneracy is visible between T_{rh} and H_{end} : higher reheating temperatures require correspondingly larger values of H_{end} in order to push the cutoff frequency f_{end} further towards higher frequencies, keeping the high-frequency end of the spectrum shifted back relative to the NG15 band (see eq. 2.127). The cutoff frequency f_{end} grows with both H_{end} and T_{rh} . Without this compensation, a higher T_{rh} alone would move f_{end} too close to, or below, the observed frequencies. Additionally, H_{end} and A_t^{max} are positively correlated: a higher H_{end} pushes f_{end} to higher values, moving it further into or beyond the NG15 band, which in turn requires a higher amplitude A_t^{max} to maintain a comparable signal

strength within the observed frequencies.

Fig. 4.4 compares the spectra obtained from the Bayes estimator, the maximum-likelihood sample, and the maximum-posterior parameter point to the NG15 violins described by the free spectrum. The free spectrum treats the power in each PTA frequency bin as an independent free parameter (sec. 3.6). Its posterior in each bin is shown as a violin plot: a mirrored density plot whose width reflects the posterior probability at that power value, allowing a bin-by-bin comparison with the model spectrum. While the Bayes estimator and maximum-likelihood spectra both trace the data well, the maximum-posterior spectrum overshoots the data noticeably at higher frequencies. This is a direct consequence of the unconstrained amplitude: since $A_t^{\max}$ is only bounded from below by the data, while perturbativity sets the upper limit, the single highest-posterior-density point favors a large amplitude close to that upper bound. Interestingly, the preferred spectra all have f_{end} inside the NG15 band, which means that these spectra do not explain the full NG15 measurement, suggesting that this particular model is not fully adequate. Consequently, a more general analysis with general w_{rh} could in principle be a more favorable option.

The comparison to the earlier analyses in [12] and [13] is shown in fig. 4.5, where r is the tensor-to-scalar ratio at CMB scales from eq. 2.139. Here, the cutoff frequency f_{end} is computed with eq. 2.127. The new model reproduces the results of [12] and [13] reasonably well: the same negative degeneracy between r and n_t is recovered, again reflecting that a line of constant spectral amplitude at the pivot frequency must run through the region favored by the NG15 data, so a lower r (i.e. a lower amplitude) has to be compensated by a higher n_t . The main difference between the three analyses is that both very low and very high T_{rh} regions are disfavored in the new model, which is not the case in [12] and [13], where an “S”-like shape appears in the n_t - and r - T_{rh} posteriors. This originates from the additional perturbativity constraint applied here: at high T_{rh} , the cutoff frequency f_{end} moves further beyond the NG15 band, requiring a large amplitude that starts to violate perturbativity. At low T_{rh} , the $n_t \simeq 4$ branch requires long reheating and correspondingly large amplitudes to fit the data, again pushing towards the perturbativity bound.

A similar analysis for this case has been done in [30] with DR2 from EPTA. A comparison including f_{end} using eq. 2.127 is shown in fig. 4.6. The new model again shows a preference for both the $n_t \simeq 2$ and $n_t \simeq 4$ branches, with the transition between them occurring around $T_{\text{rh}} \sim 1 \text{ GeV}$, consistent with the frequency argument given above. The posteriors for T_{rh} , r , and n_t agree well with the unconstrained (“green”) EPTA posteriors of [30]. However, unlike the analysis presented here, [30] similarly to [13] does not impose a perturbativity check on the amplitude, which allows it to reach considerably higher values of f_{end} than the new model, in which the perturbativity bound $A_t^{\max} < 1$ effectively caps how far the cutoff frequency can be pushed beyond the NG15 band.

4.3.2 Without backreaction

In the general reheating scenario, w_{rh} is treated as a free parameter, while backreaction is not yet included. The resulting posterior distributions for the full numerical model are shown in fig. 4.7–4.9 for $q = 0$, $q = -1$ and $q = 1$ (tables C.2–C.4).

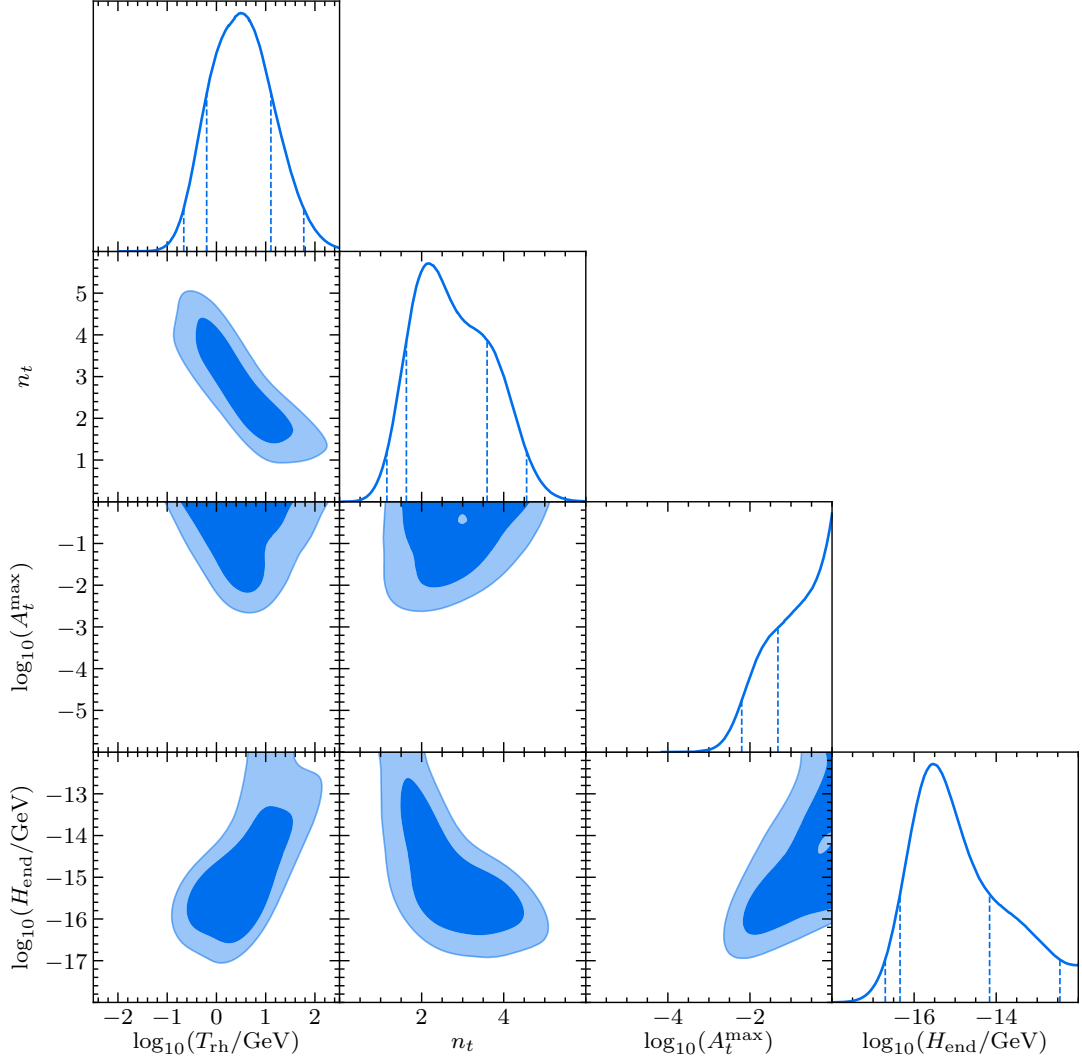


Figure 4.3: Posterior plots for the standard reheating case $w_{\text{rh}} = q = 0$ for the new numerical model.

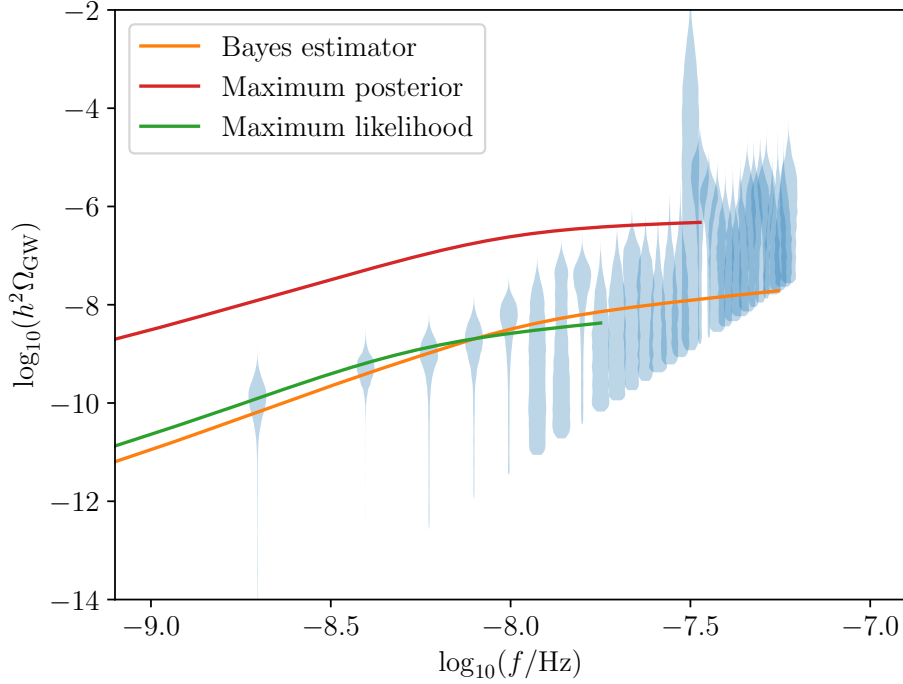


Figure 4.4: Spectra for the standard reheating case $w_{\text{rh}} = q = 0$ with the Bayes estimator values, with the highest posterior values and the maximum likelihood compared to the NG15 violins.

Across all three values of q , n_t and H_{end} remain well constrained, with $n_t \approx 1.8$ and $H_{\text{end}} \approx 10^{-15}$ GeV, values very similar to the $w_{\text{rh}} = 0$ case discussed in sec. 4.3.1. Notably, the second branch at $n_t \simeq 4$ as well as much of the transition region, which was present when w_{rh} was fixed to zero, vanishes once w_{rh} is allowed to vary. This branch had originated from the need to compensate the additional reheating-induced tilt k^{-2} at low T_{rh} purely through a shallower primordial spectrum. With w_{rh} now free, the data can instead adjust the reheating tilt itself, and radiation-domination-like reheating scenarios are consistently preferred within the NG15 band. Notably, the reheating model with this data cannot differentiate between a RD-like reheating, instantaneous reheating or a scenario where f_{rh} is at much higher frequencies than the NG15 band.

Both A_t^{max} and w_{rh} are pushed towards the upper edge of their respective priors, with Bayes estimators around $A_t^{\text{max}} \approx 10^{-1.8}$ (similarly unconstrained as in sec. 4.3.1) and $w_{\text{rh}} \approx 0.37$, while T_{rh} prefers comparatively low values close to the lower end of the explored range, with $T_{\text{rh}} \approx 10^{-0.5}$ GeV. As in the $w_{\text{rh}} = 0$ case, a degeneracy between A_t^{max} and H_{end} persists: higher H_{end} pushes f_{end} to higher frequencies, requiring a correspondingly higher amplitude to maintain the signal strength within the NG15 band.

A further, negative degeneracy is observed between A_t^{max} and w_{rh} : larger values of w_{rh} are associated with smaller preferred amplitudes. This can be traced back to the shape of the reheating transfer function itself, $T_{\text{RH}}(k) \propto k^{-(1-3w_{\text{rh}})/(1+3w_{\text{rh}})}$. For $w_{\text{rh}} > 1/3$ the exponent becomes positive, so that $T_{\text{RH}}(k)$ grows with k instead of being suppressed, meaning that a stiffer equation of state during reheating actively amplifies the spectrum

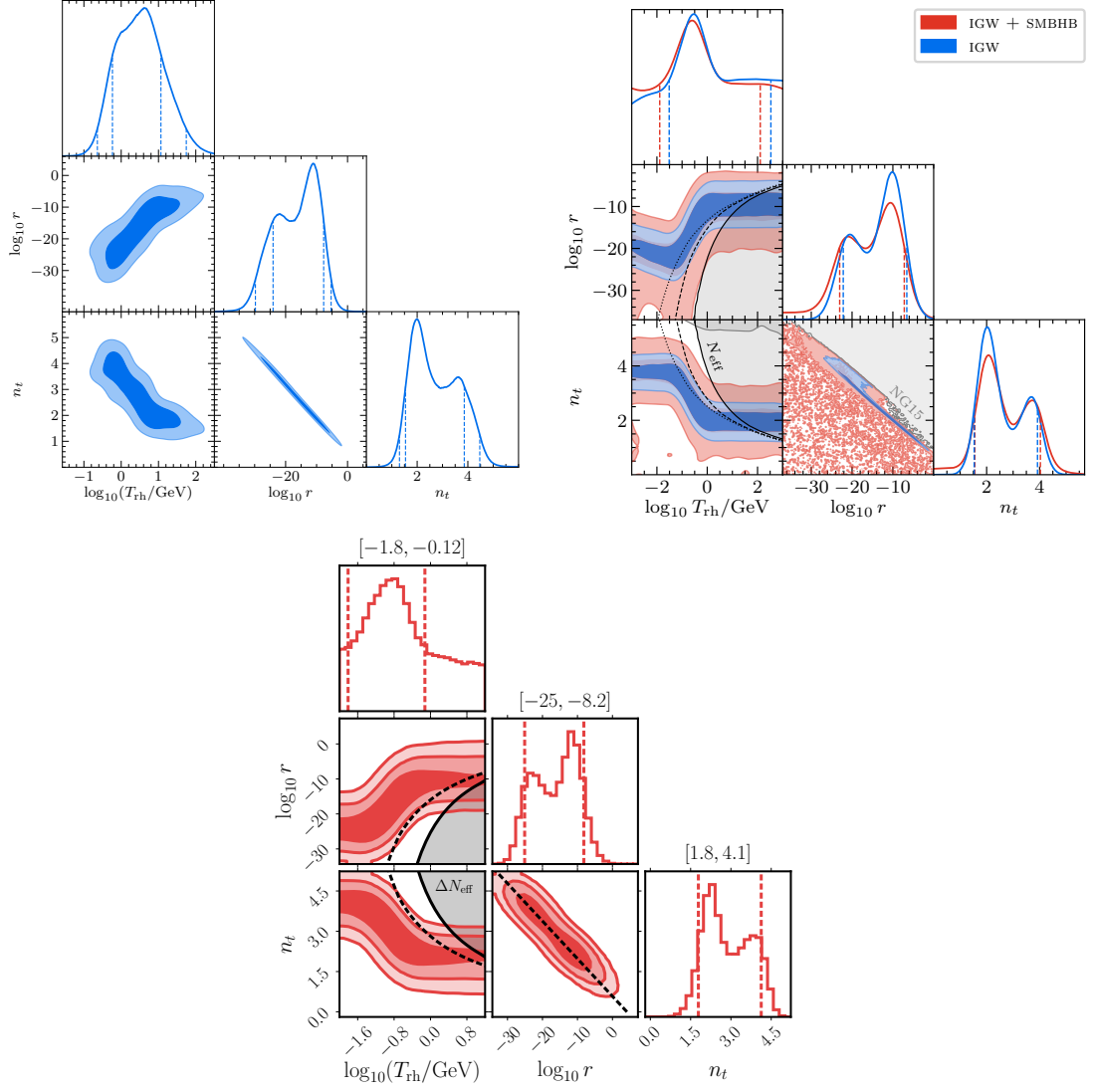


Figure 4.5: Comparison of the posterior plots for the standard reheating case $w_{\text{rh}} = q = 0$ of the new numerical model (top left) with [12] (top right) and [13] (bottom).

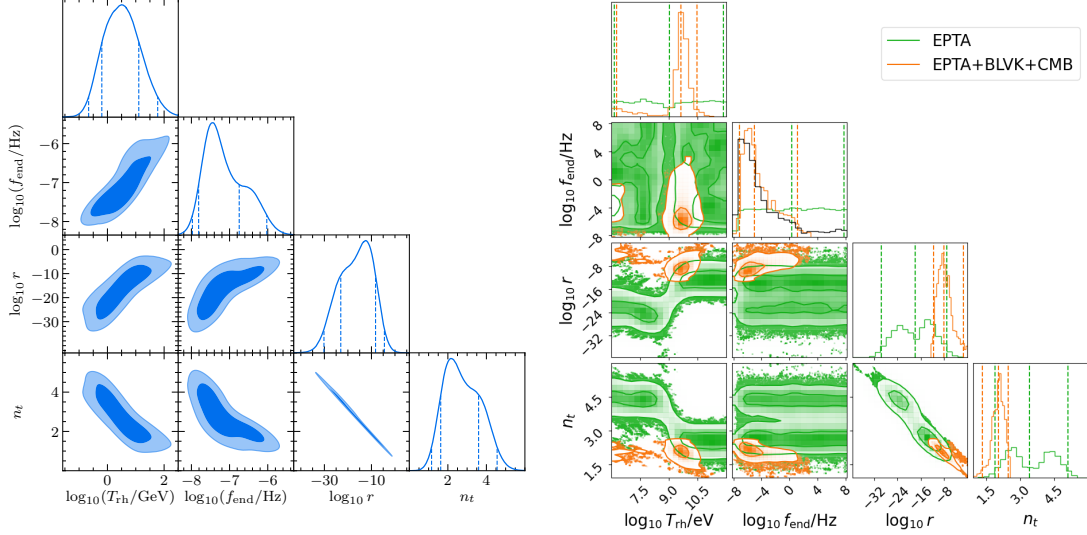


Figure 4.6: Comparison of the posterior plots for the standard reheating case $w_{\text{rh}} = q = 0$ of the new numerical model (left) with [30] (right). In the right plot, the green posteriors are unconstrained by other experiments than EPTA, similar to the new numerical model in this case.

at frequencies that re-enter the horizon during this era, rather than damping it as in the case $w_{\text{rh}} < 1/3$. Since T_{rh} is preferentially low, placing f_{rh} inside or below the NG15 band, the observed frequencies probe exactly this reheating-dominated part of the transfer function. A larger w_{rh} therefore provides an additional, reheating-induced enhancement of the spectrum on top of the primordial blue tilt, so that a smaller primordial amplitude A_t^{max} suffices to match the data, producing the observed negative correlation.

The posteriors for $q = -1, 0, 1$ show large overall agreement, with all constrained parameters occupying similar regions of parameter space, and the corresponding best-fit spectra shown in fig. 4.10 lie close to one another across the NG15 band. Small deviations between the three cases are nevertheless visible. However, these may simply reflect marginal convergence of the respective MCMC chains rather than a genuine physical effect of q , so a more detailed comparison is left for future work. Additionally, since a radiation-domination-like reheating scenario is preferred, it is not possible to distinguish between the different q , since their difference lies mainly in the transition between reheating and RD (see fig. 2.4). As of now, the sensitivity of the NG15 data cannot clearly distinguish between different temperature dependencies of the decay rate.

Since the evaluation of the full numerical model is computationally expensive, it is only computed on a grid for the MCMC analysis. For this reason, a comparison with the analytical fit function is performed in order to assess its validity as a viable approximation. The posterior distributions of the analytical fit is shown in fig. 4.11 (table C.5). Overall, the two sets of posteriors show a large degree of agreement: the constraints on T_{rh} , n_t and A_t^{max} are essentially indistinguishable between the two models, and the general shape of the $H_{\text{end}}-A_t^{\text{max}}$ and $A_t^{\text{max}}-w_{\text{rh}}$ degeneracies discussed above is reproduced by the fit function as well. This confirms that, at the level of the overall parameter constraints, the analytic fit function of eq. 3.7 is a good approximation to the full numerical reheating

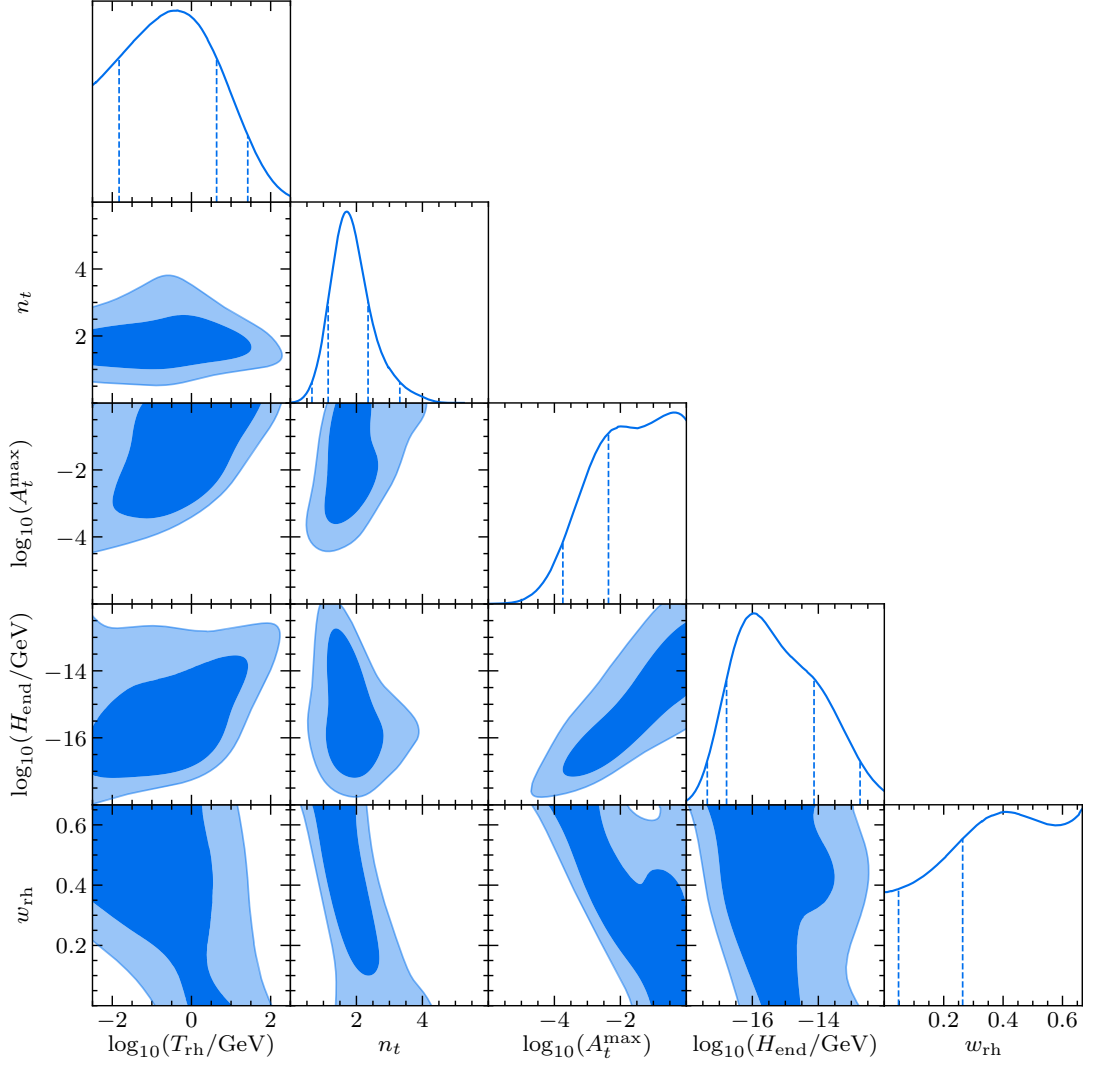


Figure 4.7: Posterior distributions for the full numerical model without backreaction for $q = 0$.

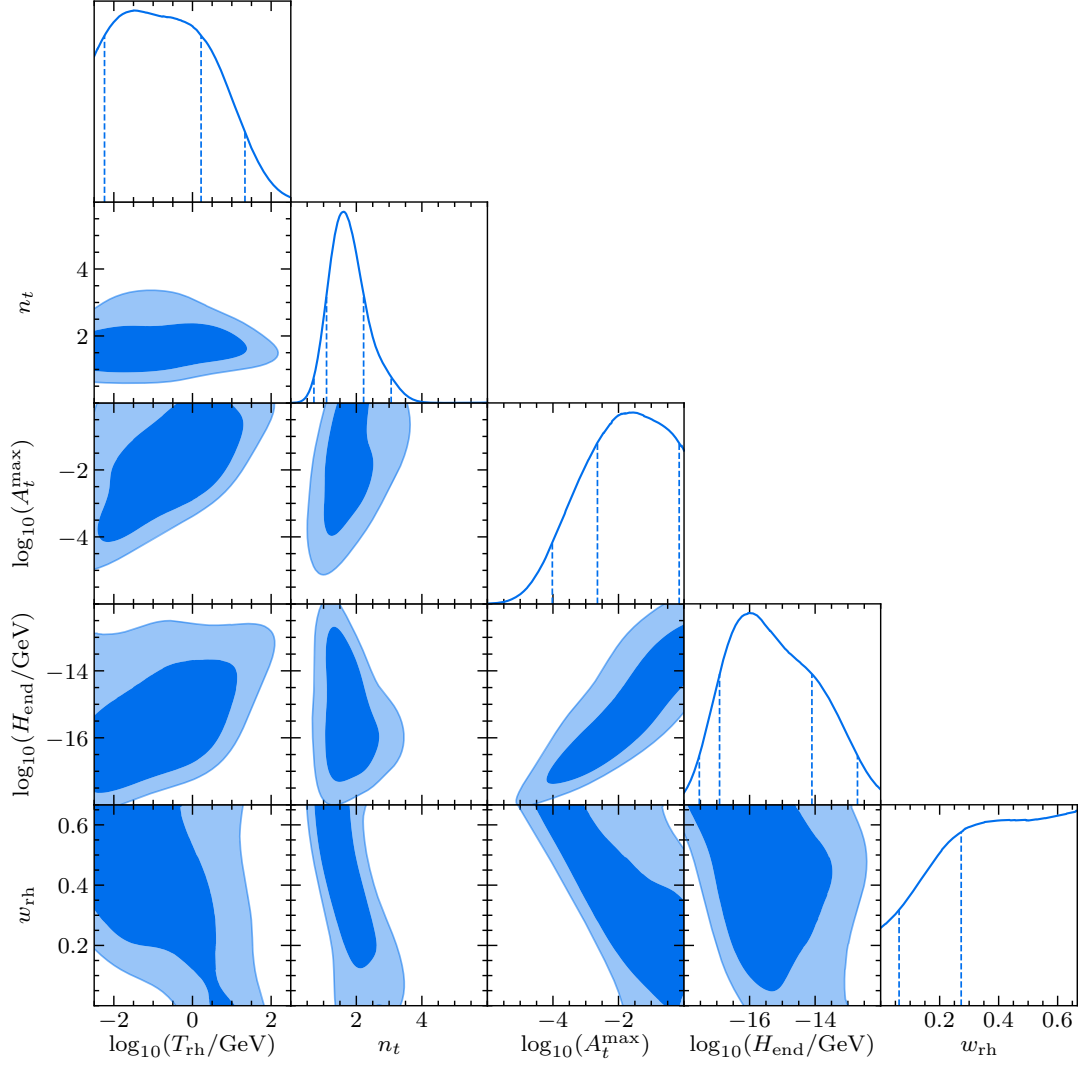


Figure 4.8: Posterior distributions for the full numerical model without backreaction for $q = -1$.

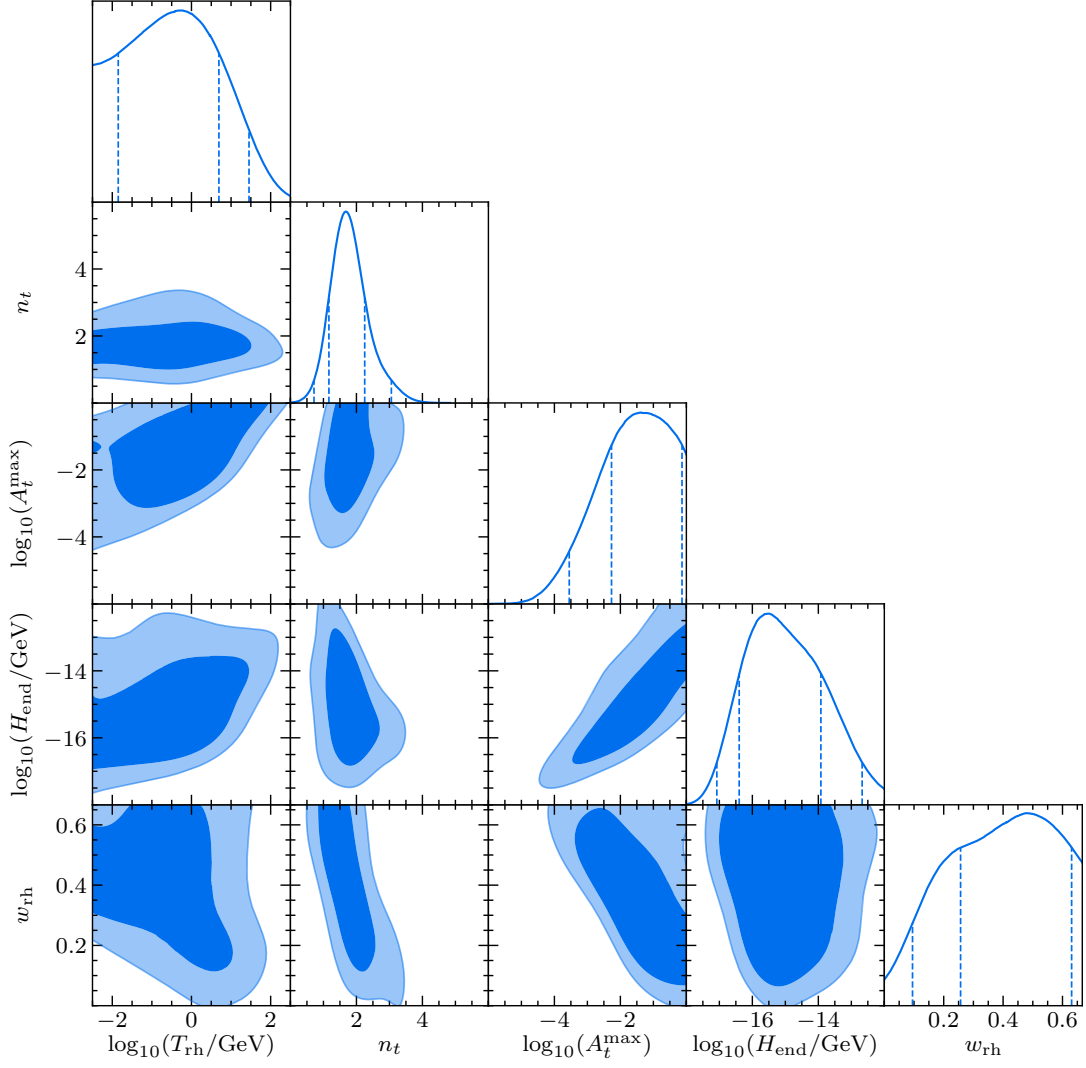


Figure 4.9: Posterior distributions for the full numerical model without backreaction for $q = 1$.

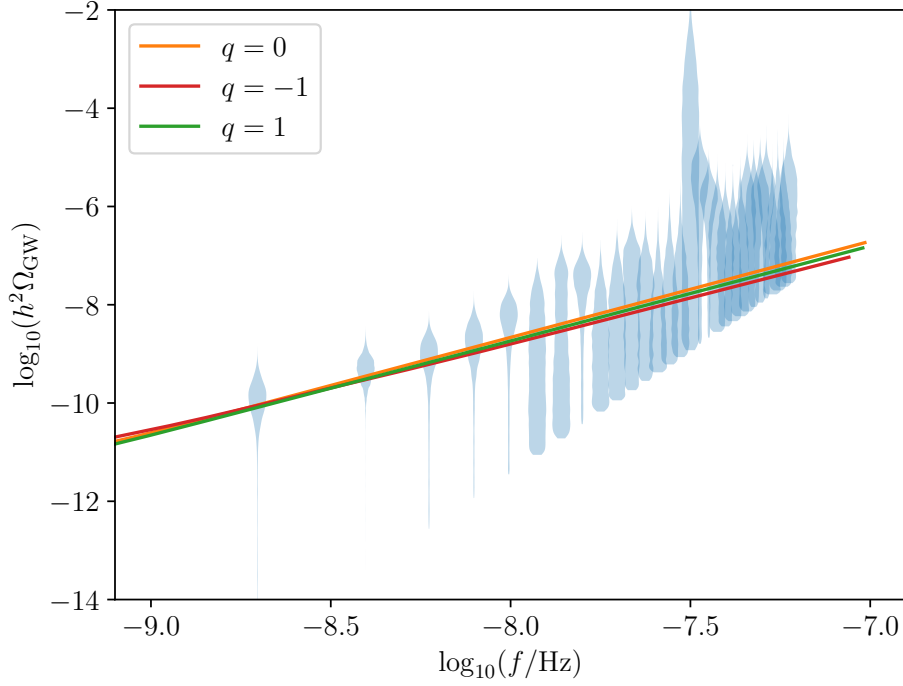


Figure 4.10: Spectra with the Bayes estimator values of the full numerical model for $q = 0, -1, 1$ compared to the NG15 violins.

transfer function.

However, a closer inspection of the H_{end} posterior reveals a small but noticeable difference: the second peak at $\log_{10}(H_{\text{end}}/\text{GeV}) \simeq -14.5$, present in the full numerical model, is suppressed in the fit-function result. Other small differences are visible in the other posteriors as well. These could, however, equally originate from the marginal convergence of the respective MCMC chains rather than from the approximations underlying the fit function.

To identify the origin of this discrepancy, the differences between the fit and the numeric model are investigated. Two main approximations enter the fit function: (i) a constant radiation equation-of-state parameter $w_R = 1/3$, and (ii) the use of analytic estimates for f_{rh} and f_{end} (sec. 2.3.5) in place of their numerically determined values. These are tested separately against the full numerical model in figs. 4.12–4.14 (tables C.6–C.8). Fixing $w_R = 1/3$ in the full numerical model, thereby neglecting the effect of phase transitions on the relativistic degrees of freedom, leaves the resulting posteriors essentially unchanged compared to the full numerical model. This indicates that the NG15 data is not sensitive to the details of the phase transitions encoded in $g_*(T)$ and $g_{*,s}(T)$ at the current level of precision.

In contrast, replacing the analytic estimates of f_{rh} and f_{end} with their numerically determined counterparts produces a significant shift in the posteriors of A_t^{max} , w_{rh} and H_{end} : the combination of a small amplitude with a large w_{rh} is no longer favored, and the second peak in the H_{end} posterior disappears, shifting the two-dimensional degeneracy between H_{end} and A_t^{max} towards larger amplitudes. The same behavior is found when

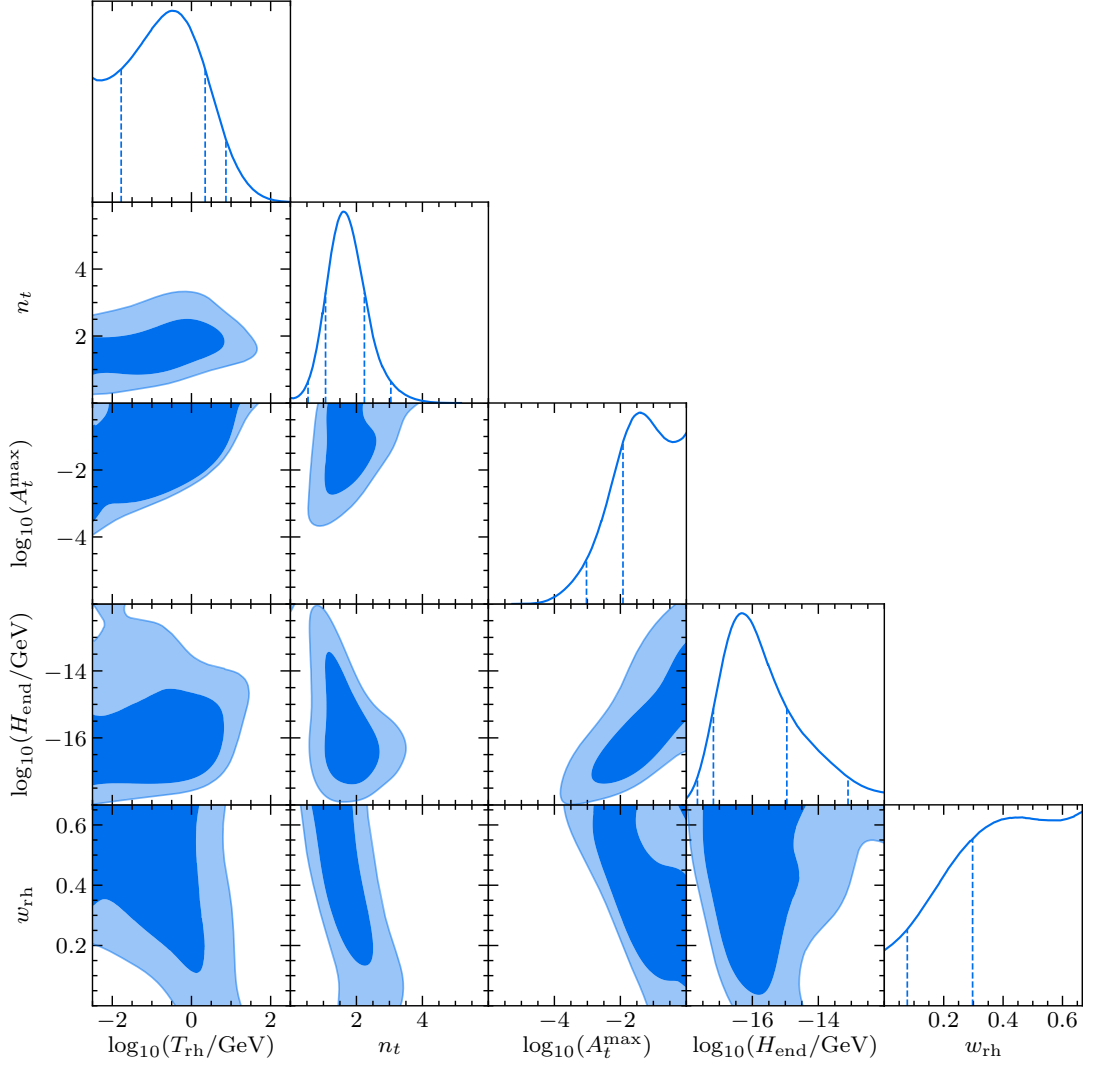


Figure 4.11: Posterior distributions of the analytical fit for $q = 0$.

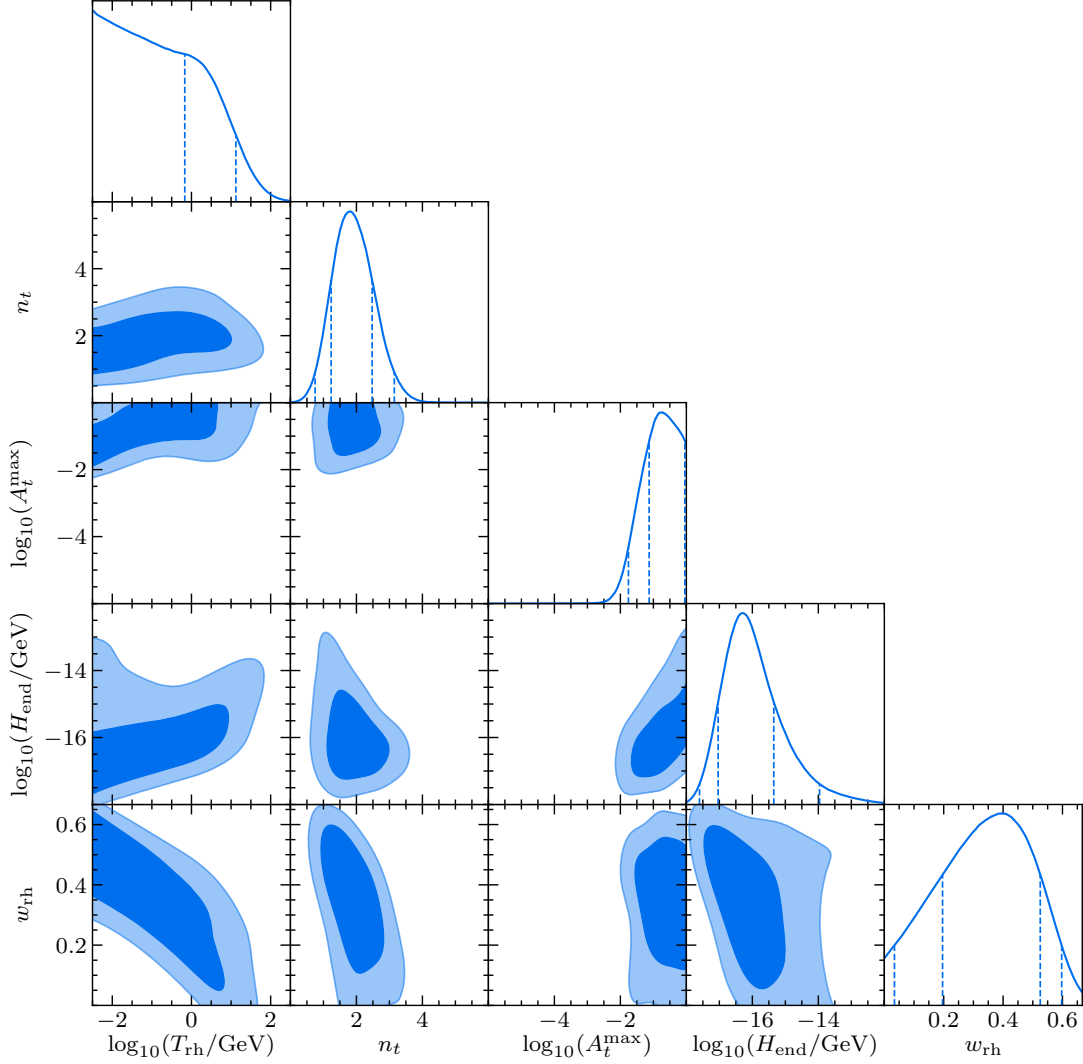


Figure 4.12: Posterior distributions for the fit function with numerically calculated f_{rh} and f_{end} with constant $w_R = 1/3$.

both approximations—constant w_R and numerically determined f_{rh} , f_{end} —are relaxed simultaneously. This confirms that it is specifically the analytic approximation of f_{rh} and f_{end} , rather than the constant- w_R assumption that is responsible for the suppression of the second H_{end} peak observed when comparing the fit function to the full numerical model above.

The disappearance of the large- w_{rh} , small- A_t^{max} region is, however, not reproduced when using the actual fit function. This discrepancy may again be attributable to insufficient convergence of the MCMC chains, but needs further investigation in future work.

4.3.3 With backreaction

Following the discussion of sec. 4.2, since high values of w_{rh} are preferred by the data, and backreaction becomes relevant precisely for stiff equations of state ($w_{\text{rh}} > 1/3$), the effect of backreaction on the reheating dynamics could in principle be strong enough to noticeably

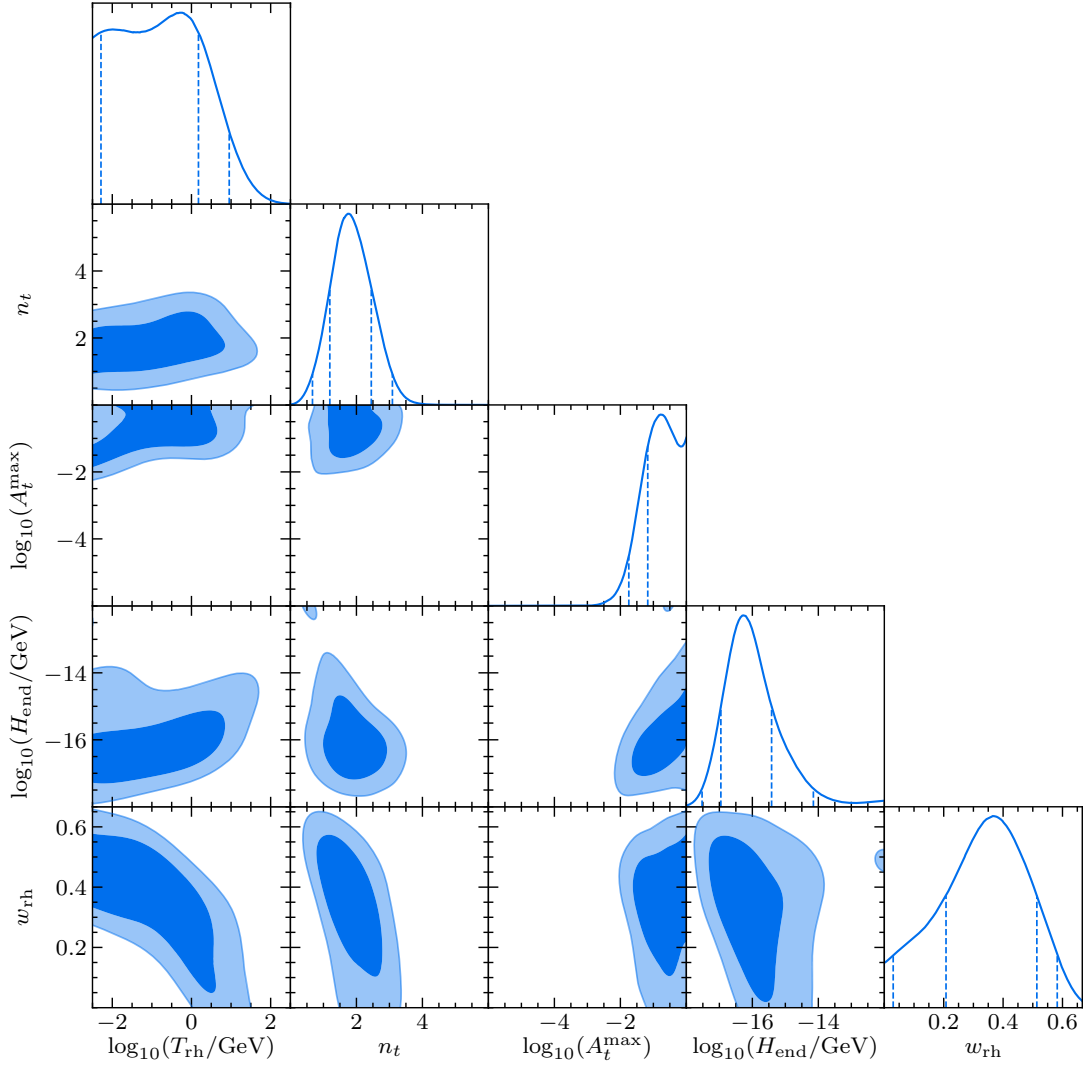


Figure 4.13: Posterior distributions for the fit function with numerically calculated f_{rh} and f_{end} .

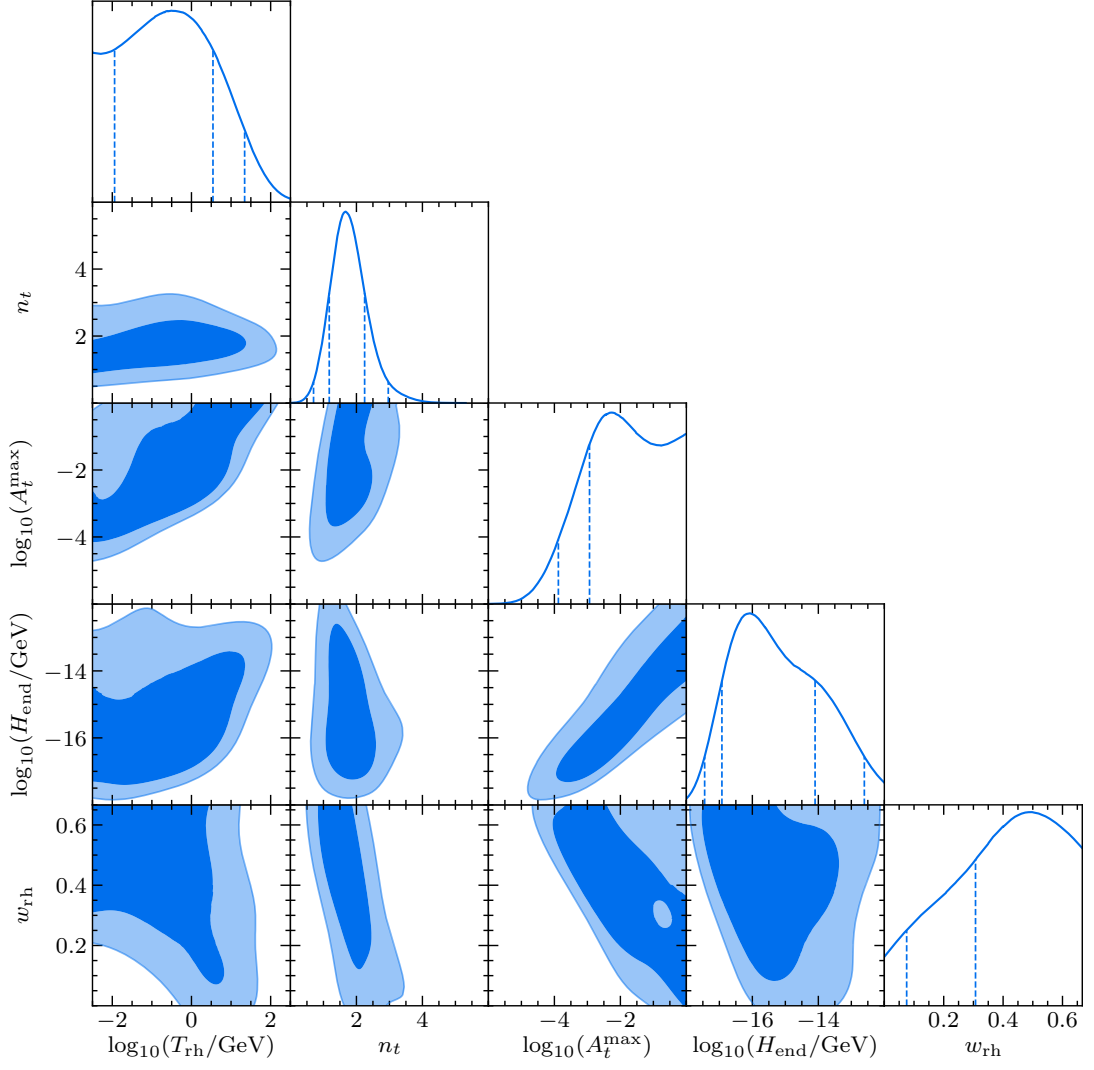


Figure 4.14: Posterior distributions for the fully numerical model with constant $w_R = 1/3$.

impact the posterior distributions. The posterior of the full numerical model including backreaction is shown in fig. 4.15 (table C.9). The H_{end} parameter is not converged yet ($\hat{R} \approx 1.11$), but all pulsar parameters and the other model parameters are, so it is expected that the results do not differ significantly from the marginalized result.

Overall, no major differences are visible between the two cases: the constraints on T_{rh} , n_t , A_t^{max} and w_{rh} remain very similar with and without backreaction, and the general shape of the degeneracies discussed in sec. 4.3.2 is preserved. Some small differences can nevertheless be observed. In particular, the second peak in the H_{end} posterior, already discussed in the context of the fit-function comparison in sec. 4.3.2, is again less pronounced once backreaction is included. Beyond this, no further systematic differences between the two posterior sets stand out. These small remaining differences could either indicate genuine, if minor, effects of backreaction on the fit, assuming the MCMC chains are sufficiently converged, or could instead simply reflect the marginal convergence of the respective chains (see fig. C.9).

This overall weak impact of backreaction can be understood from the interplay of two competing effects. On one hand, backreaction becomes energetically more important for larger w_{rh} : since $\rho_\phi \propto a^{-3(1+w_{\text{rh}})}$ falls off faster than $\rho_{\text{GW}} \propto a^{-4}$ once $w_{\text{rh}} > 1/3$, the ratio $\rho_{\text{GW}}/\rho_{\text{tot}}$ grows over the course of reheating, which is why backreaction is a concern specifically for stiff equations of state (see also [33]). On the other hand, the sensitivity of the reheating transfer function to a backreaction-induced shift of w_{tot} towards $w_{\text{GW}} = 1/3$ (eq. 2.120) itself decreases with w_{rh} : from eq. 2.108, the exponent $n(w_{\text{rh}}) = -(1 - 3w_{\text{rh}})/(1 + 3w_{\text{rh}})$ satisfies $dn/dw_{\text{rh}} = 6/(1 + 3w_{\text{rh}})^2$, which falls from 6 at $w_{\text{rh}} = 0$ to only 0.67 at $w_{\text{rh}} = 2/3$. A given shift of w_{tot} therefore translates into a considerably smaller change of the observable spectral tilt at the large values of w_{rh} favored by the NG15 fit than it would at small w_{rh} . This also explains why even the substantial gravitational-wave energy fraction of about 44% found in the example of sec. 2.3.4, evaluated at $w_{\text{rh}} = 2/3$, was found to leave only a small imprint on the resulting spectrum: the growing energy fraction and the shrinking sensitivity of the transfer function largely compensate each other, so that no strong net effect of backreaction on the posterior distributions is expected or found at the large values of w_{rh} preferred in this analysis. No corresponding statement can currently be made about how this balance would behave at lower w_{rh} , since such scenarios are disfavored by the NG15 data in the first place.

4.3.4 Constrained analysis

Finally, the theoretical and phenomenological constraints discussed in sec. 2.4 are applied to the full numerical model for $q = 0$. Since the perturbativity and ordering constraints ($A_t^{\text{max}} < 1$ and $H_{\text{end}} > H_{\text{rh}}$) are already enforced as hard priors throughout the preceding analyses, the constrained runs additionally impose the ΔN_{eff} bound of eq. 2.136 and the tensor-to-scalar-ratio bound $r < 0.034$ of eq. 2.138. The resulting constrained posteriors are shown in fig. 4.16 (table C.10).

Large differences are visible between the constrained and unconstrained runs. Whereas the unconstrained analysis favors amplitudes close to the perturbativity bound (sec. 4.3.2), the

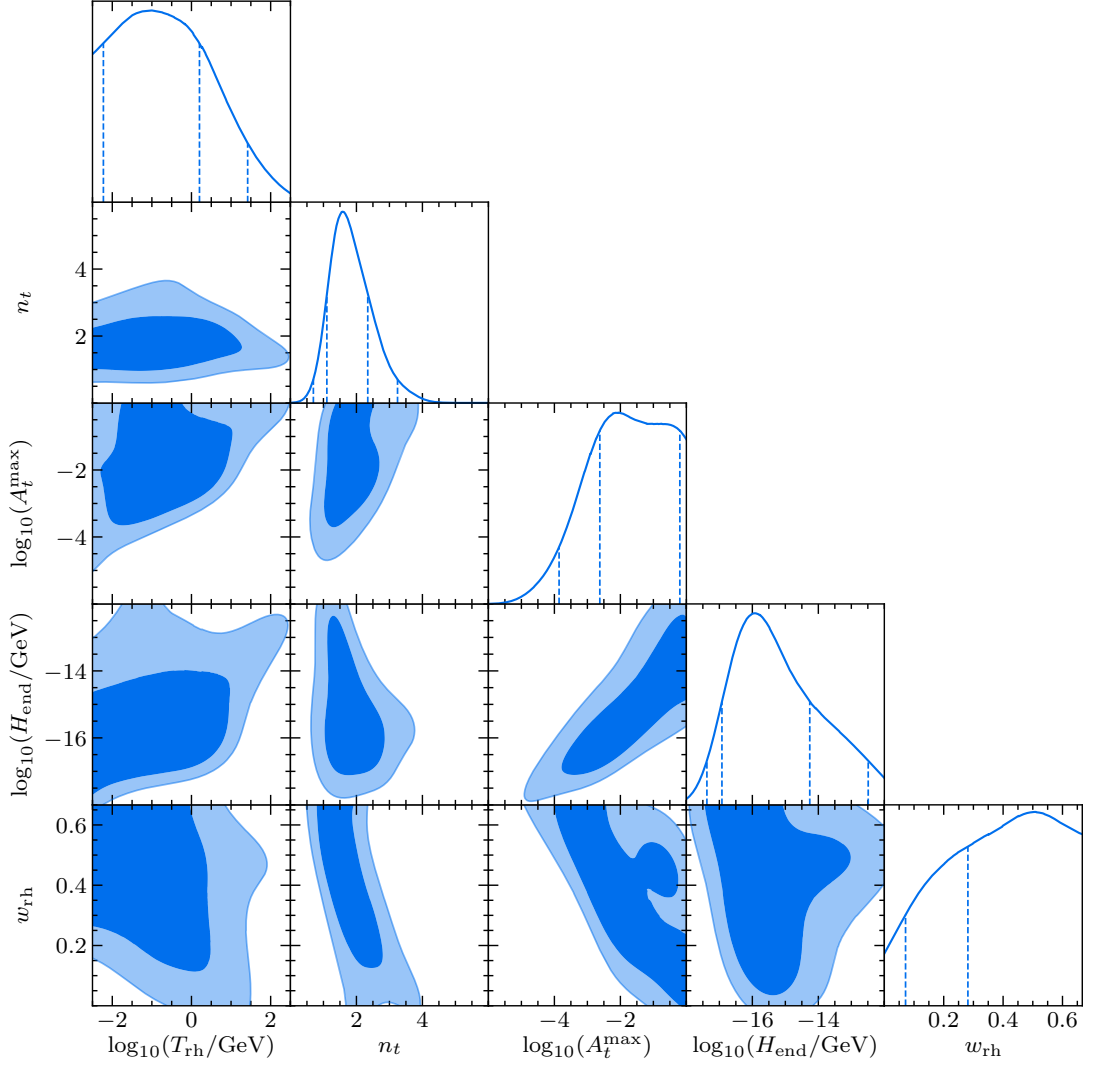


Figure 4.15: Posterior distributions for the full numerical model including backreaction.

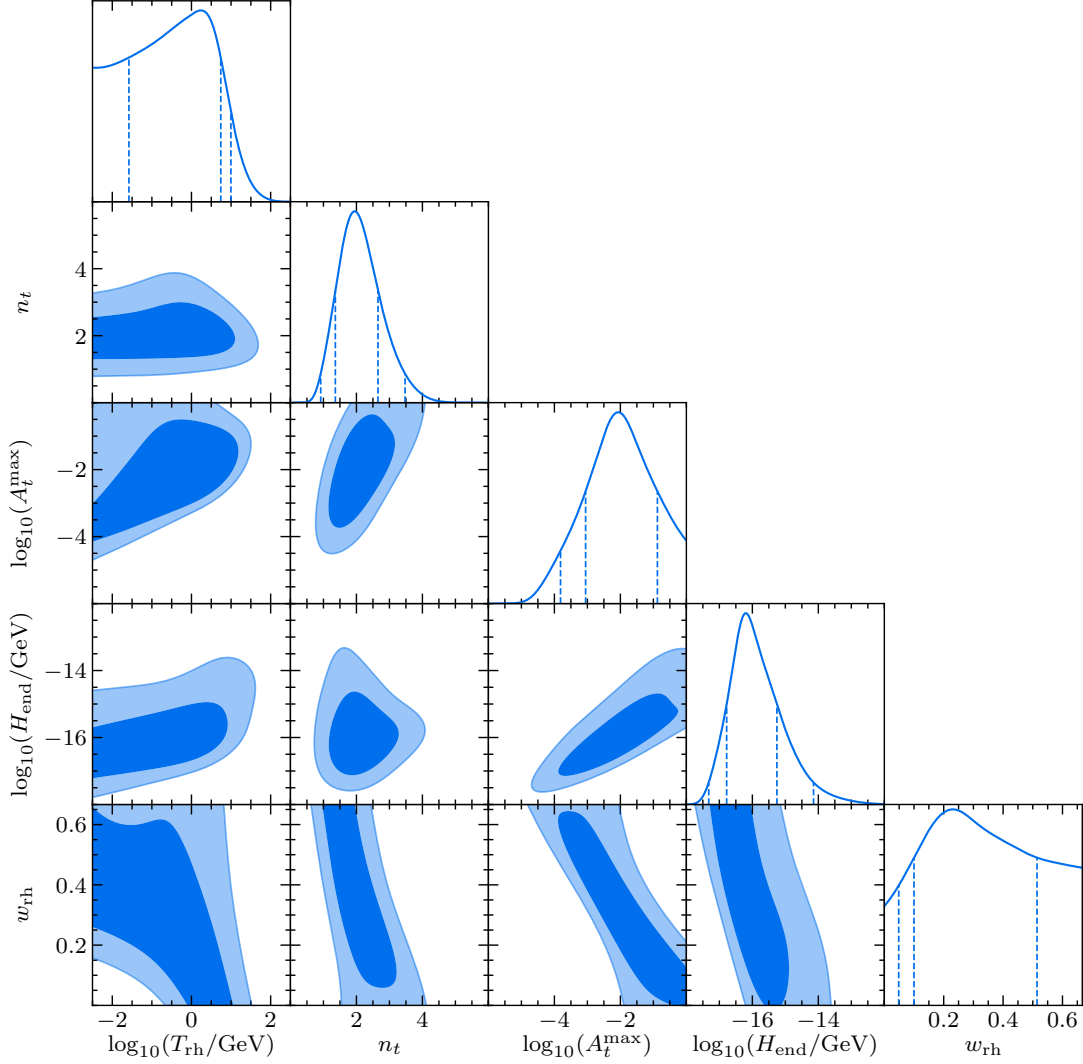


Figure 4.16: Posterior distributions for the constrained full numerical model without backreaction.

constrained posterior for $A_t^{\max}$ instead develops a pronounced peak around $A_t^{\max} \sim 10^{-2}$, well below the perturbativity limit. Similarly, the preference for large w_{rh} seen in the unconstrained case is no longer present. The maximum posterior value shifts down to $w_{\text{rh}} \simeq 1/3$. As a consequence, the second peak in the H_{end} posterior, discussed repeatedly in secs. 4.3.2 and 4.3.3, disappears entirely once the constraints are applied.

This behavior is largely driven by the ΔN_{eff} bound, which is the tightest of the two additional constraints in the relevant part of parameter space. Since both a large amplitude $A_t^{\max}$ and a large w_{rh} independently drive up the gravitational-wave energy density integrated over frequency (and, in the case of large w_{rh} , additionally enhance the reheating-era backreaction contribution discussed in sec. 4.2), both of these regions of parameter space are disfavored once the ΔN_{eff} bound is enforced, explaining the simultaneous shift observed in both $A_t^{\max}$ and w_{rh} .

Comparing the constrained posteriors with and without backreaction (fig. 4.17 and table

C.11) shows no visible differences between the two cases. This is consistent with the interpretation above: once the ΔN_{eff} bound excludes the large-amplitude, large- w_{rh} region of parameter space in which backreaction would be most relevant (sec. 4.3.3), the remaining allowed parameter space corresponds to gravitational-wave energy densities too small for backreaction to have a noticeable effect, rendering the inclusion of backreaction largely irrelevant for the constrained analysis.

It should be noted that these constrained posteriors were obtained using the `ceffyl` sampling mode of `PTArcade`, rather than the `enterprise`-based likelihood used throughout the rest of this analysis (sec. 3.2), due to technical issues encountered with the `enterprise`-based sampling when including the additional cosmological constraints. As `ceffyl` relies on a factorized, pre-computed free-spectrum likelihood rather than directly evaluating the full time-domain pulsar likelihood, the resulting posteriors should therefore be compared to the unconstrained results only qualitatively, and not treated as being directly and quantitatively comparable to the full `enterprise`-based analyses presented in the preceding sections.

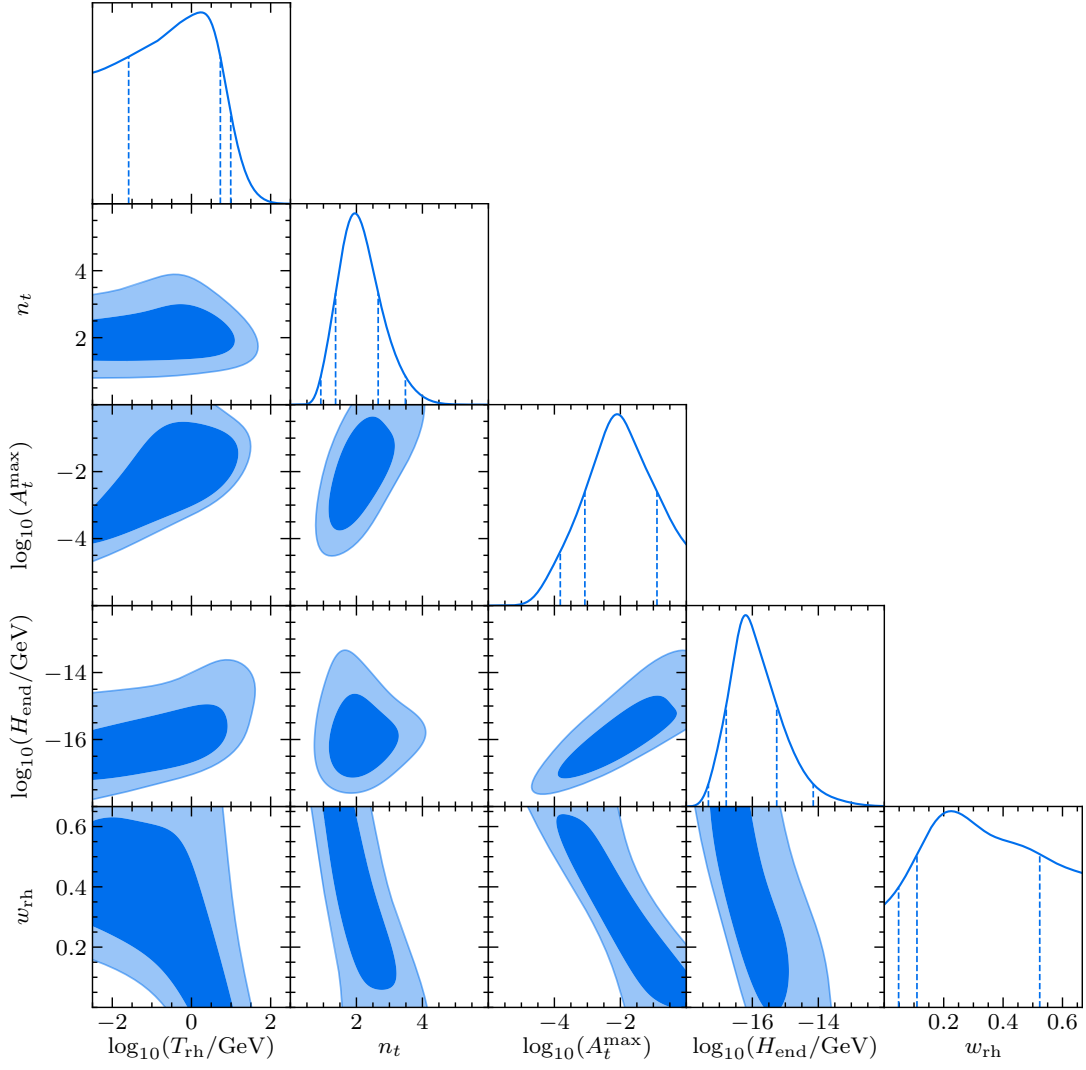


Figure 4.17: Posterior distributions for the constrained full numerical model with backreaction.

5 Conclusion and Outlook

The goal of this thesis was to construct an agnostic model for the transfer function of inflationary GWs, including the effects of reheating, and to use it to probe the reheating era with current PTA data under the assumption that reheating falls within the NANOGrav frequency band. To this end, the standard instantaneous or fixed w_{rh} treatment of reheating was generalized by introducing the averaged equation-of-state parameter w_{rh} and the temperature dependence q of the inflaton decay rate as free, model-independent parameters, and the resulting transfer function was combined with a blue-tilted, cutoff primordial tensor power spectrum characterized by A_t^{max} , n_t and the cutoff scale set by H_{end} . Compared to earlier reheating analyses of the NG15 data set, the model was constrained additionally by the perturbativity bound and the influence of the energy density of the GWs themselves on the reheating dynamics (backreaction) was investigated. To make the resulting six-dimensional parameter space tractable for MCMC sampling, an analytic fit function for the reheating transfer function was developed. It reaches sub-2% accuracy in the standard case $w_{\text{rh}} = q = 0$, improving on existing fits in the literature, and remains at a comparable level of accuracy for w_{rh} tested between 0 and $2/3$.

Fitting the new model to the NG15 data set in the standard reheating case $w_{\text{rh}} = q = 0$ reproduces the results of both the earlier NANOGrav analysis of [12, 13] and the EPTA DR2 analysis of [30] reasonably well. The main difference is that the new model is more strongly constrained at both low and high T_{rh} , since it explicitly enforces perturbativity of the tensor power spectrum, a check not performed in the three reference analyses. Once w_{rh} is allowed to vary freely, the agnostic model shows a clear preference for large values of w_{rh} and for amplitudes A_t^{max} close to the perturbativity bound, together with $n_t \simeq 1.8$, while H_{end} and T_{rh} settle around $\log_{10}(H_{\text{end}}/\text{GeV}) \simeq -15$ and $\log_{10}(T_{\text{rh}}/\text{GeV}) \simeq -0.5$, respectively, broadly similar to, but slightly shifted from, the corresponding values obtained in the standard $w_{\text{rh}} = 0$ case. No clear sensitivity of the NG15 data to the temperature dependence q of the decay rate could be established. The posteriors for $q = -1, 0, 1$ largely agree, though the small differences observed could stem either from a genuine, weak dependence on q or from insufficient convergence of the respective MCMC chains, and the present analysis cannot distinguish between these possibilities.

The analytic fit function was found to reproduce the full numerical posteriors highly accurately when applied to the NG15 data, with the constraints on T_{rh} , n_t and A_t^{max} essentially unchanged. Some differences remain, however, most notably a suppressed second peak in the H_{end} posterior. This was traced back to the analytic approximations for f_{rh} and f_{end} used in the fit function rather than to its assumption of a constant radiation equation-of-state parameter, since fixing $w_R = 1/3$ in the full numerical model on its own left the posteriors unchanged, indicating that the NG15 data is not sensitive to the relativistic degrees of freedom of the SM at the current level of precision. Some further small deviations could again be attributable to insufficient chain convergence. Backreaction, in turn, was not found to strongly affect the NG15 posteriors, either in the unconstrained or constrained runs, though the small remaining differences observed could once more reflect either a genuine, minor effect of backreaction or insufficient convergence of the chains.

This weak overall impact could be explained by the fact that a backreaction-induced shift of w_{tot} towards $1/3$ affects the observable spectral tilt less and less as w_{rh} increases, even as the underlying gravitational-wave energy fraction grows, so that the two effects largely compensate at the large w_{rh} favored by the fit. Finally, applying the phenomenological BBN (ΔN_{eff}) and tensor-to-scalar-ratio constraints of sec. 2.4 removes the preference for both large amplitudes and large w_{rh} seen in the unconstrained analysis. Correspondingly, this also removes any visible influence of backreaction on the posteriors, again with no visible difference between the backreaction and no-backreaction cases, since the region of parameter space in which backreaction would matter is largely excluded by the BBN bound. These constrained results were obtained using the `ceffyl` sampling mode of `PTArcade` rather than the `enterprise`-based likelihood used elsewhere in this thesis, due to technical issues with the latter when the additional cosmological constraints were included, so this comparison should be regarded as qualitative rather than quantitative. Taken together, these results suggest that PTAs such as NANOGrav are, in principle, a promising probe of the reheating era: the NG15 data can be used for the reheating-induced modulation of the tensor spectrum to constrain T_{rh} , n_t and, to a lesser extent, w_{rh} and H_{end} , without having to assume a specific inflationary or reheating model. It should be stressed, however, that the priors used throughout this analysis were deliberately chosen such that the reheating-induced feature falls within the NG15 frequency band. The results presented here therefore demonstrate that reheating parameters can be meaningfully constrained if reheating occurred at the corresponding energy scale, but do not constitute evidence that the NG15 signal is actually caused by reheating rather than, e.g., supermassive black-hole binaries or another cosmological source. At the same time, this sensitivity comes with important limitations: the unconstrained fits favor parameter regions bordering on theoretical inconsistency (perturbativity violation) or observational exclusion (the BBN bound), and only shrink into a physically consistent region once these phenomenological constraints are explicitly imposed. This indicates that a purely PTA-based fit, without additional cosmological input, does not yet robustly pin down the reheating parameters on its own, and that combining PTA data with independent constraints, as done here through ΔN_{eff} and r , is currently essential for a physically meaningful interpretation of any reheating signature in the data.

Several directions remain for future work. On the analysis side, longer MCMC runs would help to clarify whether the small residual differences observed between different values of q , between the fit function and the full numerical model, and between the backreaction and no-backreaction cases, reflect genuine physical effects or insufficient chain convergence. Resolving the technical issues encountered with the `enterprise`-based sampler for the constrained runs would additionally allow a direct, quantitative comparison with the unconstrained results. More broadly, adopting faster MCMC or likelihood-evaluation techniques would also allow for larger sampling sizes to compare the posteriors and make it feasible to extend the numerical grid-based approach to finer resolution and to a wider parameter range. On the modeling side, the analysis of $w_{\text{rh}} > 2/3$ remains an open task, both because the fit function has not yet been validated in this regime and because the

numerical computation of the exact reheating dynamics becomes increasingly expensive there. A fully self-consistent treatment of backreaction, for instance through a recursive numerical scheme in which the reheating background and the gravitational-wave spectrum are solved for iteratively until convergence, would remove the approximations made in sec. 3.5 and allow backreaction to be included reliably even in regimes where it is not merely a small correction. Similarly, a dedicated transfer function describing the transition from inflation to reheating would resolve the current treatment of modes close to the horizon at that transition, though at the cost of reintroducing some model dependence into an otherwise agnostic framework. Finally, extending the model to include the oscillations of the inflaton field and the associated oscillations of the Hubble rate, i.e. effects associated with preheating (see [41] for a short overview), could capture additional physics not accounted for in the averaged treatment of reheating used throughout this thesis.

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A Supplementary Derivations

This chapter offers supplementary derivations for the theory described in sec. 2. It is based on [21] and [22].

A.1 FLRW metric to first order

The FLRW metric to first order can be written as:

$$ds^2 = a^2(\eta)(\eta_{\mu\nu} + h_{\mu\nu})dx^\mu dx^\nu \quad (\text{A.1})$$

With the conformal time η with $d\eta = dt/a$ and the comoving coordinates $\mathbf{x}$. For the 4-vector one can choose $x^\mu = (\eta, \mathbf{x})$. The perturbation $h_{\mu\nu}$ is typically small in an isotropic and homogeneous universe ($|h_{\mu\nu}| \ll 1$). This thesis will stay in first order, so every term of $\mathcal{O}(h^2)$ or higher will vanish. Note that the symmetric tensor $h_{\mu\nu}$ at this moment has 10 degrees of freedom. This is because gauge transformations have not been included yet.

A.2 Helicity decomposition of $h_{\mu\nu}$

The tensor $h_{\mu\nu}$ is next decomposed into scalar, vector and tensor perturbations:

$$h_{00} = 2\psi \quad (\text{A.2})$$

$$h_{0i} = \beta_i + \partial_i \gamma \quad (\text{A.3})$$

$$h_{ij} = -2\phi\delta_{ij} + \left(\partial_i\partial_j - \frac{1}{3}\delta_{ij}\nabla^2\right)\lambda + \frac{1}{2}(\partial_i\epsilon_j + \partial_j\epsilon_i) + h_{ij}^{TT} \quad (\text{A.4})$$

With:

$$\partial_i\beta^i = 0 \quad \partial_i\epsilon^i = 0 \quad \partial^j h_{ij}^{TT} = 0 \quad \delta^{ij} h_{ij}^{TT} = 0 \quad (\text{A.5})$$

and the flat space Laplacian $\nabla^2 = \partial^i\partial_i$. The decomposition is chosen in such a way that the perturbation still has 10 degrees of freedom and the symmetry of the tensor is preserved. ψ, γ, ϕ and λ are scalar perturbations, β_i and ϵ_i are transverse vector perturbations and h_{ij}^{TT} is a traceless, transverse tensor perturbation. Notice that because h_{ij}^{TT} is symmetric, transverse and traceless, the tensor has 2 degrees of freedom left, which will later correspond to the two polarizations of the gravitational wave. Here, also the convention is used that all spatial indices can be lowered or raised with the Kronecker delta δ_{ij} , s.t. $\beta_i = \beta^i$ and so on. This is only viable for purely spatial tensors. If, for example, a four-vector is considered, this transformation is not applicable, and the full metric tensor is needed. Because this thesis only works with the first order, no mixed terms will appear in the calculations and scalar, vector and tensor perturbations will not interact. This means that the different perturbation parts can be analyzed separately.

A.3 Tensor fluctuations in curved space

Now the Einstein tensor for the traceless, transverse tensor perturbation part will be calculated. Thus, the only part of $h_{\mu\nu}$ one will look at is the h_{ij}^{TT} part. For this, the

metric is given by:

$$ds^2 = a^2[-d\eta^2 + (\delta_{ij} + h_{ij}^{TT})dx^i dx^j] \quad (\text{A.6})$$

To first order, to preserve $g^{\mu\rho}g_{\rho\nu} = \delta_\nu^\mu$, the inverse metric tensor is:

$$g^{\mu\nu} = \frac{1}{a^2}(\eta^{\mu\nu} - h^{\mu\nu}) \quad (\text{A.7})$$

First, the Christoffel symbols are calculated:

$$\Gamma_{\mu\nu}^\delta = \frac{1}{2}g^{\rho\sigma}[\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}] \quad (\text{A.8})$$

For the calculation of the specific parts of the tensor, a few properties can be used to simplify the equation: The scale factor a is only a function of (conformal) time, the Minkowski metric is independent of the spacetime coordinates, h_{ij}^{TT} is transverse and traceless and the Christoffel symbols are symmetric under commuting the lower indices.

$$\Gamma_{00}^0 = \mathcal{H} \quad (\text{A.9})$$

$$\Gamma_{0i}^0 = 0 \quad (\text{A.10})$$

$$\Gamma_{ij}^0 = \mathcal{H}\delta_{ij} + \mathcal{H}h_{ij}^{TT} + \frac{1}{2}(h_{ij}^{TT})' \quad (\text{A.11})$$

$$\Gamma_{00}^i = 0 \quad (\text{A.12})$$

$$\Gamma_{0j}^i = \mathcal{H}\delta_{ij} + \frac{1}{2}(h_{ij}^{TT})' \quad (\text{A.13})$$

$$\Gamma_{jk}^i = \frac{1}{2}(\partial_j h_{ik}^{TT} + \partial_k h_{ij}^{TT} - \partial_i h_{jk}^{TT}) \quad (\text{A.14})$$

$\mathcal{H} = \frac{a'}{a}$ is the conformal Hubble parameter. With these terms, the Ricci tensor can be calculated:

$$R_{ij} = \partial_\lambda \Gamma_{ij}^\lambda - \partial_j \Gamma_{i\lambda}^\lambda + \Gamma_{\lambda\sigma}^\lambda \Gamma_{ij}^\sigma - \Gamma_{j\sigma}^\lambda \Gamma_{i\lambda}^\sigma \quad (\text{A.15})$$

The second summand will vanish, so the spatial Ricci tensor is given by:

$$R_{ij} = (\mathcal{H}' + 2\mathcal{H}^2)\delta_{ij} + (\mathcal{H}' + 2\mathcal{H}^2)h_{ij}^{TT} + \mathcal{H}(h_{ij}^{TT})' + \frac{1}{2}(h_{ij}^{TT})'' - \frac{1}{2}\nabla^2 h_{ij}^{TT} \quad (\text{A.16})$$

For determining the Ricci scalar, R_{00} is also needed:

$$R_{00} = -3\mathcal{H}' \quad (\text{A.17})$$

Next, the Ricci scalar can be computed:

$$R = g^{\mu\nu}R_{\mu\nu} = \frac{1}{a^2}(-R_{00} + R_{ii}) = \frac{6}{a^2}(\mathcal{H}' + \mathcal{H}^2) \quad (\text{A.18})$$

$$Rg_{ij} = 6(\mathcal{H}' + \mathcal{H}^2)\delta_{ij} + 6(\mathcal{H}' + \mathcal{H}^2)h_{ij}^{TT} \quad (\text{A.19})$$

Subsequently, combining everything for the Einstein tensor:

$$G_{ij} = R_{ij} - \frac{1}{2}Rg_{ij} = -(2\mathcal{H}' + \mathcal{H}^2)\delta_{ij} - (2\mathcal{H}' + \mathcal{H}^2)h_{ij}^{TT} + \mathcal{H}(h_{ij}^{TT})' + \frac{1}{2}(h_{ij}^{TT})'' - \frac{1}{2}\nabla^2 h_{ij}^{TT} \quad (\text{A.20})$$

Moving up one of the indices yields:

$$G_j^i = g^{ik}G_{kj} = \frac{1}{a^2}[-(2\mathcal{H}' + \mathcal{H}^2)\delta_{ij} + \mathcal{H}(h_{ij}^{TT})' + \frac{1}{2}(h_{ij}^{TT})'' - \frac{1}{2}\nabla^2 h_{ij}^{TT}] \quad (\text{A.21})$$

$$\bar{G}_j^i = -\frac{1}{a^2}(2\mathcal{H}' + \mathcal{H}^2)\delta_{ij} \quad (\text{A.22})$$

$$\delta G_j^i = \frac{1}{2a^2}[(h_{ij}^{TT})'' + 2\mathcal{H}(h_{ij}^{TT})' - \nabla^2 h_{ij}^{TT}] \quad (\text{A.23})$$

Here, $\bar{G}_j^i$ is the Einstein tensor of the background metric (zeroth order) and δG_j^i is the Einstein tensor of the transverse, traceless tensor perturbation (first order) with $G_j^i = \bar{G}_j^i + \delta G_j^i$.

Similarly to $h_{\mu\nu}$, the energy-momentum tensor can also be decomposed into a scalar, vector and tensor part:

$$T_0^0 = -\rho \quad T_0^i = S^i + \partial S \quad T_j^i = p\delta_j^i + \Sigma_j^i \quad (\text{A.24})$$

One can obtain the components of T_i^0 with $T_i^0 = g^{0\mu}g_{i\nu}T_\mu^\nu$. As before one can choose the conventions:

$$S^i = S_i \quad \partial^i S = \partial_i S \quad \Sigma^{ij} = \Sigma_j^i = \Sigma_{ij} \quad (\text{A.25})$$

Σ_{ij} can also be decomposed in a scalar, transverse vector and transverse, traceless tensor part:

$$\Sigma_{ij} = \Sigma_{ij}^s + \Sigma_{ij}^v + \Sigma_{ij}^t \quad (\text{A.26})$$

$$\Sigma_{ij}^s = \left(\partial_i \partial_j - \frac{1}{3} \delta_{ij} \nabla^2 \right) \sigma \quad (\text{A.27})$$

$$\Sigma_{ij}^v = \frac{1}{2}(\partial_i \sigma_j + \partial_j \sigma_i) \quad (\text{A.28})$$

$$\Sigma_{ij}^t = \sigma_{ij}^{TT} \quad (\text{A.29})$$

Where:

$$\partial_i \sigma^i = 0 \quad \partial_i \sigma_{ij}^{TT} = 0 \quad \delta^{ij} \sigma_{ij}^{TT} = 0 \quad (\text{A.30})$$

In an isotropic and homogeneous universe, the energy-momentum tensor takes the form of a perfect fluid and the energy density and pressure can be split into a background part $\bar{\rho}, \bar{p}$ and a perturbation part $\delta\rho, \delta p$. In addition, $S, S_i, \sigma, \sigma_i, \sigma_{ij}^{TT}$ describe perturbations of the perfect fluid. As with the metric perturbation, the perturbations can be treated separately to first order. For instance, if one just looks at the perturbed, transverse, traceless part of the energy-momentum tensor:

$$\delta T_j^i = \sigma_{ij}^{TT} \quad (\text{A.31})$$

To finally obtain the EOM, one can look at the perturbed, transverse, traceless part of

the field equation:

$$\delta G_j^i = 8\pi G \delta T_j^i \quad (\text{A.32})$$

Using eq. (A.23) and (A.31):

$$(h_{ij}^{TT})'' + 2\mathcal{H}(h_{ij}^{TT})' - \nabla^2 h_{ij}^{TT} = 16\pi G a^2 \sigma_{ij}^{TT} \quad (\text{A.33})$$

One can split the two degrees of freedom in h_{ij}^{TT} and σ_{ij}^{TT} into two polarization parts by switching to momentum space (where $-\nabla^2$ will change to k^2):

$$h_{ij}^{TT}(\eta, \mathbf{k}) = \sum_{A=+, \times} e_{ij}^A(\hat{\mathbf{k}}) h_A(\eta, \mathbf{k}) \quad (\text{A.34})$$

$$\sigma_{ij}^{TT}(\eta, \mathbf{k}) = \sum_{A=+, \times} e_{ij}^A(\hat{\mathbf{k}}) \sigma_A(\eta, \mathbf{k}) \quad (\text{A.35})$$

$\hat{\mathbf{k}}$ is the normal vector of momentum $\mathbf{k}$, and the two polarization tensors e_{ij}^A are orthogonal to $\hat{\mathbf{k}}$ and each other. With this, the two degrees of freedom can be decoupled into:

$$h_A'' + 2\mathcal{H}h_A' + k^2 h_A = 16\pi G a^2 \sigma_A \quad (\text{A.36})$$

B Fitting the reheating transfer function

The fit function in eq. 3.7 was fitted individually 100 times in the range $p = 2-6$ for each $q = -1, 0, 1$, yielding the results shown in fig. B.1. To get an overall fit for the parameters $a(p, q), c(p, q), \alpha(p, q)$, the dependence of these parameters on p is examined (see fig. B.2). There one sees that there is a divergence around $p = 4$, which corresponds to the case in which $T_{\text{RH}} \rightarrow 1$. Here, the fitting procedure requires extreme parameter values to produce a noticeable effect, making this feature a numerical artifact rather than a physical one. To make an overall refit, the divergence is excluded and the following functions to model the parameters is chosen:

$$a(p) = d_0 + \frac{d_1}{p} + \frac{d_2}{p^2} + \frac{d_3}{p^3} + \frac{d_4}{p^4} \quad (\text{B.1})$$

$$c(p) = b_0 \cdot \ln(b_1 \cdot p + b_2) + b_3 \cdot \sqrt{p} + b_4 \quad (\text{B.2})$$

$$\alpha(p) = c_0 + \frac{c_1}{p} + \frac{c_2}{p^2} + \frac{c_3}{p^3} + \frac{c_4}{p^4} \quad (\text{B.3})$$

Refitting gives a general reheating transfer function, while still achieving sub-2% accuracy (see fig. B.3). The refits were tested up to $w_{\text{rh}} = 2/3$ or equivalently $p = 10$. The values of c_i, d_i and b_i for the refitting are shown in table B.1.

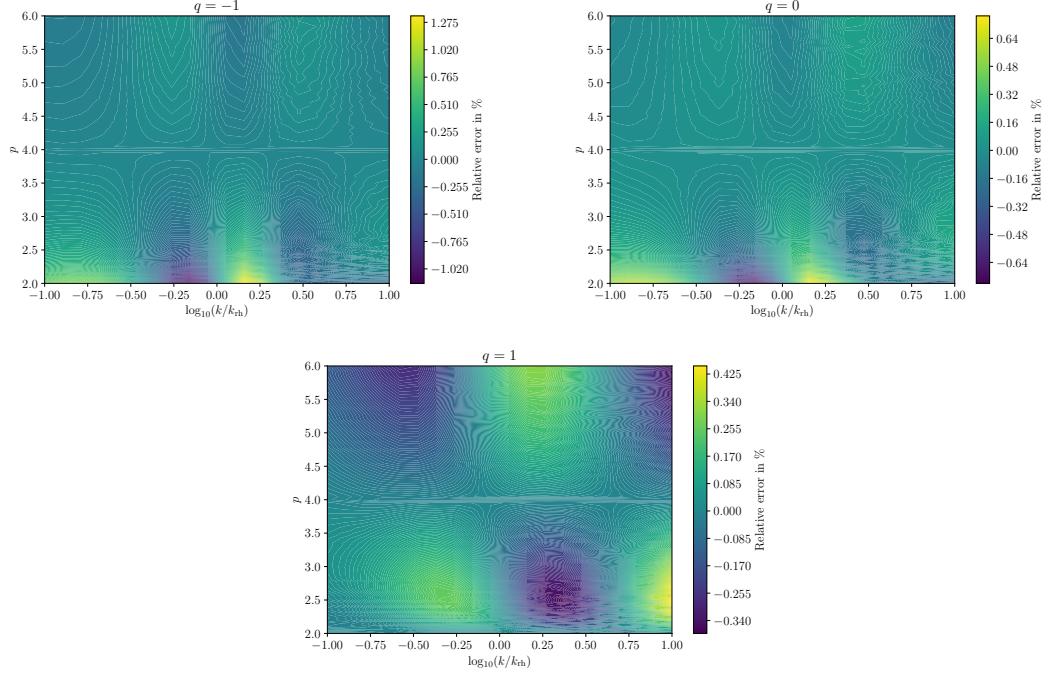


Figure B.1: Relative error of the individual fits to the numerical result of the reheating transfer function. Fits were made for 100 individual p from 2 to 6 for different $q = -1, 0, 1$.

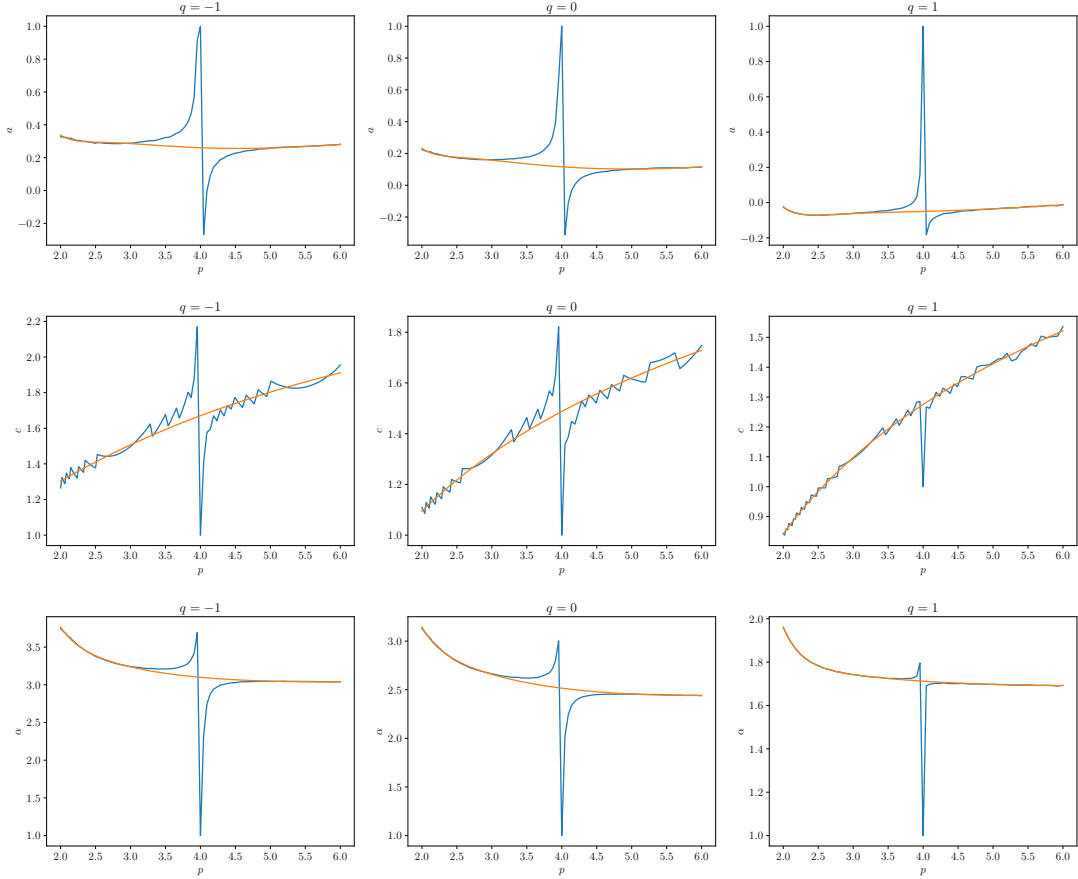


Figure B.2: Values of $a(p, q)$, $c(p, q)$ and $\alpha(p, q)$ dependent on p and q . The blue lines show the individual fit values, while the orange line shows the overall refit of the parameters, excluding the divergence around $p = 4$.

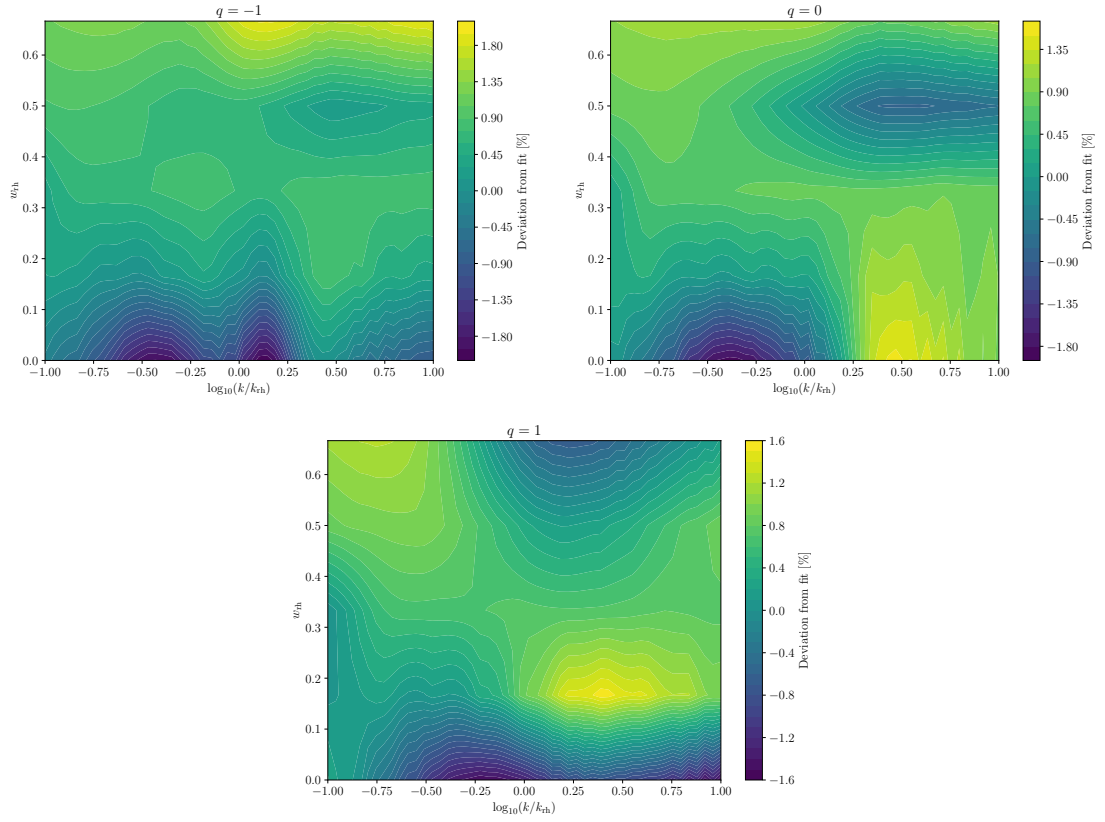


Figure B.3: Relative error of the refitted reheating transfer function compared to the numerical result.

q	Function	Coeff. 0	Coeff. 1	Coeff. 2	Coeff. 3	Coeff. 4
0	$a(p)$	-13.821	68.660	-141.946	106.273	1.081
	$c(p)$	-10.259	0.413	2.171	4.951	5.332
	$\alpha(p)$	-12.960	69.377	-147.180	118.711	3.243
1	$a(p)$	-5.681	28.480	-64.710	54.205	0.397
	$c(p)$	0.969	2.707	1.016	-0.279	-0.571
	$\alpha(p)$	-1.776	14.928	-43.713	45.458	1.729
-1	$a(p)$	-15.113	71.984	-145.331	106.861	1.384
	$c(p)$	-15.875	0.432	1.667	8.595	3.851
	$\alpha(p)$	-13.328	66.886	-135.188	106.636	3.905

Table B.1: Refitted coefficients of the parameter functions $a(p)$, $c(p)$, and $\alpha(p)$ (eqs. B.1–B.3) for $q = -1, 0, 1$. The columns Coeff. 0 ... Coeff. 4 denote the respective fit parameters $d_0, \dots, d_4$ for $a(p)$, $b_0, \dots, b_4$ for $c(p)$, and $c_0, \dots, c_4$ for $\alpha(p)$.

C NG15 tables and diagnostics

C.1 NG15 tables

This chapter shows the tables of the NG15 analysis described in sec. 4.3 including the priors, credible intervals, Bayes estimator, parameter values for maximum posterior and the parameters with the maximum likelihood. If a credible interval boundary cannot be determined, it is denoted with a "...". The Bayes estimator describes the mean value of the posterior with the standard deviation as its uncertainty.

Parameter	Priors	68% CI	95% CI	Bayes estimator	Max posterior	Max likelihood
$\log_{10} A_t^{\max}$	[-6.000,0.000]	[-1.331,...]	[-2.194,...]	-0.998 ± 0.692	0.000	-2.095
$\log_{10}(H_{\text{end}}/\text{GeV})$	[-18.000,-12.000]	[-16.338,-13.806]	[-16.557,...]	-14.942 ± 1.114	-15.675	-16.125
$\log_{10}(T_{\text{rh}}/\text{GeV})$	[-2.500,2.500]	[-0.235,1.065]	[-0.639,1.745]	0.509 ± 0.631	0.620	0.254
n_t	[0.000,6.000]	[1.540,3.859]	[1.302,4.471]	2.746 ± 0.926	1.995	2.676

Table C.1: Priors, credible intervals, Bayes estimator, parameter with maximum posterior value and sample with maximal likelihood for the numerical model with $w_{\text{rh}} = q = 0$.

Parameter	Priors	68% CI	95% CI	Bayes estimator	Max posterior	Max likelihood
$\log_{10} A_t^{\max}$	[-6.000,0.000]	[-2.663,-0.013]	[-3.736,...]	-1.819 ± 1.130	-0.566	-0.220
$\log_{10}(H_{\text{end}}/\text{GeV})$	[-18.000,-12.000]	[-16.821,-13.974]	[-17.258,-12.797]	-15.230 ± 1.248	-16.205	-14.265
$\log_{10}(T_{\text{rh}}/\text{GeV})$	[-2.500,2.500]	[-1.836,0.629]	[...,1.425]	-0.464 ± 1.116	-0.411	1.531
n_t	[0.000,6.000]	[1.161,2.309]	[0.684,3.298]	1.874 ± 0.658	1.675	2.107
w_{rh}	[0.000,0.667]	[0.261,...]	[0.048,...]	0.362 ± 0.184	0.491	0.598

Table C.2: Priors, credible intervals, Bayes estimator, parameter with maximum posterior value and sample with maximal likelihood for the full numerical model with $q = 0$.

Parameter	Priors	68% CI	95% CI	Bayes estimator	Max posterior	Max likelihood
$\log_{10} A_t^{\max}$	[-6.000,0.000]	[-2.467,-0.001]	[-4.086,...]	-1.966 ± 1.185	-1.292	-2.922
$\log_{10}(H_{\text{end}}/\text{GeV})$	[-18.000,-12.000]	[-16.930,-14.144]	[-17.438,-12.756]	-15.266 ± 1.309	-16.236	-16.541
$\log_{10}(T_{\text{rh}}/\text{GeV})$	[-2.500,2.500]	[-2.183,0.276]	[...,1.320]	-0.592 ± 1.126	-1.570	-0.219
n_t	[0.000,6.000]	[1.088,2.185]	[0.745,3.075]	1.773 ± 0.600	1.559	1.710
w_{rh}	[0.000,0.667]	[0.273,...]	[0.060,...]	0.371 ± 0.180	0.664	0.270

Table C.3: Priors, credible intervals, Bayes estimator, parameter with maximum posterior value and sample with maximal likelihood for the full numerical model with $q = -1$.

Parameter	Priors	68% CI	95% CI	Bayes estimator	Max posterior	Max likelihood
$\log_{10} A_t^{\max}$	[-6.000,0.000]	[-2.123,...]	[-3.571,...]	-1.696 ± 1.042	-1.098	-1.489
$\log_{10}(H_{\text{end}}/\text{GeV})$	[-18.000,-12.000]	[-16.408,-13.950]	[-17.044,-12.711]	-15.019 ± 1.173	-15.744	-15.319
$\log_{10}(T_{\text{rh}}/\text{GeV})$	[-2.500,2.500]	[-1.900,0.648]	[...,1.476]	-0.449 ± 1.143	-0.255	0.189
n_t	[0.000,6.000]	[1.166,2.200]	[0.826,3.110]	1.799 ± 0.574	1.676	2.157
w_{rh}	[0.000,0.667]	[0.183,0.632]	[0.103,...]	0.381 ± 0.168	0.467	0.326

Table C.4: Priors, credible intervals, Bayes estimator, parameter with maximum posterior value and sample with maximal likelihood for the full numerical model with $q = 1$.

Parameter	Priors	68% CI	95% CI	Bayes estimator	Max posterior	Max likelihood
$\log_{10} A_t^{\max}$	[-6.000,0.000]	[-1.953,...]	[-3.028,...]	-1.434 ± 0.887	-1.502	-2.485
$\log_{10}(H_{\text{end}}/\text{GeV})$	[-18.000,-12.000]	[-17.134,-15.064]	[-17.520,-13.174]	-15.699 ± 1.186	-16.485	-17.033
$\log_{10}(T_{\text{rh}}/\text{GeV})$	[-2.500,2.500]	[-1.760,0.358]	[...,0.864]	-0.704 ± 0.968	-0.425	-0.761
n_t	[0.000,6.000]	[1.069,2.211]	[0.601,3.041]	1.729 ± 0.616	1.605	1.743
w_{rh}	[0.000,0.667]	[0.297,...]	[0.075,...]	0.385 ± 0.174	0.667	0.429

Table C.5: Priors, credible intervals, Bayes estimator, parameter with maximum posterior value and sample with maximal likelihood for the analytic fit model with $q = 0$.

Parameter	Priors	68% CI	95% CI	Bayes estimator	Max posterior	Max likelihood
$\log_{10} A_t^{\max}$	[-6.000,0.000]	[-1.133,-0.054]	[-1.745,...]	-0.853 ± 0.527	-0.854	-1.369
$\log_{10}(H_{\text{end}}/\text{GeV})$	[-18.000,-12.000]	[-17.053,-15.431]	[-17.544,-14.020]	-15.966 ± 0.926	-16.335	-15.966
$\log_{10}(T_{\text{rh}}/\text{GeV})$	[-2.500,2.500]	[..., -0.161]	[...,1.116]	-0.736 ± 1.107	-2.500	-0.233
n_t	[0.000,6.000]	[1.235,2.475]	[0.751,3.147]	1.912 ± 0.615	1.819	1.230
w_{rh}	[0.000,0.667]	[0.201,0.527]	[0.036,0.594]	0.338 ± 0.152	0.412	0.351

Table C.6: Priors, credible intervals, Bayes estimator, parameter with maximum posterior value and sample with maximal likelihood for the fit model with numeric parameters f_{rh} , f_{end} and constant EOS parameter for radiation $w_R = 1/3$ with $q = 0$.

Parameter	Priors	68% CI	95% CI	Bayes estimator	Max posterior	Max likelihood
$\log_{10} A_t^{\max}$	[-6.000,0.000]	[-1.126,...]	[-1.753,...]	-0.845 ± 0.516	-0.747	-1.416
$\log_{10}(H_{\text{end}}/\text{GeV})$	[-18.000,-12.000]	[-16.969,-15.527]	[-17.479,-14.227]	-15.975 ± 0.883	-16.312	-16.694
$\log_{10}(T_{\text{rh}}/\text{GeV})$	[-2.500,2.500]	[-2.308,0.229]	[...,0.945]	-0.726 ± 1.030	-0.292	-1.157
n_t	[0.000,6.000]	[1.199,2.448]	[0.671,3.079]	1.859 ± 0.622	1.677	1.920
w_{rh}	[0.000,0.667]	[0.228,0.530]	[0.039,0.579]	0.335 ± 0.145	0.356	0.385

Table C.7: Priors, credible intervals, Bayes estimator, parameter with maximum posterior value and sample with maximal likelihood for the fit model with numeric parameters f_{rh} , f_{end} with $q = 0$.

Parameter	Priors	68% CI	95% CI	Bayes estimator	Max posterior	Max likelihood
$\log_{10} A_t^{\max}$	[-6.000,0.000]	[-2.919,...]	[-3.882,...]	-1.973 ± 1.151	-2.226	-3.115
$\log_{10}(H_{\text{end}}/\text{GeV})$	[-18.000,-12.000]	[-16.959,-13.916]	[-17.320,-12.691]	-15.241 ± 1.322	-16.279	-16.682
$\log_{10}(T_{\text{rh}}/\text{GeV})$	[-2.500,2.500]	[-2.000,0.484]	[...,1.361]	-0.526 ± 1.119	-0.449	-2.469
n_t	[0.000,6.000]	[1.178,2.233]	[0.711,2.949]	1.788 ± 0.567	1.662	1.724
w_{rh}	[0.000,0.667]	[0.306,...]	[0.074,...]	0.389 ± 0.173	0.511	0.348

Table C.8: Priors, credible intervals, Bayes estimator, parameter with maximum posterior value and sample with maximal likelihood for the numerical model with constant EOS parameter for radiation $w_R = 1/3$ with $q = 0$.

Parameter	Priors	68% CI	95% CI	Bayes estimator	Max posterior	Max likelihood
$\log_{10} A_t^{\max}$	[-6.000,0.000]	[-2.483,-0.070]	[-3.876,...]	-1.894 ± 1.139	-1.752	-0.330
$\log_{10}(H_{\text{end}}/\text{GeV})$	[-18.000,-12.000]	[-16.910,-14.358]	[-17.247,-12.511]	-15.246 ± 1.296	-16.122	-13.796
$\log_{10}(T_{\text{rh}}/\text{GeV})$	[-2.500,2.500]	[-2.150,0.274]	[...,1.413]	-0.597 ± 1.136	-1.237	-0.260
n_t	[0.000,6.000]	[1.106,2.297]	[0.762,3.245]	1.856 ± 0.650	1.434	1.996
w_{rh}	[0.000,0.667]	[0.282,...]	[0.068,...]	0.377 ± 0.177	0.520	0.203

Table C.9: Priors, credible intervals, Bayes estimator, parameter with maximum posterior value and sample with maximal likelihood for the backreaction model with $q = 0$.

Parameter	Priors	68% CI	95% CI	Bayes estimator	Max posterior	Max likelihood
$\log_{10} A_t^{\max}$	[-6.000,0.000]	[-3.075,-0.887]	[-3.836,...]	-2.056 ± 1.033	-2.100	-1.356
$\log_{10}(H_{\text{end}}/\text{GeV})$	[-18.000,-12.000]	[-16.794,-15.258]	[-17.367,-14.190]	-15.849 ± 0.834	-16.280	-16.001
$\log_{10}(T_{\text{rh}}/\text{GeV})$	[-2.500,2.500]	[-1.536,0.769]	[...,0.987]	-0.587 ± 1.037	0.460	-0.375
n_t	[0.000,6.000]	[1.368,2.662]	[0.896,3.470]	2.139 ± 0.670	1.952	3.163
w_{rh}	[0.000,0.667]	[0.112,0.523]	[0.048,...]	0.333 ± 0.181	0.214	0.143

Table C.10: Priors, credible intervals, Bayes estimator, parameter with maximum posterior value and sample with maximal likelihood for the constrained full numerical model with $q = 0$.

Parameter	Priors	68% CI	95% CI	Bayes estimator	Max posterior	Max likelihood
$\log_{10} A_t^{\max}$	[-6.000,0.000]	[-3.048,-0.870]	[-3.829,...]	-2.041 ± 1.032	-2.066	-1.594
$\log_{10}(H_{\text{end}}/\text{GeV})$	[-18.000,-12.000]	[-16.767,-15.239]	[-17.339,-14.164]	-15.841 ± 0.832	-16.193	-16.105
$\log_{10}(T_{\text{rh}}/\text{GeV})$	[-2.500,2.500]	[-1.658,0.804]	[...,0.990]	-0.584 ± 1.040	0.413	-0.403
n_t	[0.000,6.000]	[1.358,2.654]	[0.891,3.459]	2.132 ± 0.668	1.939	2.958
w_{rh}	[0.000,0.667]	[0.102,0.531]	[0.048,...]	0.332 ± 0.181	0.245	0.181

Table C.11: Priors, credible intervals, Bayes estimator, parameter with maximum posterior value and sample with maximal likelihood for the constrained backreaction model with $q = 0$.

C.2 Convergence diagnostics

The convergence of the MCMC chains is assessed using the split- $\hat{R}$ diagnostic [42, 43]. It uses an estimation of the variance of the target distribution and compares it to the within-chain variance. For each parameter, every chain is split into two halves after a burn-in fraction of 25% is removed, yielding M sub-chains of length N . Denoting the sample mean of sub-chain m by $\bar{\theta}_m$ and the overall mean by $\bar{\theta}$, the between-chain variance is computed as

$$B = \frac{N}{M-1} \sum_{m=1}^M (\bar{\theta}_m - \bar{\theta})^2, \quad (\text{C.1})$$

and the within-chain variance is given by

$$W = \frac{1}{M} \sum_{m=1}^M s_m^2, \quad s_m^2 = \frac{1}{N-1} \sum_{n=1}^N (\theta_{m,n} - \bar{\theta}_m)^2. \quad (\text{C.2})$$

The marginal posterior variance is then estimated as a weighted combination of W and B ,

$$\widehat{\text{Var}}^+(\theta) = \frac{N-1}{N} W + \frac{1}{N} B, \quad (\text{C.3})$$

from which the potential scale reduction factor is obtained as

$$\hat{R} = \sqrt{\frac{\widehat{\text{Var}}^+(\theta)}{W}}. \quad (\text{C.4})$$

By splitting each chain in half before $\hat{R}$ is computed, the diagnostic is made sensitive not only to disagreement between independently initialized chains, but also to non-stationarity within a single chain, so that a stricter test of convergence is obtained than with the original formulation. A value of $\hat{R} - 1$ close to zero indicates that the between-chain and within-chain variances are in agreement, i.e., that the chains have converged to a common target distribution and are well mixed. Following common practice in the PTA literature, a weak convergence threshold of $\hat{R} - 1 < 10^{-1}$ and a strong convergence threshold of $\hat{R} - 1 < 10^{-2}$ are adopted. This means that a parameter is fully converged if it is below the strong threshold and marginally converged if it is above the strong threshold and below the weak threshold. Parameters which are above the weak threshold are considered as not converged.

In the following figures, the histograms of the $\hat{R}$ values of the red noise pulsar parameters for **enterprise** and the $\hat{R}$ values of the model parameters of the different analyses discussed in sec. 4.3 are shown. For the **ceffy1** runs, because the parameters are fitted to the free spectrum, only the $\hat{R}$ values of the model parameters are shown.

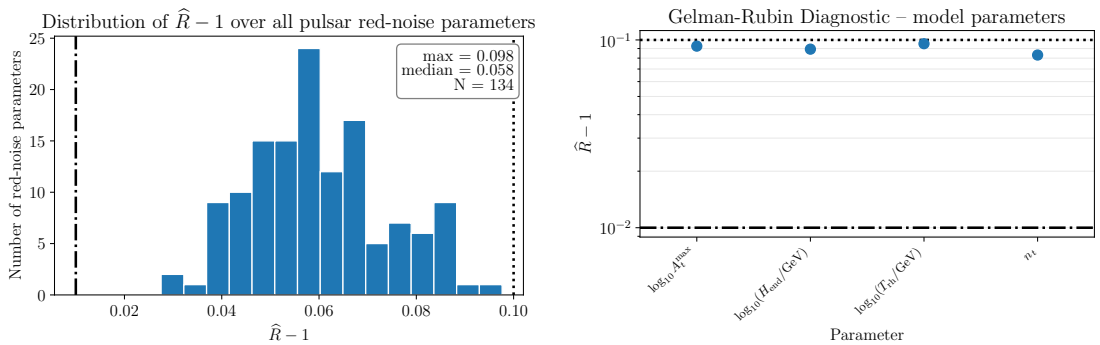


Figure C.1: $\hat{R} - 1$ values of the full numerical model without backreaction in the standard case $w_{\text{rh}} = q = 0$.

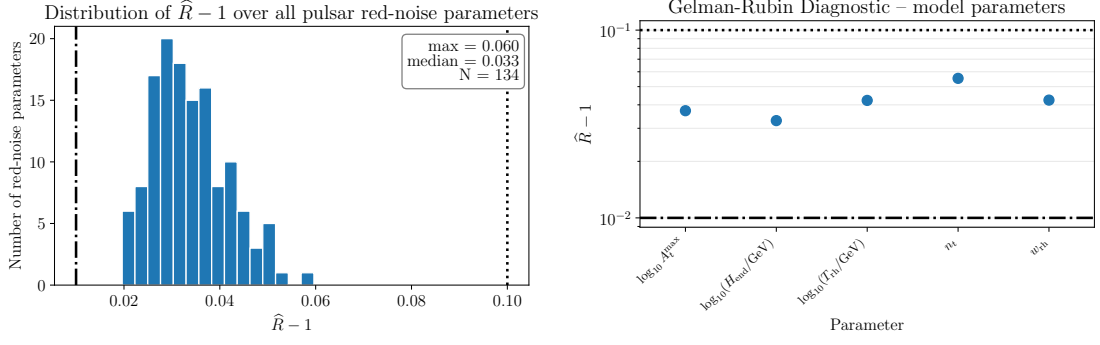


Figure C.2: $\hat{R} - 1$ values of the full numerical model without backreaction with $q = 0$.

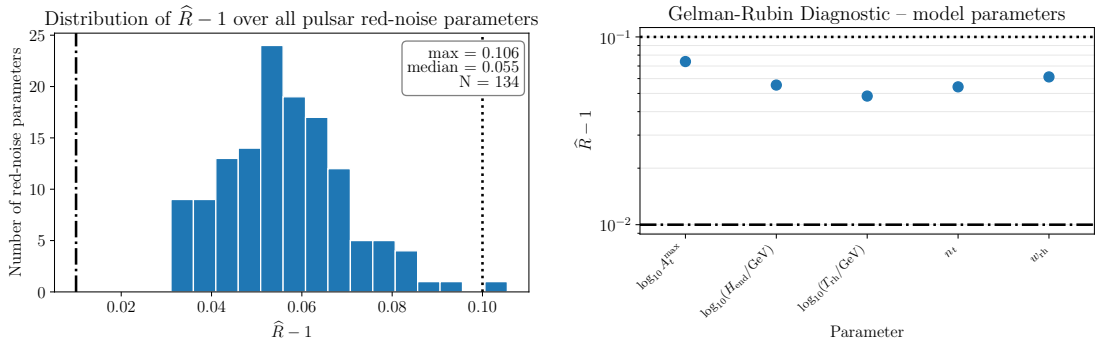


Figure C.3: $\hat{R} - 1$ values of the full numerical model without backreaction with $q = -1$.

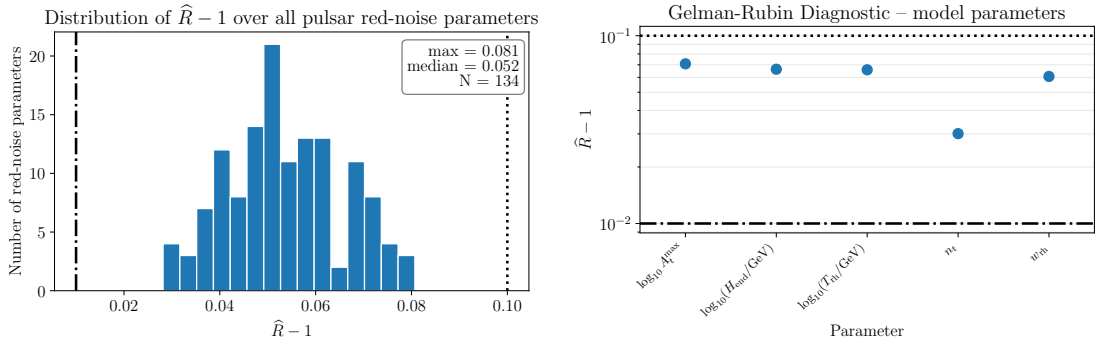


Figure C.4: $\hat{R} - 1$ values of the full numerical model without backreaction with $q = 1$.

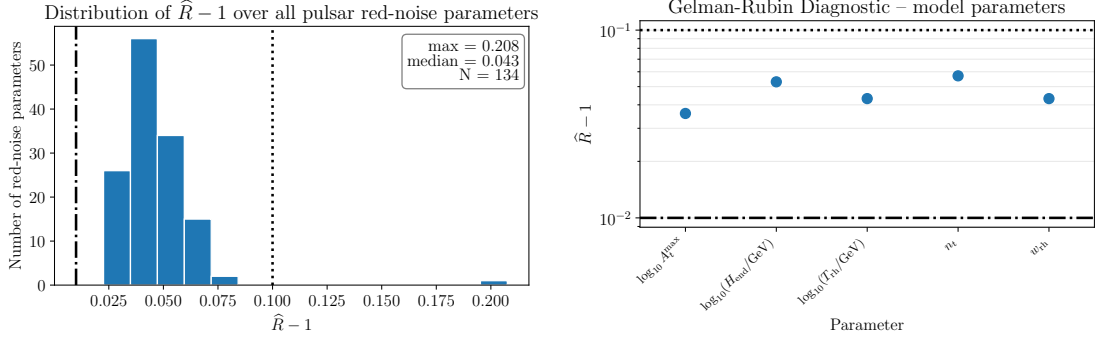


Figure C.5: $\hat{R} - 1$ values of the analytic fit model with $q = 0$.

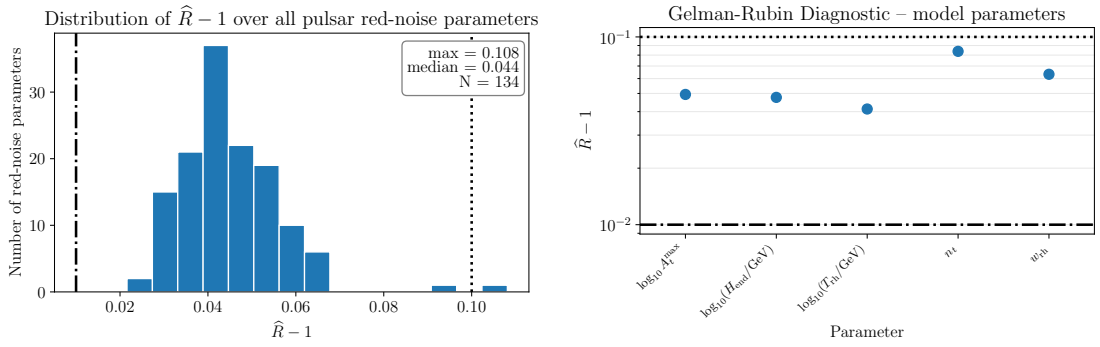


Figure C.6: $\hat{R} - 1$ values of the fit model with numeric parameters f_{rh} , f_{end} and constant EOS parameter for radiation $w_R = 1/3$ with $q = 0$.

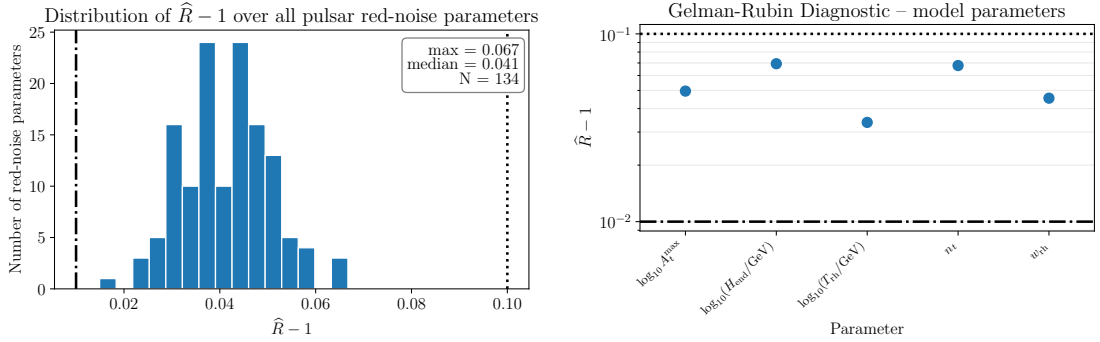


Figure C.7: $\hat{R} - 1$ values of the fit model with numeric parameters f_{rh} , f_{end} with $q = 0$.

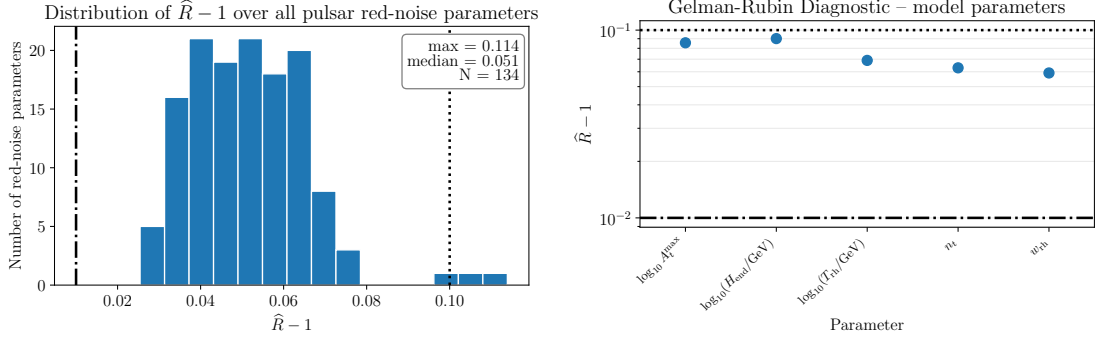


Figure C.8: $\hat{R} - 1$ values of the numerical model with constant EOS parameter for radiation $w_R = 1/3$ with $q = 0$.

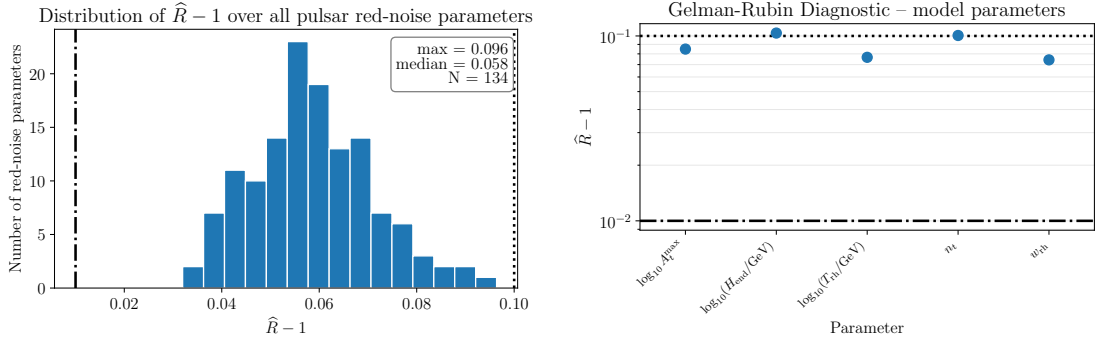


Figure C.9: $\hat{R} - 1$ values of the backreaction model with $q = 0$.

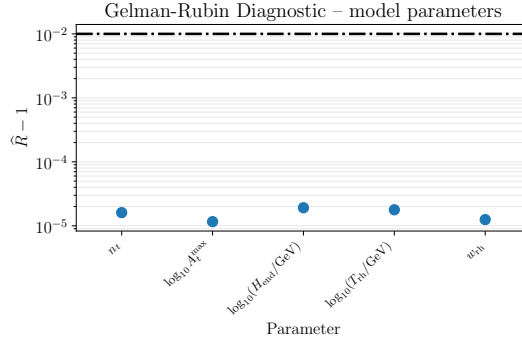


Figure C.10: $\hat{R} - 1$ values of the constrained full numerical model without backreaction with $q = 0$. Because the run was performed with `ceffy1`, only the model parameters are available.

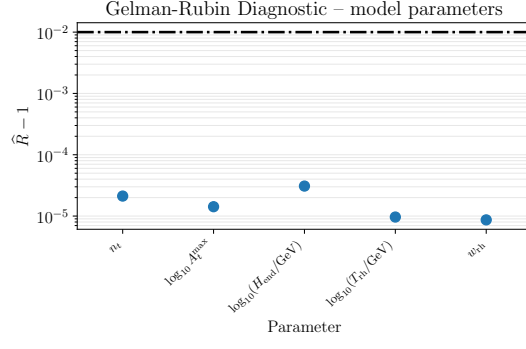


Figure C.11: $\hat{R} - 1$ values of the constrained backreaction model with $q = 0$. Because the run was performed with `ceffy1`, only the model parameters are available.

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