



UNIVERSITÀ DEGLI STUDI DI PADOVA

Dipartimento di Fisica e Astronomia “Galileo Galilei”

MASTER DEGREE IN PHYSICS

Final Dissertation

Comparison of different computations of backreaction in axion inflation

Thesis supervisor

Prof. Marco Peloso

Thesis co-supervisor

Prof. Kai Schmitz

Candidate

Paolo Pianizzola

Academic Year 2025/2026

Abstract

Axion inflation coupled to a gauge field can give rise to a rich phenomenology: the motion of the axion can strongly amplify the gauge field, which in turn can produce observable density perturbations and gravitational waves. In the observationally interesting cases, the amplification of the gauge field is so strong that it significantly backreacts on the motion of the axion. Obtaining the precise evaluation of this coupled system, which is necessary for accurate phenomenological predictions, is still an open problem, which has been addressed with different analytical and numerical techniques in the literature. Two widely used schemes are: (1) a numerical computation of the evolution of the Fourier modes of the gauge field, coupled to an axion field approximated to be exactly homogeneous axion, and (2) the reformulation of the equations of motion in position space in terms of a tower of equations for two-point correlators of the gauge field with an increasing number of spatial derivatives acting on them. Although these two methods are formally equivalent, their implementations are based on different approximations and truncations. Works that have employed these two methods appear to be in good qualitative agreement, but a detailed quantitative comparison between them is still lacking. The thesis has the goal to perform such a comparison, to assess the accuracy and the performance of the two different implementations.

Contents

1	Introduction	1
2	A brief introduction to inflation theory	3
2.1	The Hot Big Bang model	3
2.1.1	Historical introduction	3
2.1.2	Mathematical description of our universe within General Relativity	4
2.2	The horizon and flatness problems	6
2.2.1	Horizon problem	6
2.2.2	Flatness problem	7
2.2.3	Solution by accelerated expansion	8
2.3	Single-field slow-roll inflation	9
2.3.1	de Sitter spacetime	10
2.4	Cosmological perturbations	12
2.4.1	Gauge choice	12
2.4.2	SVT decomposition	12
2.4.3	Inflaton and metric perturbations	12
2.4.4	Gauge choices and gauge-invariant variables	13
2.4.5	Statistics of cosmological perturbations	14
2.5	Quantum fluctuations in de Sitter spacetime	15
2.6	Quantum fluctuations during slow-roll	17
2.6.1	Scalar perturbations	17
2.6.2	Tensor perturbations	18
2.6.3	The energy scale of inflation	18
2.6.4	Primordial spectra during slow-roll	18
2.7	Models for inflation	19
2.7.1	Small-field inflation	19
2.7.2	Large-field inflation	19
2.7.3	Other possible extension of slow-roll	20
3	Axion inflation with U(1) gauge fields	21
3.1	The model	23
3.1.1	Derivation of the classical equations of motion	23
3.1.2	Derivation of the equations of motion in the homogeneous approximation	24
3.2	Anber-Sorbo solution	30
3.3	Gravitational waves production	32
3.3.1	Green functions	36
4	Numerical techniques in axion inflation	39
4.1	Mode-by-mode approach and <i>BackHom</i>	40
4.1.1	Rescaling of the variables	40
4.1.2	Equations of motion in the code	40
4.1.3	Initial conditions	41

4.1.4	Definition of the grid in momentum space	41
4.1.5	The cutoffs	42
4.1.6	Comments on the computational strategy	44
4.2	Gradient expansion formalism and GEF	45
4.2.1	The boundary terms	47
4.2.2	Details on the numerical implementation of the boundary terms	49
4.2.3	Truncation of the chain of equations	50
4.2.4	Details on the code implementation	51
5	<i>BackHom</i> vs GEF: the comparison	55
5.1	Benchmark scenario no. 1	55
5.1.1	Phenomenological remarks on the first benchmark scenario	56
5.1.2	The comparison	59
5.2	Benchmark scenario no. 2	65
5.2.1	The sections of the potential	65
5.2.2	Parameter choice	67
5.2.3	Phenomenological remarks on the second benchmark scenario	67
5.2.4	The comparison	69
5.3	Comments on the run time	72
5.4	GEF and its comparison with lattice simulations	73
6	Numerical exploration of the Anber-Sorbo stability region	77
6.1	The stability analysis	77
6.1.1	The toy model	77
6.1.2	The linearized gradient expansion formalism	79
6.1.3	Stability in the strong backreaction regime	81
6.2	Numerical exploration of the stability region	84
6.2.1	Results	85
7	Conclusions	93

Notation and conventions

In the present work we adopt the following notations and conventions:

- Natural Units: $c = \hbar = k_B = 1$;
- Reduced Planck mass: $M_p := (8\pi G)^{-1/2}$, where G is Newton's constant;
- Mostly plus metric signature: $(- + + +)$;
- Greek indices $(\mu, \nu, \dots)$ run from 0 to 3, while Latin ones $(i, j, \dots)$ from 1 to 3;
- Einstein summation convention by which repeated indices are implicitly summed over;
- The subscript 0 denotes evaluation at the present cosmic time t_0 , e.g. $H_0 = H(t_0)$;
- Differentiation with respect to cosmic time t is denoted with a dot, e.g. $\dot{\rho} := \partial\rho/\partial t$, while differentiation with respect to conformal time τ is denoted with a prime, e.g. $\rho' := \partial\rho/\partial\tau$. If the dependence of a quantity is stated, then the prime indicates differentiation with respect to its argument, e.g. $V'(\phi) := \partial V/\partial\phi$;
- The Fourier transform and its inverse are defined as

$$g(t, \mathbf{x}) := \int \frac{d^3k}{(2\pi)^{3/2}} e^{i\mathbf{k}\cdot\mathbf{x}} \tilde{g}(t, \mathbf{k}) \quad \tilde{g}(t, \mathbf{k}) := \int \frac{d^3x}{(2\pi)^{3/2}} e^{-i\mathbf{k}\cdot\mathbf{x}} g(t, \mathbf{x}) ;$$

- The expression “h.c.” stands for the Hermitian conjugate of the preceding term.
- The expressions “R.H.S.” and “L.H.S.” mean *right hand side* and *left hand side*, respectively, and are used to reference what is written to the right or left of the equal sign, =, in an equation.

Chapter 1

Introduction

A primordial epoch of accelerated expansion [1] provides a solution to the issue of initial conditions present in Standard Big Bang cosmology, namely the horizon and flatness problems, and large-scale measurements [2, 3] are in strong agreement with this paradigm. The so-called *inflation theory*, in addition to predicting a homogeneous and isotropic universe, offers a framework for producing primordial perturbations, that were measured for the first time by COBE [4, 5].

The single-field slow-roll mechanism is a well-known system able to induce an inflationary epoch. In this framework, accelerated expansion is realized by a scalar field ϕ , called the inflaton, rolling on an almost flat potential $V(\phi)$ in such a way that the content of the universe is effectively described by a cosmological constant inducing inflation. This mechanism is just a low-energy description and a complete particle physics realization of this era is still missing.

The full UV theory should be able to generate this primordial epoch of accelerated expansion free of any fine tuning, but the slow-roll regime actually relies on the smallness of two parameters ε , $|\eta| \ll 1$, that can be related, respectively, to the first and second derivative of the inflaton potential. Therefore, the inclusion of irrelevant operators of the form $\delta V \propto \phi^{4+n}/M^n$ in the low-energy effective description can easily spoil the flatness of the potential leading to ε , $\eta \approx \mathcal{O}(1)$, unless $M \gg M_p$ which is unnatural.

In the proposal of *natural inflation* [6], this problem is addressed by considering an approximate shift symmetry $\phi \rightarrow \phi + \text{const}$, preserving the smallness of the slow-roll parameters ε , η from large quantum corrections. As is the case for axion-like particles (ALP), the continuous shift symmetry breaks down to a discrete one $\phi \rightarrow \phi + 2\pi f$, where f is the axion decay constant, because of instantons or non-perturbative effects in string theories. The potential used in [6] is $V = \Lambda^4[1 - \cos(\frac{\phi}{f})]$, but it is excluded by observations [2] if we respect the theoretical bound $f < M_p$, which is imposed to neglect quantum gravity effects and from limits coming from string theory [7]. If the value of f is too small, the slope of the potential steepens, since $V'(\phi)/V(\phi) \propto 1/f$, and, therefore, the slow-roll regime would not last long enough.

Ref. [8] offers a way to still sustain inflation on a steep potential by considering a theory in which the axion-inflaton is coupled to $U(1)$ gauge fields via a Chern–Simons-type interaction term. In this scenario, the gauge field backreacts onto the inflaton dynamics inducing a friction term that slows down the inflaton, therefore allowing for a similar version of single-field slow-roll but on steeper slopes. Theories involving this type of coupling with $U(1)$ gauge fields are investigated in the literature via numerical and analytical techniques [9–11]. The two widely used schemes we analyze and compare in this thesis are the *mode-by-mode* approach (MbM), in which a one dimensional grid in momentum space is considered and a discrete number of gauge modes $A(k)$ is evolved, and the *gradient expansion formalism* (GEF), in which a tower of equations for two point functions in the “electric” and “magnetic” fields, e.g. $\langle \mathbf{E} \cdot [(\nabla \times)^n \mathbf{B}] \rangle$, with increasing insertions of spatial derivatives is employed. These two techniques are based on the approximation of *homogeneous backreaction*, in which the axion has no dependence on the spatial coordinates, i.e. $\phi = \phi(t)$. Hence, the discussion in this thesis is valid in the limit of small axion gradients, which is not guaranteed to persist for all models and values of

the parameters.

We remark that in [8] an analytical solution to the gauge field mode equation is examined within the approximations of constant Hubble rate H and constant inflaton velocity $\dot{\phi}$. Numerical studies [12–14] show that the inflaton velocity does not evolve in a slow adiabatic manner, but rather oscillations take place and Ref. [15] proves analytically the appearance of these oscillations. In the recent study of [16], the existence of a region in parameter space where it is possible to adiabatically realize this analytical solution is suggested. This region of parameter space is explored via both the MbM approach and the GEF, giving further insight in the relation between the two techniques.

Chapter 2 focuses on the inflationary paradigm introducing the problems of Standard Cosmology, the single-field slow-roll mechanism and how primordial perturbations are featured within this framework. In Chapter 3, we discuss in detail the model of axion inflation with $U(1)$ gauge fields deriving the equations of motion, reviewing the results of [8] and giving an overview of the production of gravitational waves in this scenario. Chapter 4 is dedicated to presenting the way the MbM approach is numerically implemented in a code named *BackHom* [10] and how the GEF is realized in GEFF, its code implementation. In Chapter 5, we compare these two codes in two benchmark scenarios commenting on how the output changes slightly depending on the level of approximation with which the simulations are set. In Chapter 6, we explore the region of parameter space in which the analytical solution proposed in [8] can be realized and we bring forward the analysis employing both codes in order to further study the differences in the two approaches in this peculiar regime.

Chapter 2

A brief introduction to inflation theory

2.1 The Hot Big Bang model

2.1.1 Historical introduction

In 1913 American astronomer Vesto Slipher first measured the radial velocity of galaxies with respect to us [17]. He noticed that light coming from them is redshifted, implying that distant galaxies are receding from the Earth. This result was first interpreted in 1922 by Soviet physicist Alexander Friedmann as the expansion of the universe [18,19]: according to the Copernican principle, by which we do not occupy a special position in the sky, the observation of distant galaxies receding from us is not tied to our specific location in the sky, but it could be done by any other observer living in other galaxies, hence two points in the sky are always departing one from the other, on large scales of at least 100 Mpc.

This experimental result gained solid recognition in 1929 thanks to the work of Edwin Hubble, American astronomer to whom is dedicated the well-known *Hubble law*, relating linearly the recessional velocity of galaxies and their distance [20]. On the theoretical side instead, in 1927, Georges Lemaître, a Belgian physicist, astronomer and priest, arrived independently at Friedmann's result and in the framework of Einstein's General Relativity they described the universe with the following metric [21]:

$$ds_{FLRW}^2 = -dt^2 + a^2(t) \left[\frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \right] \quad (2.1)$$

where $a(t)$ is called the scale factor and k is the curvature parameter. For $k = -1$ the universe is said to be open or hyperbolic, for $k = 0$ it is flat and for $k = +1$ it is closed or spherical. If the scale factor $a(t)$ was not a scalar, but a vector, we would clearly lose isotropy, while if it had a dependence on the spatial coordinates, the metric would not be homogeneous anymore.

In 1935 Howard P. Robertson and Arthur Geoffrey Walker, respectively an American and British mathematical physicists, managed to prove that this is the only available choice for a metric to describe a spatially homogeneous and isotropic spacetime, hence it is now known as the Friedmann-Lemaître-Robertson-Walker (FLRW) metric [22-25].

The claim of homogeneity, isotropy and flatness of our universe is strongly supported by the observation of the Cosmic Microwave Background (CMB). The CMB is radiation characterized by a thermal black body spectrum filling up our universe and it was generated when the universe transitioned from opaque to transparent during the recombination epoch, roughly 380 000 years from the Big Bang. Before recombination, the universe was filled up with a hot plasma and then, after cooling down to a temperature such that neutral atoms could form, photons could no longer undergo Thomson scattering with free electrons so that, from this moment on, this thermal radiation has been traveling to us in a

more or less unimpeded manner, giving us a picture of the universe at that moment of the evolution of the universe.

The first detection of the CMB happened accidentally in 1964 by the two astronomers, Arno Penzias and Robert Woodrow Wilson, who built a radiometer for radio astronomy and satellite communication experiments [26]. They were awarded the Nobel prize in 1978. The NASA COsmic Background Explorer (COBE) satellite was launched in 1989 and it was able to measure for the first time the anisotropies of the CMB [4, 5]. They found that the CMB spectrum is characterized by a black body silhouette with temperature $T \simeq 2.73\text{K}$ and the deviations from it are very small, $\frac{\delta T}{T} \approx 10^{-5}$. Since the CMB is characterized by the same temperature and the same spectrum no matter where the telescope was pointing, this experiment is stringent proof of the isotropy of our observable universe, whereas we deduce homogeneity from the Copernican principle. A Nobel prize was awarded for this mission in 2006. After that, one of the best experimental probes of the cosmological parameters characterizing the ΛCDM model, which is the model describing at the best our universe, is given by ESA PLANCK satellite that operated from 2009 to 2013 [27].

2.1.2 Mathematical description of our universe within General Relativity

The mathematical description of our universe is built upon three pillars [28, 29]:

1. Cosmological principle: we do not live in a privileged location in the universe (Copernican principle) and on large scales the observable universe is spatially homogeneous and isotropic, hence we can choose the FLRW metric to describe it;
2. Weyl postulate: the world lines of particles meet only at a singular point in the finite or infinite past, meaning that there is a unique world line going through each point in the spacetime and we can describe the matter content of the universe as a perfect fluid;
3. General relativity: via the Einstein equations we can relate the evolution of the spacetime to its matter content.

General Relativity is the geometric theory of gravity characterizing the dynamics of the metric $g_{\mu\nu}$ in the vacuum through the *Einstein-Hilbert action*:

$$S_{EH} := \int d^4x \sqrt{-g} \frac{M_p^2}{2} R \quad (2.2)$$

where $M_p := (8\pi G)^{-1/2}$ is the reduced Planck mass, G the Newton constant and R the scalar curvature. Varying S_{EH} with respect to the metric, we get the *Einstein tensor*:

$$G_{\mu\nu} := \frac{2}{M_p^2 \sqrt{-g}} \frac{\delta S_{EH}}{\delta g^{\mu\nu}} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \quad (2.3)$$

where $R_{\mu\nu}$ is the Ricci tensor. The stress-energy tensor of a perfect fluid is given by

$$T_{\mu\nu} := \frac{-2}{\sqrt{-g}} \frac{\delta S_m}{\delta g^{\mu\nu}} = \rho u_\mu u_\nu + p \gamma_{\mu\nu} \quad (2.4)$$

where S_m is the action of the matter fields, ρ and p are the energy density and pressure in the fluid rest frame, respectively, u_μ the 4-velocity and $\gamma_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu$ is the projector onto 3-dimensional hypersurfaces orthogonal to u_μ . Through the variation of the full action $S = S_{EH} + S_m$, we deduce the Einstein equations:

$$G_{\mu\nu} = 8\pi G T_{\mu\nu} \quad (2.5)$$

relating the tensors (2.4) and (2.3).

For a perfect fluid in the geometry (2.4), the Einstein equations have only two nontrivial components, that can be rewritten as

$$H^2 := \left(\frac{\dot{a}}{a}\right)^2 = \frac{\rho}{3M_p^2} - \frac{k}{a^2} \quad (2.6)$$

$$\frac{\ddot{a}}{a} = -\frac{1}{6M_p^2}(\rho + 3p) \quad (2.7)$$

where H is called the Hubble rate and we will refer to eq. (2.6) as the *Friedmann equation*. From now on (with the exception of Section 2.2.2) we set the curvature parameter k to 0, assuming spatial flatness.

By exploiting the Bianchi identity,

$$\nabla^\mu G_{\mu\nu} = 0 \stackrel{(2.5)}{=} \nabla^\mu T_{\mu\nu} = 0, \quad (2.8)$$

where ∇_μ is the covariant derivative, we can find a third equation:

$$\dot{\rho} + 3H(\rho + p) = 0, \quad (2.9)$$

to which we will refer to as the *continuity equation*. Since the continuity equation follows from Bianchi identity ($\nabla^\mu G_{\mu\nu} = 0$) and the Einstein equations, it is not an independent equation, and we have an undetermined system of three unknowns (the energy density, the pressure, and the scale factor) governed by two equations. We close the system by assuming that the fluid is barotropic, with a linear relation between the energy density and the pressure,

$$p = w \rho, \quad (2.10)$$

where the constant w is called the *equation of state* of the fluid. This assumption is motivated by the fact that non-relativistic matter and radiation in the universe are well modeled by a barotropic fluid of equation of state $w = 0$ and $w = 1/3$, respectively. Another case of interest for the description of our universe is that of a vacuum energy, that behaves as a barotropic fluid with equation of state $w = -1$.

By solving the continuity equation, (2.9), we can relate the energy density of the universe to the scale factor:

$$\rho(t) = \rho_* \left(\frac{a(t)}{a_*}\right)^{-3(1+w)} \quad \text{for } w \neq -1 \quad (2.11)$$

$$\rho(t) = \text{const} \quad \text{for } w = -1$$

where the quantities labeled with the star are evaluated at a generic reference time t_* . Thanks to the Friedmann equation (2.6) we can write in general the scale factor as a function of time:

$$\begin{aligned} a(t) &= a_* \left[\frac{3(1+w)H_*}{2} t \right]^{\frac{2}{3(1+w)}} \quad \text{for } w \neq -1 \\ a(t) &= a_* e^{H_*(t-t_*)} \quad \text{for } w = -1 \end{aligned} \quad (2.12)$$

where $H_*^2 = \frac{\rho_*}{3M_p^2}$ and is constant in the second case. To derive the relation in the first row, the reference time has been chosen so that $a = 0$ at $t = 0$. Since for $k = 0$ the scale factor enters eqs. (2.6) and (2.7) only as a ratio with its time derivatives, the constant factor a_* in the formulae above gets always simplified. If $a(t)$ is a solution of the Einstein equations, then also $C a(t)$ solves them, where

C is some real constant. Therefore, the normalization of the scale factor, i.e. the specific value of a_* , is arbitrary and unphysical.

Observations are consistent with a universe that evolves from the Big Bang singularity located at a finite time in the past, when the scale factor vanishes. In this framework we distinguish three main epochs: first an epoch of domination of radiation (RD), with $w = 1/3$, lasting until about 50 000 years from the Big Bang, a matter dominated one (MD), with $w = 0$, ending at around 9.8 billion years and finally a dark energy era (ΛD), with $w = -1$, in which the Cosmological constant is dominating, lasting until today at about 14 billion years. The main dependencies of our variables are summarized in the following table.

RD	$w = 1/3$	$\rho \propto a^{-4}$	$a \propto t^{\frac{1}{2}}$	$H \propto t^{-1}$
MD	$w = 0$	$\rho \propto a^{-3}$	$a \propto t^{\frac{2}{3}}$	$H \propto t^{-1}$
ΛD	$w = -1$	$\rho = const$	$a \propto e^{Ht}$	$H = const$

We can relate the w parameter to the acceleration of the expansion of the universe via eq. (2.7):

$$\frac{\ddot{a}}{a} = -\frac{1}{6M_p^2}(1 + 3w) \cdot \rho. \quad (2.13)$$

This equation is valid only for a single species, while for more species w is replaced by

$$w_{eff} = \frac{\sum_i p_i}{\sum_i \rho_i} \quad (2.14)$$

where p_i and ρ_i are the pressure and energy density of the various components of the universe. Hence, we notice that an accelerated expansion requires $w < -1/3$, which is not achieved if the universe is dominated by non-relativistic matter or radiation.

For the majority of its lifetime the universe has been undergoing decelerated expansion. Two international collaborations, the High- z Supernova Search Team and the Supernova Cosmology Project, discovered that we are living in a ΛD epoch by analyzing Type Ia supernovae in 1998 and 1999 [30,31]. Adam Riess, Brian P. Schmidt and Saul Perlmutter were awarded the Nobel prize in physics for this discovery in 2011.

2.2 The horizon and flatness problems

The model we just described, commonly denoted as the *standard hot big-bang model* is affected by two complications: the horizon and flatness problems, that were first pointed out respectively by Wolfgang Rindler [32] and Robert Dicke [33]. We will follow the exposition of these issues and their solutions in the remainder of the chapter as in [34,35].

2.2.1 Horizon problem

Let us introduce now the notion of horizon. Considering a null radial trajectory in our coordinate system, we can define the particle horizon, i.e. the maximum distance that can separate two causally connected events:

$$R_h(t) := a(t) \int_{t_i}^t dr(t') = a(t) \int_{t_i}^t \frac{1}{a(t')} dt' \quad (2.15)$$

where t_i is the time at which the universe began and for a universe with $w > -1$ we can always set $t_i = 0$. We highlight the difference between comoving, r , and physical distance, R . The quantity R_h is the maximum length that can separate two causally related events.

In a universe made of standard matter, without a cosmological constant, we can find the following expression:

$$R_h(t) = \frac{3(1+w)}{1+3w} \cdot t = \frac{2}{1+3w} H^{-1} \quad \text{with} \quad w > -\frac{1}{3} \quad (2.16)$$

where in the last equality $H = 2/[2(1+w)t]$ has been used. If we define the Hubble radius $r_H := H^{-1}$, we see that if we choose the integration boundaries of eq. (2.15) to be within a specific domination epoch, $R_h \propto r_H$. Hence, r_H is a quantity that up to $\mathcal{O}(1)$ factors approximates well the particle horizon locally, within an era, but has no memory of what happened before. Intuitively, by momentarily reintroducing the speed of light c , the Hubble time $t_H := r_H/c$ represents the time in which the universe roughly doubles its size.

Physical length scales grow proportionally to $a(t)$, but the cosmological horizon of a universe dominated by non-relativistic matter or radiation increases faster than that. Instead, the cosmological horizon remains constant for $w = -1$.

$$RD \implies R_h \propto t \propto a^2 \propto H^{-1}$$

$$MD \implies R_h \propto t \propto a^{\frac{3}{2}} \propto H^{-1}$$

$$\Lambda D \implies R_h = H^{-1} = \text{const}$$

This is not compatible with the idea of a homogeneous and isotropic universe, i.e. a system which was able to thermalize, since with the passing of time the horizon uncovers regions of our universe that were not in causal contact with one another, but despite this they still look in perfect thermal equilibrium. To be more quantitative, we can give an estimate of the angle that patches of thermalized radiation could cover on the CMB. By neglecting the first RD and the last ΛD eras which are significantly shorter than the MD epoch, we can estimate the angular size of the patches in the following way¹:

$$\Delta\theta \approx \frac{2 \times \text{horizon at rec.} \times \text{redshift}}{\text{horizon now}} = 2 \times \frac{2/H_{rec} \times (1+z_{rec})}{2/H_0} \approx 2(1+z_{rec})^{-1/2} \approx 3^\circ \quad (2.17)$$

where z is the redshift defined as $z(t) := \frac{a_0}{a(t)} - 1$ and we considered $z_{rec} = 1100$; but instead of 3° , as stated before, we measure that the entirety of the CMB has the same temperature to a very high degree.

2.2.2 Flatness problem

Let us consider again the Friedmann equation and study the contribution of the curvature term. Let us first define the critical energy density today as the energy density our universe would have assuming exact flatness: $\rho_{c,0} = 3M_p^2 H_0^2$, and then work with the fractional energy densities $\Omega_i := \rho_{i,0}/\rho_{c,0}$ and $\Omega_k = -k/(a_0 H_0)^2$, so that eq. (2.6), evaluated today, acquires the form:

$$\Omega_k = 1 - \sum_i \Omega_i \quad (2.18)$$

The measurement [36, 37], $\Omega_k = (0.0007 \pm 0.0019)$, shows that the relative curvature contribution cannot exceed a few parts per mill today, hence our universe is flat to a very high degree. The problem lies in the fact that the solution $\Omega_k = 0$ is unstable: in a universe filled with only matter and radiation, the quantity $k/(aH)^2$ grows with time, while the Ω_i variables decrease.

¹In the computation we consider $R_h(t) \simeq 2/H(t)$ at both recombination and now, we consider the largest possible size of the patches to be double the size of the horizon at that time and $H_0/H_{rec} = t_{rec}/t_0 = (a_{rec}/a_0)^{3/2} = (1+z_{rec})^{3/2}$.

Indeed, defining $\rho_k := k/a^2$ and assuming the universe has been dominated by radiation from the Planck time, $t_p = \sqrt{\hbar G/c^5} \approx 5 \times 10^{-44} s$, until today, we find the following:

$$\frac{\rho_k}{\rho_{rad}} \propto a^{-2} \times a^4 = a^2 \implies \frac{\rho_k}{\rho} \Big|_{t=t_0} = \frac{\rho_k}{\rho} \Big|_{t=t_p} \left(\frac{a(t_0)}{a(t_p)} \right)^2 \approx \frac{\rho_k}{\rho} \Big|_{t=t_p} \left(\frac{T_p}{T_0} \right)^2 \approx \frac{\rho_k}{\rho} \Big|_{t=t_p} 10^{64}, \quad (2.19)$$

where in the second to last equality we used the fact that $a \propto T^{-1}$, $T_p = 10^{19}$ GeV is the Planck temperature and $T_0 \approx 10^{-13}$ GeV is the temperature of the universe today. Hence, we should reasonably expect the curvature term of the Friedmann equation to be dominating. Without changing the evolution history of the universe, we can justify this only via a huge amount of fine tuning.

2.2.3 Solution by accelerated expansion

Both of these problems are implicitly tied to the acceleration of the universe. We can easily see it by noticing that

$$R_h \propto \int_{t_i}^t dt' (t')^{-\alpha} \quad (2.20)$$

with $\alpha = 2/[3(1+w)]$. This integral is IR dominated if $\alpha > 1$, namely if $w < -1/3$. Moreover, it diverges if $t_i \rightarrow 0$. This means that, for $w < -1/3$, the particle horizon can always be made sufficiently large to guarantee thermalization over any arbitrarily large physical distance, provided that t_i is taken sufficiently small, so that the accelerated stage lasts long enough.

We can also prevent the flatness problem by requiring the existence of an epoch in which the universe was filled up with some kind of matter that depleted less quickly than the curvature term ($\propto a^{-2}$) so that during this era the curvature contribution could have become negligible: since $\rho \propto a^{-3(1+w)}$ our requirement is also satisfied for $w < -1/3$. Hence, both problems are solved by including in the evolution of the universe an epoch of accelerated expansion. The theoretical framework that aims at solving these issues by introducing an era of accelerated expansion is called *inflation theory*, a name given for the first time by Alan H. Guth in [1].

We now know that we need an inflationary epoch, but what is its minimum duration? Let us start by defining the number of e-folds:

$$N := \log \left(\frac{a_f}{a_i} \right) = \int_i^f H dt \quad (2.21)$$

where a_f is the value of the scale factor at the end of inflation and a_i at the beginning. The condition we need to ask to make this estimate is that the comoving Hubble radius today is less or at most equal to the comoving Hubble radius at the beginning of the inflationary era:

$$r_H(t_i)/a(t_i) \geq r_H(t_0)/a(t_0), \quad (2.22)$$

i.e. we require two comoving observers that just entered the particle horizon, which locally is always proportional to the Hubble radius, to have also been within the particle horizon at some time in the far past, that we call the beginning of inflation. Let us estimate the number of e-folds inflation lasted as follows:

$$e^N = \frac{a_f}{a_i} \geq a_f \frac{r_H(t_0)}{a_0 r_H(t_i)} = \frac{a_f H_i}{a_0 H_0} \approx \left(\frac{H_0}{H_f} \right)^{1/2} \frac{H_f}{H_0} = \left(\frac{H_f}{H_0} \right)^{1/2} \approx e^{63 + \frac{1}{2} \ln(H_f/10^{13} \text{ GeV})}. \quad (2.23)$$

In the third to last equality, we are assuming $H_i \approx H_f$, since it stays almost constant throughout inflation, and that only an era of radiation domination interurred from the end of inflation until now,

letting us employ $a \propto t^{1/2} \propto H^{-1/2}$; instead, in the last equality, we are substituting $H_0 \approx 10^{-42} \text{GeV}$. To avoid overproduction of gravitational waves, we have that $H_f \lesssim 10^{13} \text{ GeV}$, hence we can conclude that at least ≈ 63 e-folds of inflation are needed. Dropping all of the approximations we made, introduces an error $\Delta N \approx 15$.

We remark that condition (2.22), while solving the horizon problem, solves the flatness problem as well. We can see it in the following:

$$\begin{aligned} \left. \frac{\rho_k}{\rho} \right|_{t_0} \leq \left. \frac{\rho_k}{\rho} \right|_{t_i} &\iff \left. \frac{\Omega_k}{1 - \Omega_k} \right|_{t_0} \leq \left. \frac{\Omega_k}{1 - \Omega_k} \right|_{t_i} \iff \Omega_k|_{t_0} \leq \Omega_k|_{t_i} \iff \\ &\iff \left. \frac{1}{a^2 H^2} \right|_{t_0} \leq \left. \frac{1}{a^2 H^2} \right|_{t_i} \iff \left. \frac{1}{aH} \right|_{t_0} \leq \left. \frac{1}{aH} \right|_{t_i}, \end{aligned} \quad (2.24)$$

where we start from the condition solving the flatness problem until we arrive at the definition of comoving Hubble radius.

2.3 Single-field slow-roll inflation

Let us see what happens if in the universe there was only a single scalar field, called Inflaton, dominating the total energy density. Let us consider the standard action for a real scalar field in a FLRW spacetime (minimally) coupled to the metric:

$$S = \int d^4x \sqrt{-g} \left[\frac{M_p^2}{2} R - \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right] = S_{EH} + S_\phi \quad (2.25)$$

where R is the scalar curvature, S_{EH} is the Einstein-Hilbert action, S_ϕ is the action for a canonically normalized scalar field ϕ , $g_{\mu\nu}$ is the FLRW metric and $\sqrt{-g} = a^3$. We can derive the classical equation of motion for the scalar field by varying the action with respect to $\delta\phi$:

$$\ddot{\phi} + 3H\dot{\phi} + \frac{\partial V}{\partial \phi} = 0, \quad (2.26)$$

and the Friedmann equation takes the following form:

$$H^2 = \frac{1}{3M_p^2} \left(\frac{1}{2} \dot{\phi}^2 + V(\phi) \right) \quad (2.27)$$

where we have assumed in both equations the scalar field to be homogeneous, $\phi(t, \mathbf{x}) = \phi(t)$. We can then derive the energy density and the pressure from the energy momentum tensor under the same assumption:

$$\begin{aligned} \rho_\phi &= \frac{1}{2} \dot{\phi}^2 + V(\phi), \quad p_\phi = \frac{1}{2} \dot{\phi}^2 - V(\phi) \\ \implies w_\phi &:= \frac{p_\phi}{\rho_\phi} = \frac{\frac{1}{2} \dot{\phi}^2 - V(\phi)}{\frac{1}{2} \dot{\phi}^2 + V(\phi)}. \end{aligned} \quad (2.28)$$

We notice that in the limit of negligible kinetic energy the w_ϕ parameter approaches -1, hence it can yield accelerated expansion while the Hubble parameter stays approximately constant, since in the Friedmann equation the potential would play the role of an effective cosmological constant.

To better describe this regime of accelerated expansion we firstly introduce the slow-roll parameter ε , which is directly related to the second derivative of the scale factor:

$$\frac{\ddot{a}}{a} = H^2 + \dot{H} = H^2(1 - \varepsilon) \implies \varepsilon := -\frac{\dot{H}}{H^2} = \frac{1}{2M_p^2} \frac{\dot{\phi}^2}{H^2} \quad (2.29)$$

where in the last step we applied the time derivative to eq. (2.27) and substituted the Klein-Gordon (KG) equation, (2.26), in. We have accelerated expansion if $\varepsilon < 1$ and the limit $\varepsilon \rightarrow 0$ corresponds to the de Sitter limit in which the potential energy density dominates

$$\dot{\phi}^2 \ll V(\phi). \quad (2.30)$$

hence yielding $w_\phi \approx -1$. Then we also need to make sure that this regime lasts long enough to solve the horizon and flatness problems, so we define a second slow-roll parameter:

$$\eta := -\frac{\ddot{\phi}}{H\dot{\phi}} \quad (2.31)$$

and by asking for $\eta \ll 1$, we are imposing for the acceleration of the inflaton to be small with respect to its velocity (times H for dimensional reasons) so that if the kinetic energy is small, it will stay for some time.

Given the slow-roll conditions

$$\varepsilon, |\eta| \ll 1, \quad (2.32)$$

we can simplify the Friedmann and KG equations in the following way:

$$H^2 \approx \frac{V}{3M_p^2} \approx \text{const}, \quad \dot{\phi} \approx -\frac{1}{3H} \frac{\partial V}{\partial \phi}. \quad (2.33)$$

In order to connect the slow-roll regime to a model, i.e. a particular potential, it is useful to define the following potential-dependent parameters:

$$\varepsilon_V := \frac{M_p^2}{2} \left(\frac{1}{V} \frac{\partial V}{\partial \phi} \right)^2, \quad \eta_V := M_p^2 \frac{1}{V} \frac{\partial^2 V}{\partial \phi^2}. \quad (2.34)$$

These potential-dependent parameters are different from ε and η , but in the slow-roll approximation it is possible to relate them.

$$\varepsilon_V \approx \frac{M_p^2}{2} \left(\frac{1}{3M_p^2 H^2} (-3H\dot{\phi}) \right)^2 = \frac{1}{2M_p^2} \frac{\dot{\phi}^2}{H^2} = \varepsilon, \quad \eta \approx \frac{\partial_t(-V'/3H)}{H\dot{\phi}} = -\frac{V''}{3H^2} - \frac{V'}{3H\dot{\phi}} \varepsilon = \eta_V - \varepsilon_V. \quad (2.35)$$

2.3.1 de Sitter spacetime

If the Hubble parameter is perfectly constant, then the spacetime is said to be de Sitter and since in the slow-roll regime we induce a quasi de Sitter epoch, we dedicate this section to characterizing the de Sitter solution. As we obtained before:

$$a(t) = a_* e^{H(t-t_*)} \implies ds^2 = -dt^2 + e^{2Ht} (dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2)), \quad (2.36)$$

where we chose t_* so that $a(t_*) = 1$. In general, a d -dimensional de Sitter spacetime is defined as the embedding of a $(d+1)$ -dimensional Minkowski spacetime:

$$ds^2 = -dX_0^2 + \sum_{i=1}^d dX_i^2 \quad \text{with the condition} \quad X^\mu X_\mu = L^2 \quad (2.37)$$

for some constant L . We can choose a set of coordinates such that the condition is automatically satisfied and choosing the coordinates called *flat slicing* we can get back the metric in FLRW form:

$$\begin{aligned} X_0 &= L \sinh\left(\frac{t}{L}\right) + \frac{r^2}{2L} e^{\frac{t}{L}}, \\ X_1 &= L \cosh\left(\frac{t}{L}\right) - \frac{r^2}{2L} e^{\frac{t}{L}}, \\ X_i &= e^{\frac{t}{L}} y_i \quad \text{with } i = 2, \dots, d \quad \text{and } r^2 = \mathbf{y}^2 \\ \implies ds^2 &= -dt^2 + e^{\frac{2t}{L}} d\mathbf{y}^2 = -dt^2 + e^{\frac{2t}{L}} (dr^2 + d\Omega_{d-2}^2). \end{aligned} \quad (2.38)$$

In the last step we chose $d\mathbf{y}^2$ to describe a flat euclidean metric in $\mathbb{R}^{d-1}$. We notice that they cover only half of the spacetime since $X_0 + X_1 > 0$ and that for $d = 4$ and $L = H^{-1}$ we get (2.36). Another choice we could have made is called the *static patch*:

$$\begin{aligned} X_0 &= \sqrt{L^2 - r_s^2} \sinh\left(\frac{t_s}{L}\right), \\ X_1 &= \sqrt{L^2 - r_s^2} \cosh\left(\frac{t_s}{L}\right), \\ X_i &= r_s z_i \quad \text{with } i = 2, \dots, d \quad \text{and } \mathbf{z}^2 = 1 \\ \implies ds^2 &= -\left(1 - \frac{r_s^2}{L^2}\right) dt_s^2 + \frac{dr_s^2}{1 - \frac{r_s^2}{L^2}} + r_s^2 d\Omega_{d-2}^2 \end{aligned} \quad (2.39)$$

where we chose $d\mathbf{z}^2$ to describe the metric in $\mathbb{S}^{d-2}$. We used the subscript s to highlight the fact that the time and radial coordinates are different from the *flat slicing* ones. This is an equivalent set of coordinates for which the metric does not depend on time, hence the name *static*, but more notably we see that there is a coordinate singularity at $r_s = L$, meaning that this choice of coordinates is valid up to this horizon.

The physical meaning behind this singularity is that two objects separated by a physical distance greater than L will never causally interfere with each other in the future, although they could have in the past when they were within the horizon. A light-like observer cannot keep up with the rate of expansion of the universe at this stage. Even though the presence of the horizon is explicit only in static coordinates, it is still there for any choice of coordinates. For example, we can compare the scaling of a length with how much a radially moving light ray travels in a spacetime described by (2.36):

$$ds^2 = 0 \iff dt = \pm e^{Ht} dr \xrightarrow{+} \int_{t_0}^t e^{-Ht'} dt' = \int_0^r dr' \implies r = -\frac{1}{H} (e^{-Ht} - e^{-Ht_0}), \quad (2.40)$$

then, defining $\Delta t = t - t_0$, we find that the physical distance $R(t)$ traveled by the light ray is:

$$R(t) = a(t)r = e^{Ht}r = -\frac{1}{H} (1 - e^{H\Delta t}) = \frac{e^{H\Delta t} - 1}{H}. \quad (2.41)$$

So, if A is at a finite distance l from B , we can equate the evolution of the distance between the two observers with the distance that a light ray coming from A travels to reach B (or viceversa) and check if the equation has solutions for a finite time interval Δt :

$$le^{H\Delta t} = R(t) = \frac{e^{H\Delta t} - 1}{H} \implies e^{H\Delta t} = \frac{1}{1 - Hl} \implies \Delta t = \frac{1}{H} \log\left(\frac{1}{1 - Hl}\right). \quad (2.42)$$

Hence, we can conclude that for $l < H^{-1}$ the equation is solvable and for $l = H^{-1}$ it would take an infinite amount of time to reach B , so there is still a kind of horizon even though it is not explicit in the coordinate choice.

2.4 Cosmological perturbations

Single-field slow-roll inflation other than solving the horizon and flatness problems offers as well a mechanism for generating primordial perturbations [38]. We proceed to give a brief overview of cosmological perturbations still following [34, 35].

We define the perturbation of a given quantity $X(t, \mathbf{x})$ as

$$\delta X(t, \mathbf{x}) := X(t, \mathbf{x}) - \bar{X}(t) \quad (2.43)$$

where $\bar{X}(t)$ is a homogeneous background depending only on cosmic time. For small fluctuations, $\delta X(t, \mathbf{x}) \ll X(t, \mathbf{x})$, we can expand the Einstein equations at linear order in order to approximate the full non-linear solution

$$\delta G_{\mu\nu} = 8\pi G \delta T_{\mu\nu}. \quad (2.44)$$

By choosing FLRW as a background, this equation inherits all of the symmetries of the background metric, such as homogeneity and isotropy.

2.4.1 Gauge choice

When we are dealing with cosmological perturbation we are implicitly defining two manifolds: the physical one, featuring the perturbations, and the background one, that is exactly homogeneous and isotropic. In order to introduce perturbations, we need to define a map relating a point of the background to a specific point of the physical one. Since this map is not unique, changing it associates a different physical point with the same background point, so the perturbation assigned to that background point takes a different value, even though the background itself, being homogeneous and isotropic, is left unchanged.

This freedom in the association can be considered a gauge freedom and, depending on the map we choose, we would compute different perturbations.

2.4.2 SVT decomposition

The high degree of symmetry in a FLRW universe (homogeneity and isotropy) allows us to decompose any perturbation in scalar (S), vector (V) and tensor (T) components that evolve independently at the linearized level. We can see this clearly in Fourier space

$$X_{\mathbf{k}}(t) = \int \frac{d^3x}{(2\pi)^{3/2}} X(t, \mathbf{x}) e^{-i\mathbf{k}\cdot\mathbf{x}} \quad (2.45)$$

where X can be any kind of fluctuation, *e.g.* of the inflaton or of the metric. Due to translational invariance of the linearized Einstein equations, each Fourier mode evolves independently and we can study them one by one. Invariance under rotations around a wave vector $\mathbf{k}$ instead implies that perturbations of different helicity m evolve independently of one another.

We say that a given quantity X has helicity m if it is multiplied by a factor $e^{im\psi}$ when we rotate the coordinate system by an angle ψ :

$$X_{\mathbf{k}} \longrightarrow e^{im\psi} X_{\mathbf{k}}. \quad (2.46)$$

The helicity of scalars, vectors and tensors is respectively 0, ± 1 and ± 2 .

2.4.3 Inflaton and metric perturbations

We decompose the inflaton field and the metric into a homogeneous background solution and its perturbations in the following way

$$\phi(t, \mathbf{x}) = \bar{\phi}(t) + \delta\phi(t, \mathbf{x}), \quad g_{\mu\nu}(t, \mathbf{x}) = \bar{g}_{\mu\nu}(t) + \delta g_{\mu\nu}(t, \mathbf{x}), \quad (2.47)$$

where

$$\begin{aligned} ds^2 &= g_{\mu\nu} dx^\mu dx^\nu \\ &= -(1 + 2\Phi)dt^2 + 2aB_i dx^i dt + a^2[(1 - 2\Psi)\delta_{ij} + E_{ij}]dx^i dx^j. \end{aligned} \quad (2.48)$$

The SVT decomposition of the metric perturbations is given by

$$B_i := \partial_i B - S_i \quad \text{where} \quad \partial_i S^i = 0, \quad (2.49)$$

and

$$E_{ij} := 2\partial_{(i} E + 2\partial_{(i} F_{j)} + h_{ij} \quad \text{where} \quad \partial_i F^i = 0, \quad h_i^i = 0 = \partial^i h_{ij}. \quad (2.50)$$

We highlight the fact that vector perturbations, such as S_i and F_i , cannot be excited by inflation. We can see this by counting the available degrees of freedom: being a two-index symmetric tensor, the metric has 10 degrees of freedom, but taking into account our gauge freedom we lose 4 of them. Then, it is possible to show that there is a residual gauge, implying that other 4 degrees of freedom can be fixed via a gauge choice, so we are left with just 2 that are of tensorial origin. The inflaton field is a scalar, providing an extra scalar degree of freedom.

2.4.4 Gauge choices and gauge-invariant variables

A generic infinitesimal gauge transformation takes the following form:

$$\begin{aligned} t &\longrightarrow t + \alpha \\ x^i &\longrightarrow x^i + \beta^i + \delta^{ij} \partial_j \beta \end{aligned} \quad (2.51)$$

where α , β and β^i are infinitesimal transformation parameters and β^i is transverse, i.e. $\partial_i \beta^i = 0$. By these means, we are clearly free to set to zero two scalars and a vector perturbations. Some gauge choices we can make are for example:

- the *spatially flat* gauge in which we set to zero the scalar and vector degrees of freedom of the spatial part of the metric;
- the *synchronous* gauge in which we set to zero δg_{00} and δg_{0i} so that the observers on the physical spacetime are characterized by the same time as the observers defined on the background;
- the *uniform energy density* gauge in which we set to zero the perturbation of the energy density.

Another approach to the study of perturbations is to consider quantities that do not depend on the gauge choice, i.e. quantities that are gauge-invariant. We proceed now to list some of them:

- the *curvature perturbation on uniform energy density hypersurfaces*

$$-\zeta := \Psi + \frac{H}{\dot{\rho}} \delta \rho \quad (2.52)$$

that takes into account the spatial curvature of constant-density hypersurfaces, $R^{(3)} = 4\nabla^2 \Psi / a^2$. During slow-roll inflation, we see that

$$-\zeta \approx \Psi + \frac{H}{\dot{\phi}} \delta \phi \quad (2.53)$$

- one of the *Bardeen potentials*:

$$\Psi_B := \Psi + a^2 H (\dot{E} - B/a). \quad (2.54)$$

In the Poisson gauge, in which E and B are set to zero, this potential plays the role of the Newtonian gravitational potential.

- the *comoving curvature perturbation*:

$$\mathcal{R} := \Psi - \frac{H}{\bar{\rho} + \bar{p}} \delta q \quad (2.55)$$

where δq is the scalar part of the 3-momentum density $T_i^0 = \partial_i \delta q$ and during inflation we have that $T_i^0 = -\dot{\phi} \partial_i \delta \phi$ so that

$$\mathcal{R} = \Psi + \frac{H}{\dot{\phi}} \delta \phi \quad (2.56)$$

$\mathcal{R}$ and ζ can be related in full generality via the linearized Einstein equations:

$$-\zeta = \mathcal{R} + \frac{k^2}{a^2 H^2} \frac{2\bar{p}}{3(\bar{\rho} + \bar{p})} \Psi_B \quad (2.57)$$

where k is the comoving momentum, so that on super-horizon scales ($k \ll aH$) these two perturbations actually coincide.

2.4.5 Statistics of cosmological perturbations

The theory of inflation is not predictive in a classical sense (namely, it does not predict the location in the universe in which a given overdensity forms), but it predicts the statistical properties of the primordial perturbations. The most important statistical quantity that characterizes them is their power spectrum P_X which is related to the variance of the perturbation modes in Fourier space by

$$\langle X_{\mathbf{k}} X_{\mathbf{k}'} \rangle = \delta^{(3)}(\mathbf{k} + \mathbf{k}') P_X(k), \quad (2.58)$$

where $\langle \dots \rangle$ is the ensemble average of classical fluctuations in the classical case or the quantum expectation value in the case of Quantum Field Theory (QFT). We expect the $\delta^{(3)}(\mathbf{k} + \mathbf{k}')$ factor to always be present due to homogeneity of the background

$$\langle X(t, \vec{x}), X(t, \vec{x}) \rangle = \int \frac{d^3 k}{(2\pi)^{3/2}} \frac{d^3 k'}{(2\pi)^{3/2}} e^{i\mathbf{x} \cdot (\mathbf{k} + \mathbf{k}')} \langle X_{\mathbf{k}} X_{\mathbf{k}'} \rangle = \int \frac{d^3 k}{(2\pi)^3} P_X(k), \quad (2.59)$$

so that any dependence on spatial coordinates disappears. It is useful to define the dimensionless power spectrum with

$$\Delta_X^2(k) := \frac{k^3}{2\pi^2} P_X(k), \quad (2.60)$$

which in the scalar case, $X = \mathcal{R}$, acquires the form

$$\Delta_s^2 := \Delta_{\mathcal{R}}^2 = \frac{k^3}{2\pi^2} P_{\mathcal{R}}(k). \quad (2.61)$$

To encode scale dependence we define the spectral index (or tilt) as

$$n_s - 1 := \frac{d \ln \Delta_s^2}{d \ln k}, \quad (2.62)$$

so that scale invariance corresponds to $n_s = 1$. The running of the spectral index is defined as

$$\alpha_s := \frac{dn_s}{d \ln k}. \quad (2.63)$$

Hence, we can parametrize the dimensionless power spectrum with respect to a reference scale k_* as follows:

$$\Delta_s^2(k) = A_s(k_*) \left(\frac{k}{k_*} \right)^{n_s(k_*) - 1 + \frac{1}{2} \alpha_s(k_*) \ln(k/k_*)}. \quad (2.64)$$

If the perturbations are Gaussian, all the relevant information is encoded by the power spectrum. Deviations from this behavior would be encoded by higher-order correlation functions.

We define the power spectrum of the two polarization modes of h_{ij} , that are $h \equiv h^+, h^\times$, as

$$\langle h_{\mathbf{k}} h_{\mathbf{k}'} \rangle = \delta(\mathbf{k} + \mathbf{k}') P_h(k), \quad \Delta_h^2 = \frac{k^3}{2\pi^2} P_h(k). \quad (2.65)$$

The tensorial power spectrum is defined as the sum of the power spectra of the two polarizations

$$\Delta_t^2 = 2\Delta_h^2. \quad (2.66)$$

In this case the tilt is defined without the -1 for historical reasons

$$n_t := \frac{d \ln \Delta_t^2}{d \ln k}. \quad (2.67)$$

Therefore, we parametrize

$$\Delta_t^2(k) = A_t \left(\frac{k}{k_*} \right)^{n_t(k_*)}. \quad (2.68)$$

2.5 Quantum fluctuations in de Sitter spacetime

In this section, we consider a massless scalar field in de Sitter spacetime and we quantize it in order to outline the essence of the quantization procedure.

Working in conformal time, defined as $d\tau = dt/a(t)$, the action of a scalar field on a de Sitter background reads

$$\begin{aligned} S &= - \int d^4x \sqrt{-g} \frac{g^{\mu\nu}}{2} \partial_\mu \phi \partial_\nu \phi = - \int d\tau d^3x a^2 \frac{\eta^{\mu\nu}}{2} \partial_\mu \phi \partial_\nu \phi \\ &= - \int d\tau d^3x \frac{1}{2} \left[(v')^2 - (\nabla_i v)^2 + \frac{a''}{a} v^2 \right], \end{aligned} \quad (2.69)$$

where $(\dots)' = \partial_\tau(\dots)$ and in the last equality we rescaled the scalar field as $\phi = v/a$ to get a canonically normalized kinetic term. Varying the action with respect to the rescaled field v , we find the following equation of motion:

$$v'' - \nabla^2 v + \frac{a''}{a} v = 0. \quad (2.70)$$

Via a Fourier transform, $v(t, \mathbf{x}) = \int \frac{d^3k}{(2\pi)^{3/2}} v_{\mathbf{k}}(\tau) e^{i\mathbf{k} \cdot \mathbf{x}}$, we rewrite this equation in momentum space:

$$v_k'' - \left(k^2 - \frac{a''}{a} \right) v_k = 0, \quad (2.71)$$

where we dropped the vector notation for momentum since the equation of motion in momentum space depends only on the magnitude k and not its angular components.

Quantization

To quantize the field we first need to promote it to an operator, as well as the conjugate momentum,

$$v \longrightarrow \hat{v} = \int \frac{d^3k}{(2\pi)^{3/2}} \left[v_k(\tau) \hat{a}_{\mathbf{k}} e^{i\mathbf{k} \cdot \mathbf{x}} + v_k^*(\tau) \hat{a}_{-\mathbf{k}}^\dagger e^{-i\mathbf{k} \cdot \mathbf{x}} \right], \quad (2.72)$$

where the ladder operators need to satisfy the canonical commutation relation

$$[\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{k}'}^\dagger] = \delta^{(3)}(\mathbf{k} - \mathbf{k}') \quad (2.73)$$

and the modes must follow the subsequent condition in order for eq. (2.73) to be properly normalized:

$$\langle v_k, v_k \rangle := \frac{i}{\hbar} (v_k^* v'_k - v_k'^* v_k) = 1 . \quad (2.74)$$

The last equation is actually a boundary condition for the solutions of (2.71).

Choice of the vacuum

A second boundary condition arises from the selection of the vacuum. The choice we make is the Minkowski vacuum for a comoving observer in the far past, $\tau \rightarrow -\infty$ or $|k\tau| \gg 1$ or $k \gg aH$. In this limit all of the modes are far inside the horizon and our equation becomes

$$v_k'' + k^2 v_k \approx 0 , \quad (2.75)$$

which is the equation of motion of a harmonic oscillator. It is possible to show that we have a unique solution requiring that the vacuum is the lowest positive-energy state. Hence, we impose the following condition

$$\lim_{\tau \rightarrow -\infty} v_k = \frac{1}{\sqrt{2k}} e^{-ik\tau} \quad (2.76)$$

that is known as the Bunch-Davies vacuum.

Solution in de Sitter spacetime

On super-horizon scales, (2.71) reduces to

$$v_k'' - \frac{a''}{a} v_k \approx 0 \quad (2.77)$$

which is solved generally by

$$v_k(\tau) = A_1(k) a(\tau) + A_2(k) a(\tau)^{-2} , \quad (2.78)$$

where A_1 and A_2 are integration constants. We expect the first term to dominate while the second one to decay. By matching the sub-horizon solution with the super-horizon one at horizon crossing ($k = aH$), we can estimate A_1 :

$$|v_k^{sup}| = |A_1(k) a(\tau)| = \frac{1}{\sqrt{2k}} = |v_k^{sub}| \implies |A_1(k)| = \frac{H}{\sqrt{2k^3}} . \quad (2.79)$$

The full solution, interpolating between the sub- and super-horizon limits, is given by

$$v_k = \frac{1}{\sqrt{2k}} e^{-ik\tau} \left(1 - \frac{i}{k\tau} \right) . \quad (2.80)$$

Power spectrum

Considering the Fourier decomposition $\hat{\phi}_{\mathbf{k}}$ of the quantized field operator $\hat{\phi} = \hat{v}/a$, its power spectrum acquires the following form:

$$\begin{aligned}\langle \hat{\phi}_{\mathbf{k}} \hat{\phi}_{\mathbf{k}'} \rangle &= \delta^{(3)}(\mathbf{k} + \mathbf{k}') \frac{|v_{\mathbf{k}}(\tau)|^2}{a^2} \\ &= \delta^{(3)}(\mathbf{k} + \mathbf{k}') \frac{H^2}{2k^3} (1 + k^2 \tau^2) .\end{aligned}\tag{2.81}$$

On super-horizon scales ($-k\tau \ll 1$), we notice that it tends to a constant, recovering (2.79):

$$\langle \hat{\phi}_{\mathbf{k}} \hat{\phi}_{\mathbf{k}'} \rangle \longrightarrow \delta^{(3)}(\mathbf{k} + \mathbf{k}') \frac{H^2}{2k^3} , \quad \Delta_{\phi}^2 = \left(\frac{H}{2\pi} \right)^2 .\tag{2.82}$$

Since the power spectrum of a mode does not depend on time, it remains basically untouched from the moment the mode exits the horizon until it re-enters after inflation has ended.

2.6 Quantum fluctuations during slow-roll

Let us consider single-field slow-roll models described by the action

$$S = \int d^4x \sqrt{-g} \left[\frac{M_p^2}{2} R - \frac{1}{2} (\partial\phi)^2 - V(\phi) \right]\tag{2.83}$$

and make the following gauge choices

$$\delta\phi = 0, \quad g_{ij} = a^2[(1 - 2\mathcal{R})\delta_{ij} + h_{ij}], \quad \partial^i h_{ij} = 0 = h_i^i,\tag{2.84}$$

so that, while the inflaton field is not perturbed, all scalar degrees of freedom are taken into account by fluctuations of the metric $\mathcal{R}(t, \mathbf{x})$.

2.6.1 Scalar perturbations

It can be shown that (2.83), expanded to the second order in $\mathcal{R}$, is

$$S_{(2)} = -\frac{1}{2} \int d^4x \, a^3 \frac{\dot{\phi}^2}{H^2} \left[\dot{\mathcal{R}}^2 - a^{-2} (\nabla_i \mathcal{R})^2 \right] = -\frac{1}{2} \int d\tau d^3x \left[(v')^2 - (\nabla_i v)^2 + \frac{z''}{z} v^2 \right],\tag{2.85}$$

where we have defined the Sasaki-Mukhanov variable $v := z\mathcal{R}$ with $z^2 := a^2 \dot{\phi}^2 / H^2 = 2a^2 M_p^2 \varepsilon$. Note that this action is analogous to the one found in the de Sitter case. Through a procedure similar to the one outlined for de Sitter, it is possible to show that the dimensionless power spectrum of scalar perturbation acquires the following form on super-horizon scales:

$$\Delta_{\mathcal{R}}^2(k) = \left(\frac{H^2}{2\pi \dot{\phi}} \right)^2 \left(\frac{k}{aH} \right)^{2\eta_V - 6\varepsilon} .\tag{2.86}$$

We observe that here, perfect scale invariance is broken by a power law in the slow-roll parameters. Inverting the relation $\mathcal{R} = \frac{H}{\dot{\phi}} \delta\phi$, which is valid in the spatially flat gauge, and taking the limit $\eta_V, \varepsilon \rightarrow 0$, we note that the de Sitter result $\Delta_{\delta\phi}^2 = H^2/(2\pi)^2$ is clearly recovered.

2.6.2 Tensor perturbations

Let us now briefly focus on tensor perturbations. We highlight that they are automatically gauge-invariant, since gauge transformations alter only scalar and vector degrees of freedom. By expanding the Einstein-Hilbert action to the second order in tensor fluctuations, we find

$$S_{(2)} = -\frac{M_p^2}{8} \int d\tau d^3x a^2 [(h'_{ij})^2 - (\partial_l h_{ij})^2]. \quad (2.87)$$

Then, defining the Fourier transform

$$h_{ij} = \int \frac{d^3k}{(2\pi)^{3/2}} \sum_{s=+, \times} \epsilon_{ij}^s(k) h_{\mathbf{k}}^s(\tau) e^{i\mathbf{k} \cdot \mathbf{x}}, \quad (2.88)$$

where $\epsilon_{ii} = 0 = k^i \epsilon_{ij}$ and $\epsilon_{ij}^s(k) \epsilon_{ij}^{s'}(k) = 2\delta_{ss'}$, the action becomes

$$S_{(2)} = - \sum_s \int d\tau d\mathbf{k} \frac{a^2}{4} M_p^2 [h_{\mathbf{k}}^{s'} h_{\mathbf{k}}^{s'} - k^2 h_{\mathbf{k}}^s h_{\mathbf{k}}^s] = - \sum_s \frac{1}{2} \int d\tau d\mathbf{k} \left[(v_{\mathbf{k}}^{s'})^2 - \left(k^2 - \frac{a''}{a} \right) (v_{\mathbf{k}}^s)^2 \right], \quad (2.89)$$

where we have defined the canonically normalized field $v_{\mathbf{k}}^s = a M_p h_{\mathbf{k}}^s / 2$. Analogously to the previous subsection, the power spectrum of tensor perturbations reads:

$$\Delta_t^2 = \frac{2}{\pi^2} \frac{H^2}{M_p^2} \left(\frac{k}{aH} \right)^{-2\varepsilon}. \quad (2.90)$$

2.6.3 The energy scale of inflation

We define the *tensor-to-scalar* ratio as

$$r := \frac{\Delta_t^2(k)}{\Delta_s^2(k)}. \quad (2.91)$$

Given that Δ_s^2 is fixed and that $\Delta_t^2 \propto H^2 \approx V$, r can be considered a measure of the energy scale of inflation:

$$V^{1/4} \approx \left(\frac{r}{0.01} \right)^{1/4} 10^{16} \text{GeV}. \quad (2.92)$$

We note that the values of $r > 0.01$ correspond to energy scales related to Grand Unification Theory (GUT).

2.6.4 Primordial spectra during slow-roll

The results for the scalar and tensor power spectra at horizon crossing ($k = aH$) are

$$\Delta_s^2(k = aH) := \Delta_{\mathcal{R}}^2(k = aH) = \frac{1}{8\pi^2} \frac{H^2}{M_p^2} \frac{1}{\varepsilon} \bigg|_{k=aH} \approx \frac{1}{24\pi^2} \frac{V}{M_p^4} \frac{1}{\varepsilon_V} \bigg|_{k=aH}, \quad (2.93)$$

$$\Delta_t^2(k = aH) := 2\Delta_h^2(k = aH) = \frac{2}{\pi^2} \frac{H^2}{M_p^2} \bigg|_{k=aH} \approx \frac{2}{3\pi^2} \frac{V}{M_p^4} \bigg|_{k=aH}, \quad (2.94)$$

where in both lines the last passage consists in applying the slow-roll approximation. After fixing a pivot scale k_* , we can relate the power spectrum of scalar and tensor perturbations of momentum k to the one computed at this pivot scale:

$$\Delta_s^2(k) = \Delta_s^2(k_*) \left(\frac{k}{k_*} \right)^{n_s-1}, \quad \Delta_t^2(k) = \Delta_t^2(k_*) \left(\frac{k}{k_*} \right)^{n_t}. \quad (2.95)$$

The *tensor-to-scalar ratio* is

$$r = 16\varepsilon_* , \quad (2.96)$$

where the $*$ indicates that the quantity is computed at the pivot scale. To first order in Hubble slow-roll parameters, the tilts of the spectra are

$$n_s - 1 = 2\eta_V^* - 6\varepsilon_V^* , \quad n_t = -2\varepsilon_V^* , \quad (2.97)$$

but we also have the consistency relation

$$r = -8n_t , \quad (2.98)$$

thanks to which we can find a specific trait marking single-field models against other types of inflationary models.

We highlight the fact that slow-roll models allow us to relate the power spectra directly to the potential, so that we can derive information on its shape directly from experimental measurements.

2.7 Models for inflation

There are two main categories of single-field inflation: small-field and large-field inflation.

2.7.1 Small-field inflation

In these models the field travels over a sub-Planckian distance $\Delta\phi \ll M_p$. They predict that the amplitude of gravitational waves generated during inflation is too small to be detected. The potential can usually be approximated by

$$V(\phi) = V_0 \left[1 - \left(\frac{\phi}{\mu} \right)^p \right] + \dots \quad (2.99)$$

where the dots represent terms that become relevant only near the end of inflation and during reheating.

Usually the mechanisms that generate small-field inflation are connected to the pattern of spontaneous symmetry breaking and the Coleman-Weinberg potential is one of the famous examples used in this context [39, 40]:

$$V(\phi) = V_0 \left[\left(\frac{\phi}{\mu} \right)^4 \left(\ln \left(\frac{\phi}{\mu} \right) - \frac{1}{4} \right) + \frac{1}{4} \right] . \quad (2.100)$$

2.7.2 Large-field inflation

In this other set of models the field variation is super-Planckian ($\Delta\phi \gg M_p$) and the gravitational waves produced should be observable.

A basic model of this kind is *chaotic inflation* [41] in which there is a single monomial term dominating:

$$V(\phi) = \lambda_q M_p^4 \left(\frac{\phi}{M_p} \right)^q . \quad (2.101)$$

Requiring $\lambda_p \ll 1$ is needed in order to get a small amplitude of density fluctuations, and consequently this guarantees that the potential energy stays sub-Planckian, $V \ll M_p^4$, even though the field takes on large values, so that we can neglect quantum gravity effects.

2.7.3 Other possible extension of slow-roll

There are many other ways to induce inflation other than single-field slow-roll. Here we list some broad categories.

- **Non minimal coupling to gravity**

In the action in (2.25), the metric and the inflaton field are minimally coupled, but we could introduce new terms mixing them in other ways [42, 43], such as $R\phi^2$. Usually, these modifications can be traced back to minimally coupled models by redefining the fields.

- **Modified gravity**

Another possibility is to modify the Einstein-Hilbert action so that it changes behavior in the UV [44–48]. Also, this kind of theories can be reduced to minimally coupled ones. An example is given by Starobinsky:

$$R \rightarrow R + \frac{R^2}{6M_p^2} \quad (2.102)$$

that by Weyl rescaling becomes equivalent to a scalar field minimally coupled to gravity with a potential $V(\phi) \propto \left(1 - e^{-\frac{2\phi}{\sqrt{6}M_p}}\right)^2$.

- **Non canonical kinetic terms**

The idea here is basically to substitute the canonical kinetic term with something with a more exotic behavior in the high energy limit [49, 50] by which inflation is driven even in the presence of a steep potential:

$$\mathcal{L} = -\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi) \longrightarrow \mathcal{L}' = F(\phi, g_{\mu\nu}, \partial_\mu\phi) - V(\phi). \quad (2.103)$$

- **More than one field**

Considering scenarios in which inflation is driven by more than a single field [41, 51, 52] is obviously another prospect, but the problem here is that the possibilities increase exponentially, hence the theory loses predictive power.

Fig. 2.1, taken from the final data release of the Planck collaboration [2], shows how some models are favored by data while others are basically discarded. For example, we can notice that according to this release Starobinsky inflation is greatly favored, whereas natural inflation, discussed in the next chapter, is dismissed.

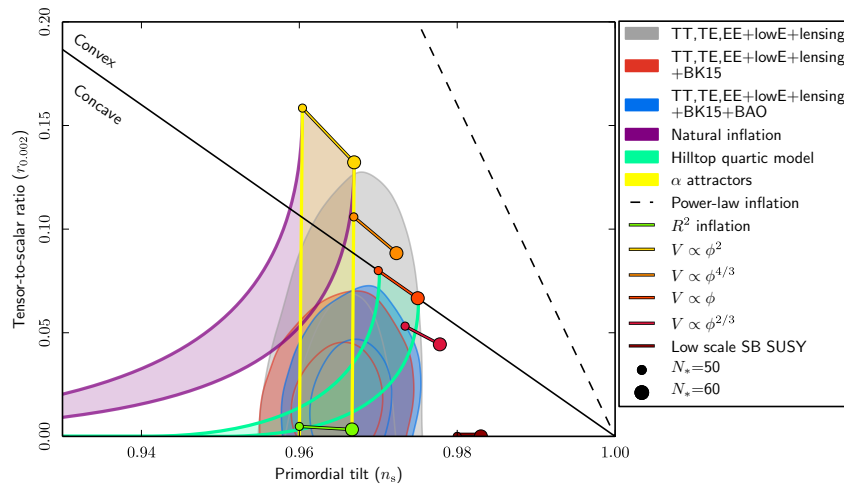


Figure 2.1: Marginalized joint 68% and 95% CL regions for n_s and $r_{0.002}$ from Planck in combination with other data sets compared to the theoretical predictions of selected inflationary models.

Chapter 3

Axion inflation with U(1) gauge fields

As we have seen in the first chapter, a primordial era of accelerated expansion is currently the best option we have to explain cosmological observations and we can realize this epoch thanks to a scalar field slowly rolling along an almost flat potential. In building this model, we split the inflaton field into a classical homogeneous background and a space-dependent perturbation, which we quantize:

$$\phi(t, \vec{x}) = \bar{\phi}(t) + \delta\phi(t, \vec{x}) , \quad (3.1)$$

then we want the classical contribution to be dominating over the quantum corrections for both phenomenological, i.e. homogeneity and isotropy of CMB radiation, and theoretical reasons, i.e. the slow-roll construction relies on the classical equations of motion. The requirement of a Classical-beats-Quantum (CbQ) regime is expressed by the condition

$$\dot{\phi} \gg \langle \delta\phi^2 \rangle \approx \left(\frac{H}{2\pi} \right)^2 \quad (3.2)$$

which must be satisfied throughout inflation. Moreover, the potential $V(\phi)$ we consider is actually an effective potential taking into account quantum corrections coming from physics at energy scales higher than the cutoff Λ_{UV} , so we need to make sure that its flatness is not spoiled at loop level. This is not a given, especially in the case of a scalar, because the square of its mass parameter, m^2 , i.e. the second derivative of the potential, is quadratically sensitive to the new physics scale. By naive dimensional analysis we expect the radiative correction to be

$$\delta m^2 \sim \Lambda_{UV}^2 . \quad (3.3)$$

This type of complication is famously known, in the context of the Higgs sector of the Standard Model of particle physics, as the *hierarchy problem*, whereas in the framework of cosmological inflation it takes the form of the *η problem* since the η parameter, proportional to $V''(\phi)$, must be small in order to have a successful slow-roll regime. Generically, the tree level potential would get loop corrections such as

$$\delta V(\phi) \propto \frac{\phi^{4+n}}{M^n} \quad \text{with} \quad n > 1 . \quad (3.4)$$

These are non-renormalizable terms that are generally present in an Effective Field Theory (EFT) and might arise from integrating out heavy physics at a scale M . These will generate large values of ε and η , unless $M \gg M_p$, which appear strange in a QFT context. A solution is to assume some symmetry in the UV-theory prohibiting such terms and leading to a *naturally* flat potential, in the sense of 't Hooft.

More in general, we always want a symmetry protecting small parameters, because terms breaking a symmetry get loop corrections that are also proportional to the symmetry-breaking parameter, so that in the limit of the breaking parameter going to zero, the symmetry is fully recovered also at the quantum level. This way, by slightly breaking a symmetry we can expect the loop corrections to be under control.

While it has been shown the supersymmetry is not enough to protect this parameter [53, 54], the only symmetry we can equip a scalar field with is the shift symmetry:

$$\phi \rightarrow \phi + c, \quad (3.5)$$

which was proposed for the first time by Katherine Freese, Joshua A. Frieman and Angela V. Olinto in 1990 under the name of *natural inflation* [6].

Natural inflation

Whenever a global symmetry is spontaneously broken, Nambu-Goldstone bosons (NGBs) appear and moreover, if this symmetry is only approximate due to terms in the Lagrangian breaking it explicitly, we end up with Pseudo-Nambu-Goldstone bosons (PNGBs). Due to the underlying global shift symmetry, the potential of NGBs is perfectly flat and these particles are massless, but if the symmetry gets broken by quantum corrections, they get a tiny mass becoming PNGBs and the potential is flat only approximately. In axion models, the potential arises from non-perturbative gauge field configurations (instantons) via the chiral anomaly and the loop-induced potential takes typically the following form:

$$V(\phi) = \Lambda^4 \left[1 \pm \cos \left(\frac{\phi}{f} \right) \right] \quad (3.6)$$

where Λ is the scale at which the gauge coupling becomes strong and f is called the *axion decay constant*. The shift symmetry is broken down to a discrete one: $\phi \rightarrow \phi + 2\pi f$. The axion model was initially introduced in the context of QCD in order to solve the strong CP problem [55–57], but we can take advantage of the same theoretical apparatus by considering the inflaton to be an axion so that we have a naturally flat potential.

Beyond natural inflation

As mentioned in the end of the previous chapter, Natural inflation has been excluded by Planck data since the value of n_s predicted is too small, but the fundamental concept of axion inflation has led to an entire class of models [58].

In order to neglect quantum gravity effects and due to limits imposed by string theory [7], we have that $f < M_p$, but this would lead to a potential for natural inflation too steep to ensure slow-roll dynamics, hence a model, in which coupling the axion to gauge fields leads to some friction slowing down the inflaton, has been proposed.

The main focus of this thesis is the model examined by Mohamed M. Anber and Lorenzo Sorbo [8, 59] in which the inflaton is coupled to $U(1)$ gauge fields via a Chern–Simons-type term:

$$-\frac{\phi}{4f} F \tilde{F} \quad (3.7)$$

appearing when a chiral anomaly is involved in the UV-completion. The $U(1)$ gauge symmetry we are considering does not have to be the electromagnetic one, even though we will adopt terminology associated with electromagnetism, e.g. we will define electric and magnetic fields.

3.1 The model

The model studied in detail in this work is described by the following action:

$$S[\phi, A_\mu] = \int d^4x \sqrt{-g} \left[\frac{M_p^2}{2} R - \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{\phi}{4f} F_{\mu\nu} \tilde{F}^{\mu\nu} \right], \quad (3.8)$$

where $g_{\mu\nu}$ is the spacetime metric, g its determinant, $V(\phi)$ is a generic potential for the inflaton field, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field strength tensor associated to the $U(1)$ gauge vector field A_μ , $\tilde{F}^{\mu\nu} = \varepsilon^{\mu\nu\alpha\beta} F_{\alpha\beta} / (2\sqrt{-g})$ is its dual, f is the axion decay constant and can also be written as $\frac{1}{f} = \frac{\beta}{M_p}$ where $\beta > 0$ is a dimensionless parameter¹, $\varepsilon^{\mu\nu\alpha\beta}$ is the totally antisymmetric Levi-Civita symbol normalized as $\varepsilon^{0123} = +1$.

We notice that the axion-gauge field interaction term is fully independent of the metric tensor, but thanks to the properties of the Levi-Civita symbol the integral is still invariant under coordinate transformations. This term is also shift symmetric up to boundary terms, since

$$\frac{\phi}{4} \varepsilon^{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta} = \phi \varepsilon^{\mu\nu\alpha\beta} \partial_\mu A_\nu \partial_\alpha A_\beta = -\partial_\mu \phi \varepsilon^{\mu\nu\alpha\beta} A_\nu \partial_\alpha A_\beta + (\text{total derivative}). \quad (3.9)$$

We remark that gauge invariance is still preserved even though we see A_ν appearing explicitly outside the field strength tensor $F_{\mu\nu}$.

We assume the inflationary background to be homogeneous, isotropic and spatially flat so that it is described by the FLRW metric. The line element, written in terms of cosmic (t) and conformal (τ) time is:

$$ds^2 = -dt^2 + a^2(t) d\mathbf{x}^2 = a^2(\tau) [-d\tau^2 + d\mathbf{x}^2]. \quad (3.10)$$

3.1.1 Derivation of the classical equations of motion

By taking the variational derivative of the action we can deduce the equations of motion for the matter fields:

$$\frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} g^{\mu\nu} \partial_\nu \phi) - V'(\phi) - \frac{1}{4f} F_{\mu\nu} \tilde{F}^{\mu\nu} = 0, \quad (3.11)$$

$$\frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} F^{\mu\nu}) + \frac{1}{f} \tilde{F}^{\mu\nu} \partial_\mu \phi = 0, \quad (3.12)$$

where $V'(\phi) = \partial V(\phi) / \partial \phi$ and we have used the Bianchi identity:

$$\frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} \tilde{F}^{\mu\nu}) = 0. \quad (3.13)$$

By choosing the temporal gauge, $A_0 = 0$, we can define electric and magnetic fields in the following way

$$\mathbf{E} := -\frac{1}{a} \frac{\partial \mathbf{A}}{\partial t} = -\frac{1}{a^2} \frac{\partial \mathbf{A}}{\partial \tau}, \quad \mathbf{B} := \frac{1}{a^2} \nabla \times \mathbf{A}. \quad (3.14)$$

The field strength tensor and its dual can be related to the $\mathbf{E}$ and $\mathbf{B}$ fields as follows:

¹While f represents the scale at which we expect this effective field theory to break down, β is the coupling strength of the interaction between the axion and the gauge fields. Since works adopting both conventions are analyzed in this thesis we will switch from one to the other in the following chapters.

$$F_{0i} = -aE^i, \quad F_{ij} = a^2\varepsilon_{ijk}B^k, \quad \tilde{F}_{0i} = -aB^i, \quad \tilde{F}_{ij} = -a^2\varepsilon_{ijk}E^k, \quad (3.15)$$

where ε_{ijk} is the Levi-Civita symbol in three dimensions. The energy-momentum tensor takes the following form

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi + F_\mu{}^\lambda F_{\nu\lambda} - g_{\mu\nu} \left[\frac{1}{2}\partial_\alpha\phi\partial^\alpha\phi + V(\phi) + \frac{1}{4}F_{\alpha\beta}F^{\alpha\beta} \right] \quad (3.16)$$

where the energy density is given by

$$\rho = T_{\mu\nu}u^\mu u^\nu = T_{00} = \frac{1}{2}\dot{\phi}^2 + \frac{(\nabla\phi)^2}{2a^2} + V(\phi) + \frac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2) \quad (3.17)$$

where the 4-velocity is $u^\mu = \delta^{\mu 0}$ by choosing cosmic time. The pressure can be found in the following way

$$p = \frac{1}{3}T_{\mu\nu}\gamma^{\mu\nu} = \frac{1}{2}\dot{\phi}^2 - \frac{(\nabla\phi)^2}{6a^2} - V(\phi) + \frac{1}{6}(\mathbf{E}^2 + \mathbf{B}^2), \quad (3.18)$$

where $\gamma_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu$. The equation of state and the Friedmann equation then are

$$w = \frac{p}{\rho} = \frac{\frac{1}{2}\dot{\phi}^2 - \frac{(\nabla\phi)^2}{6a^2} - V(\phi) + \frac{1}{6}(\mathbf{E}^2 + \mathbf{B}^2)}{\frac{1}{2}\dot{\phi}^2 + \frac{(\nabla\phi)^2}{2a^2} + V(\phi) + \frac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2)} \quad (3.19)$$

$$(3.20)$$

$$H^2 = \frac{\rho}{3M_p^2}. \quad (3.21)$$

The classical equations of motion describing the evolution of the axion and of gauge fields acquire the form:

$$\ddot{\phi} + 3H\dot{\phi} - \frac{\nabla^2\phi}{a^2} + V'(\phi) = \frac{1}{f}\mathbf{E} \cdot \mathbf{B}, \quad (3.22)$$

$$\dot{\mathbf{E}} + 2H\mathbf{E} - \frac{1}{a}\nabla \times \mathbf{B} + \frac{\dot{\phi}}{f}\mathbf{B} - \frac{1}{f}\nabla\phi \times \mathbf{E} = 0, \quad (3.23)$$

$$\dot{\mathbf{B}} + 2H\mathbf{B} + \frac{1}{a}\nabla \times \mathbf{E} = 0, \quad (3.24)$$

$$\nabla \cdot \mathbf{E} = -\frac{1}{f}\nabla\phi \cdot \mathbf{B}, \quad \nabla \cdot \mathbf{B} = 0. \quad (3.25)$$

3.1.2 Derivation of the equations of motion in the homogeneous approximation

The set of equations, (3.21) - (3.25), forms a self-consistent set, but it is very computationally expensive to solve numerically since they are vector valued and due to the presence of inflaton gradients that can be taken into account only by employing lattice field theory [60, 61]. Although lattice field theory is able to take fully into account the non-linearities, it is a limited approach since the evolution of the system can be sustained for only a few e-folds. In order to be more efficient, we can make the simplifying assumption of exact homogeneity of the axion field, $\phi = \phi(t)$, making it exclusively time-dependent.

Since the full treatment is quantum, we interpret the homogeneous field to be the classical background, $\phi(t) = \langle \hat{\phi}(t, \mathbf{x}) \rangle$, hence, neglecting axion perturbations the set of equations reduces to

$$H^2 = \frac{1}{3M_p^2} \left[\frac{1}{2} \dot{\phi}^2 + V(\phi) + \frac{1}{2} \langle \mathbf{E}^2 + \mathbf{B}^2 \rangle \right] \quad (3.26)$$

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = \frac{1}{f} \langle \mathbf{E} \cdot \mathbf{B} \rangle, \quad (3.27)$$

$$\dot{\mathbf{E}} + 2H\mathbf{E} - \frac{1}{a} \nabla \times \mathbf{B} + \frac{\dot{\phi}}{f} \mathbf{B} = 0, \quad (3.28)$$

$$\dot{\mathbf{B}} + 2H\mathbf{B} + \frac{1}{a} \nabla \times \mathbf{E} = 0, \quad (3.29)$$

$$\nabla \cdot \mathbf{E} = 0, \quad \nabla \cdot \mathbf{B} = 0, \quad (3.30)$$

where in equations (3.26) and (3.27) we highlight the use of the quantum expectation values $\langle \dots \rangle$.

Quantization of the vector field

In order to compute these expectation values we need to quantize the gauge vector field. Noticing that in the homogeneous approximation we can also implement the Coulomb gauge, $\nabla \cdot \mathbf{A} = 0$, on the top of the temporal one², we decompose the $U(1)$ vector field in Fourier modes characterized by just two polarizations:

$$\hat{\mathbf{A}}(t, \mathbf{x}) = \int \frac{d^3 \mathbf{k}}{(2\pi)^{3/2}} \sum_{\lambda=\pm} \left[\epsilon^\lambda(\mathbf{k}) \hat{a}_{\mathbf{k},\lambda} A_\lambda(t, k) e^{i\mathbf{k} \cdot \mathbf{x}} + \epsilon^{\lambda*}(\mathbf{k}) \hat{a}_{\mathbf{k},\lambda}^\dagger A_\lambda^*(t, k) e^{-i\mathbf{k} \cdot \mathbf{x}} \right] \quad (3.31)$$

where A_λ is the mode function, $\epsilon^\lambda(\mathbf{k})$ is the polarization operator and $\hat{a}_{\mathbf{k},\lambda}$ and $\hat{a}_{\mathbf{k},\lambda}^\dagger$ are the annihilation and creation operators, respectively, for gauge modes of momentum $\mathbf{k}$ and circular polarization $\lambda = \pm$. The ladder operators obey the following commutation relation

$$[\hat{a}_{\mathbf{k},\lambda}, \hat{a}_{\mathbf{k}',\lambda'}^\dagger] = \delta_{\lambda\lambda'} \delta^{(3)}(\mathbf{k} - \mathbf{k}'). \quad (3.32)$$

The polarization operators

The polarization operators are characterized by the following properties:

$$i\mathbf{k} \times \epsilon^\lambda(\mathbf{k}) = \lambda k \epsilon^\lambda(\mathbf{k}), \quad \mathbf{k} \cdot \epsilon^\lambda(\mathbf{k}) = 0, \quad \epsilon^{\lambda*}(\mathbf{k}) \cdot \epsilon^{\lambda'}(\mathbf{k}) = \delta^{\lambda\lambda'}, \quad \epsilon^{\lambda*}(\mathbf{k}) = \epsilon^{-\lambda}(\mathbf{k}), \quad (3.33)$$

where we observe that, by the first property, the two polarizations will not mix in eqs. (4.26) and (4.27), while the equations in (3.30) are automatically satisfied thanks to the second property. Notice that the formulas in (3.33) are not independent one from the other: by taking the dot product with $\mathbf{k}$ of the first relation we recover the second one and the $\lambda \neq \lambda'$ case of the third property can be recovered as well from the first relation, instead, by taking the dot product of the first relation with respect to $\epsilon^\lambda(\mathbf{k})$. A choice of polarization operators satisfying (3.33) reads

$$\epsilon^\lambda(\mathbf{k}) = \frac{1}{\sqrt{2}} (\cos \theta \cos \varphi - i\lambda \sin \varphi, \cos \theta \sin \varphi + i\lambda \cos \varphi, -\sin \theta)^T \quad (3.34)$$

for a given unit vector $\hat{\mathbf{k}} = \mathbf{k}/k = (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta)$, where in spherical coordinates θ is the polar angle and φ the azimuthal one.

²By taking the variation of (3.8) with respect to δA_0 and by setting $A_0 = 0$, in the homogeneous approximation we find $\partial_t(\nabla \cdot \mathbf{A}) = 0$ from which we can fix the Coulomb gauge.

We can show that the properties in (3.33) fix the polarization operators up to a phase. For each momentum $\mathbf{k}$, we can write any vector as a complex linear combination of an orthogonal triplet of unit vectors, of which $\hat{\mathbf{k}}$ is part. Then, we focus on just the $+$ polarization, since by complex conjugation we can find the other operator thanks to the fourth property:

$$\epsilon^+ = a \hat{\mathbf{k}} + b \mathbf{e}_1 + c \mathbf{e}_2 \quad \text{with} \quad a, b, c \in \mathbb{C}, \quad (3.35)$$

where $\mathbf{e}_1$ and $\mathbf{e}_2$ are orthogonal unit vectors fixed up to a $\text{SO}(2)$ rotations on the two dimensional surface orthogonal to $\hat{\mathbf{k}}$. The following applies:

$$\mathbf{k} \cdot \epsilon^+(\mathbf{k}) = 0 \implies a = 0, \quad (3.36)$$

$$i\mathbf{k} \times \epsilon^+(\mathbf{k}) = \lambda k \epsilon^+(\mathbf{k}) \implies -ib \mathbf{e}_2 + ic \mathbf{e}_1 = b \mathbf{e}_1 + c \mathbf{e}_2 \implies c = -ib, \quad (3.37)$$

$$\epsilon^{+*} \cdot \epsilon^+ = 1 \implies |b|^2 + |(-ib)|^2 = 2|b|^2 = 1, \quad (3.38)$$

where in the last line we have used both the third and fourth properties. Therefore, we conclude that the polarization operators are fixed up to a phase factor, matching the freedom under rotations we had in the choice of $\mathbf{e}_1$ and $\mathbf{e}_2$.

Equation of motion for the mode functions

The mode functions satisfy the Wronskian normalization condition

$$A_\lambda(t, k) \dot{A}_\lambda^*(t, k) - A_\lambda^*(t, k) \dot{A}_\lambda(t, k) = \frac{i}{a}, \quad (3.39)$$

so that the canonical Bohr-Heisenberg commutation relation for $\hat{\mathbf{A}}(t, \mathbf{x})$ and its conjugate momentum are properly normalized. The equation of motion for the mode functions is given with respect to cosmic and conformal time respectively by

$$\ddot{A}_\lambda(t, k) + H \dot{A}_\lambda(t, k) + \left[\frac{k^2}{a^2} - 2\lambda \frac{k}{a} H \xi \right] A_\lambda(t, k) = 0 \quad (3.40)$$

$$A_\lambda''(\tau, k) + [k^2 - 2\lambda a H \xi k] A_\lambda(\tau, k) = 0 \quad (3.41)$$

where we define the ξ variable as follows:

$$\xi := \frac{\dot{\phi}}{2fH} = \frac{\beta \dot{\phi}}{2HM_p} \quad (3.42)$$

and we will refer to it also as the *instability variable*.

Considering ξ to be positive we notice that the plus polarization mode undergoes a tachyonic instability for $k < 2aH\xi$, while if ξ were negative it would be the minus polarization to be tachyonically amplified. Consider the equation of motion of a harmonic oscillator,

$$\ddot{q} + \omega^2 q = 0. \quad (3.43)$$

Its solutions are given by a linear combination of $e^{\pm i\omega t}$ for $\omega^2 > 0$, instead we say that a tachyonic instability arises when the mass parameter ω^2 turns negative:

$$\ddot{q} - |\omega^2| q = 0. \quad (3.44)$$

We can rewrite this equation as a harmonic oscillator with imaginary frequencies: $\ddot{q} + i^2|\omega|^2 q = 0$, hence its solutions are clearly linear combinations of exponentials: $e^{\mp|\omega|t}$. Some solutions of eq. (3.41) will experience the tachyonic instability and get exponentially amplified, while all of the others will just keep oscillating.

In the limit $k \gg aH$, we notice that eq. (3.41) becomes

$$A_\lambda''(\tau, k) + k^2 A_\lambda(\tau, k) \approx 0, \quad (3.45)$$

hence we can employ the Bunch-Davies initial conditions $A_\lambda(\tau, k) = e^{-ik\tau}/\sqrt{2k}$.

By assuming $H = \text{const}$ and $\dot{\phi} = \text{const}$, which may be approximately realized during slow-roll inflation, eq. (3.41) reduces to

$$A_\lambda''(\tau, k) + \left[k^2 + \frac{2\lambda k \xi}{\tau} \right] A_\lambda(\tau, k) = 0 \quad (3.46)$$

where $\tau = -1/aH$ in de Sitter spacetime. Eq. (3.46) has the form of the Whittaker equation [62] and admits an analytical solution given by the regular and irregular Coulomb functions corresponding to the positive frequency for $-k\tau > 0$:

$$A_\lambda(\tau, k) = \frac{1}{\sqrt{2k}} [G_0(\lambda\xi, -k\tau) + iF_0(\lambda\xi, -k\tau)]. \quad (3.47)$$

In the subdomain $-k\tau > 0$ the analytical solution can be written in terms of the Whittaker W -function:

$$A_\lambda(\tau, k) = \frac{1}{\sqrt{2k}} e^{\lambda\pi\xi/2} W_{i\lambda\xi, \frac{1}{2}}(-2ik\tau). \quad (3.48)$$

Note that this solution is symmetric under the exchanges $A_+ \rightarrow A_-$ and $\xi \rightarrow -\xi$ as we expected. We will refer to (3.48) as the Anber-Sorbo solution (AS), after the names of the first two authors that implemented it in an inflationary context [8, 59]. We will discuss this solution with greater depth in the next section of the chapter.

The energy density and backreaction expectation values

The expectation value representing the gauge field energy density that appears in the Friedmann equation, (3.26), in terms of the mode functions is given by

$$\begin{aligned} \frac{\langle \mathbf{E}^2 + \mathbf{B}^2 \rangle}{2} &= \frac{1}{4\pi^2 a^4} \int dk I_1(k) \\ &= \frac{1}{4\pi^2 a^4} \int dk \sum_{\lambda=\pm} \left(k^4 |A_\lambda|^2 + k^2 a^2 |\dot{A}_\lambda|^2 \right). \end{aligned} \quad (3.49)$$

The expectation value in the axion equation, (3.27), in terms of the mode functions is

$$\begin{aligned} \langle \mathbf{E} \cdot \mathbf{B} \rangle &= -\frac{1}{4\pi^2 a^3} \int dk I_2(k) \\ &= -\frac{1}{4\pi^2 a^3} \int dk \sum_{\lambda=\pm} \lambda k^3 \frac{\partial}{\partial t} |A_\lambda(t, k)|^2 \end{aligned} \quad (3.50)$$

and we will call it backreaction integral.

Both these integrals are divergent in the UV. By substituting the Anber-Sorbo solution in the two integrals we find that the energy density integral has quartic, quadratic and logarithmic divergencies in the UV³:

$$I_1(k) \approx 2k^3 + \frac{\xi^2 k}{\tau^2} + \frac{3\xi^2(-1 + 5\xi^2)}{4\tau^2 k} + \mathcal{O}\left(\frac{1}{k^3}\right) \quad \text{for } -k\tau \rightarrow +\infty, \quad (3.51)$$

while the backreaction integral has only quadratic and logarithmic divergencies:

$$I_2(k) \approx \frac{\xi k}{\tau^2} - \frac{3\xi(1 - 5\xi^2)}{2\tau^4 k} + \mathcal{O}\left(\frac{1}{k^3}\right) \quad \text{for } -k\tau \rightarrow +\infty, \quad (3.52)$$

where we used the following approximation of the Whittaker function $W_{\kappa,\mu}(z)$ with $|\arg z| < \frac{3\pi}{2}$

$$W_{\kappa,\mu}(z) \approx e^{-z/2} z^\kappa \left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n (\mu - \kappa + \frac{1}{2})_n (-\mu - \kappa + \frac{1}{2})_n}{n! z^n} \right] \quad \text{for } |z| \rightarrow +\infty, \quad (3.53)$$

where $(a)_n = a(a+1)\dots(a+n-1)$ is the Pochhammer symbol and the following expression for the derivative of $W_{\kappa,\mu}(z)$:

$$\frac{\partial W_{\kappa,\mu}(z)}{\partial \ln z} = \left(\frac{z}{2} - \kappa\right) W_{\kappa,\mu}(z) - W_{1+\kappa,\mu}(z). \quad (3.54)$$

By looking instead at the IR limit we observe that both integrands vanish. In the limit $-k\tau \rightarrow 0$, we can approximate the Whittaker function and its derivative with respect to conformal time as follows:

$$W_{i\lambda\xi, \frac{1}{2}}(-2ik\tau) \approx \frac{e^{-ik\tau}}{\Gamma(1 - i\lambda\xi)}, \quad (3.55)$$

$$W'_{i\lambda\xi, \frac{1}{2}}(-2ik\tau) \approx -2\lambda\xi k\tau \ln(-2k\tau) \frac{e^{-ik\tau}}{\tau \Gamma(1 - i\lambda\xi)}, \quad (3.56)$$

where $\Gamma(z)$ is the Euler gamma function. Substituting in the AS solution we find

$$A_\lambda(\tau, k) \approx \frac{e^{-ik\tau}}{\sqrt{2k}} \frac{e^{\pi\xi/2}}{\Gamma(1 - i\lambda\xi)}, \quad -k\tau \rightarrow 0, \quad (3.57)$$

while its derivative clearly vanishes. Hence, we conclude that both integrals converge in the IR.

To compute this result we rewrote the Whittaker function in terms of the Kummer U -function:

$$W_{\kappa,\mu}(z) = e^{-z/2} z^{\mu+1/2} U\left(\frac{1}{2} + \mu - \kappa, 1 + 2\mu, z\right), \quad -\pi < \arg z \leq \pi, \quad (3.58)$$

and employed its logarithmic representation:

$$\begin{aligned} U(a, n+1, z) = & \frac{(-1)^{n+1}}{n! \Gamma(a-n)} \left[M(a, n+1, z) \ln z + \right. \\ & + \sum_{r=0}^{+\infty} \frac{(a)_r z^r}{(n+1)_r r!} (\psi(a+r) - \psi(1+r) - \psi(1+n+r)) \left. \right] \\ & + \frac{(n-1)!}{\Gamma(a)} z^{-n} M(a-n, 1-n, z)_n \quad \text{for } n \text{ non-negative integer,} \end{aligned} \quad (3.59)$$

³The divergent behaviour of these integrals is computed exactly by imposing a cutoff in [63]

where $\psi(z) = \Gamma'(z)/\Gamma(z)$ is the Digamma function and the last function is the sum up to n terms of Kummer M -function which is defined as follows:

$$M(a, b, z) := 1 + \frac{az}{b} + \frac{(a)_2 z^2}{(b)_2 2!} + \dots + \frac{(a)_n z^n}{(b)_n n!} + \dots \quad (3.60)$$

The appearance of UV divergencies is a common feature of QFT and it is solved by employing a renormalization scheme, therefore, to treat axion inflation with $U(1)$ gauge fields comprehensively, we should renormalize the theory. This possibility is carefully analyzed in [63] where a proper renormalization scheme is implemented. The issue lies in finding a systematic way to distinguish physically relevant modes from modes that are just vacuum fluctuations contributing to a divergent vacuum energy.

In this work, we consider a mode to be physically relevant if it has been affected by the tachyonic amplification at some point during its evolution, while all of the other oscillatory modes are considered within the vacuum and neglected. A mode is unstable when its momentum satisfies

$$k \leq k_{thr,\lambda}(t) = \max\{2\lambda a(t)H(t)\xi(t), 0\}, \quad (3.61)$$

then a mode is taken into account in an integral if it has undergone the tachyonic instability prior to the time we are evaluating the integral at, i.e. if its momentum satisfies

$$k \leq k_{h,\lambda}(t) = \max_{t' \leq t} k_{thr,\lambda}(t') = \max \left\{ \max_{t' \leq t} \{2\lambda a(t')H(t')\xi(t')\}, 0 \right\}. \quad (3.62)$$

We observe that if the instability parameter stays positive throughout the whole evolution of the system, then the cutoff of the negative polarization modes $k_{h,-}(t)$ vanishes $\forall t$. The reverse is true if $\xi(t)$ is always negative.

The final system of equations

Ultimately, the system of equations governing the dynamics of $a(t)$, $\phi(t)$ and $A_{\pm}(t, k)$ is

$$\left\{ \begin{aligned} H^2 &= \frac{1}{3M_p^2} \left[\frac{1}{2} \dot{\phi}^2 + V(\phi) + \frac{1}{2} \langle \mathbf{E}^2 + \mathbf{B}^2 \rangle \right] \\ &= \frac{1}{3M_p^2} \left[\frac{1}{2} \dot{\phi}^2 + V(\phi) + \frac{1}{4\pi^2 a^4} \sum_{\lambda=\pm} \int_0^{k_{h,\lambda}(t)} dk \left(k^4 |A_{\lambda}|^2 + k^2 a^2 |\dot{A}_{\lambda}|^2 \right) \right] \\ \ddot{\phi} + 3H\dot{\phi} + V'(\phi) &= \frac{1}{f} \langle \mathbf{E} \cdot \mathbf{B} \rangle \\ &= -\frac{1}{4\pi^2 f a^3} \sum_{\lambda=\pm} \int_0^{k_{h,\lambda}(t)} dk \lambda k^3 \frac{\partial}{\partial t} |A_{\lambda}(t, k)|^2 \\ \ddot{A}_{\lambda}(t, k) + H\dot{A}_{\lambda}(t, k) + \left[\frac{k^2}{a^2} - 2\lambda \frac{k}{a} H\xi \right] A_{\lambda}(t, k) &= 0 \end{aligned} \right. \quad (3.63)$$

where both polarizations are taken into account since we are not making any assumption on the shape of the potential.

We will refer to the R.H.S. of the axion equation as the *gauge friction* term, because it is implicitly proportional to $\dot{\phi}$ and opposes the potential gradient $V'(\phi)$. Let us intuitively show the effect of this term on the dynamics. Starting from a standard slow-roll regime in which the gauge friction and $\dot{\phi}$ are

negligible, we notice that a negative slope of the potential corresponds to $\dot{\phi} > 0$, hence the instability variable is likewise positive inducing a tachyonic amplification of the plus polarization mode. In this way, the $\lambda = +$ contribution of the gauge friction grows, resulting in a negative R.H.S. of the axion equation, which counteracts the negative slope of the potential. If instead $V'(\phi) > 0$, the minus polarization mode gets amplified leading to a positive gauge friction, hence, the opposite applies as well.

3.2 Anber-Sorbo solution

We report here the first theoretical argument for a slow-roll inflationary epoch sustained by an axion coupled to $U(1)$ gauge fields [8]. In their work, M. Anber and L. Sorbo examine a model characterized by the following axion potential and axion-vector interaction Lagrangian:

$$V(\phi) = \Lambda^4 \left[1 + \cos\left(\frac{\phi}{f}\right) \right], \quad \mathcal{L}_{int} = -\frac{\alpha}{4f} \phi F_{\mu\nu} \tilde{F}^{\mu\nu} \quad (3.64)$$

and we will maintain their notation for the interaction coupling only in the current section.

Let us focus on the case $|\xi| \gtrsim \mathcal{O}(1)$, where the instability variable is defined as $\xi_{AS \text{ paper}} := \alpha \dot{\phi} / (2fH)$. The slow-roll regime can be sustained only within a monotonic section of the potential, since if the axion crossed a local minimum, then $V' = 0$ and it would stop moving. Therefore, without loss of generality we can assume that $\dot{\phi} > 0$, implying $V' < 0$ and $\xi > 0$, throughout the whole evolution of the system. In this way, only the plus polarization mode, A_+ , will develop a tachyonic instability, hence we neglect A_- in our computations.

If in eq. (3.41) the second term in brackets is dominating over the first one

$$A''_+(\tau, k) + [k^2 - 2aH\xi k] A_+(\tau, k) \approx A''_+(\tau, k) - (2aH\xi k) A_+(\tau, k) \approx 0 \quad (3.65)$$

then the full analytical solution and its derivative can be approximated with [8]

$$A_+(\tau, k) \approx \frac{1}{\sqrt{2k}} \left(\frac{-k}{2\xi aH} \right)^{1/4} e^{\pi\xi - 2\sqrt{2\xi k/aH}} \quad (3.66)$$

$$A'_+(\tau, k) \approx \sqrt{2k\xi aH} A_+(\tau, k). \quad (3.67)$$

In order to select the physical gauge modes, we cut off the backreaction integral at $k_{thr}(\tau) = 2\xi H a(\tau)$ and employing (3.66) and (3.67) we get

$$\langle \mathbf{E} \cdot \mathbf{B} \rangle \approx - \left(\frac{H}{\xi} \right)^4 e^{2\pi\xi} \times \left[\frac{1}{2^{21}\pi^2} \int_0^{8\xi} dx x^7 e^{-x} \right] \approx 2.4 \times 10^{-4} \left(\frac{H}{\xi} \right) e^{2\pi\xi}, \quad (3.68)$$

where, to give the final numerical estimate, we sent the upper limit of the integral to infinity since $\xi \gtrsim 1$ and factor in square brackets is $\mathcal{I} := 7!/(2^{21}\pi^2) \approx 2.4 \times 10^{-4}$, meaning that the approximate expressions in (3.66) and (3.67) actually regulate the integral by themselves, without needing a cutoff.

Applying the same procedure we can compute the energy density integral:

$$\frac{\langle \mathbf{E}^2 + \mathbf{B}^2 \rangle}{2} \approx \frac{e^{2\pi\xi}}{2^{19}\pi^2} \frac{H^4}{\xi^3} \times \left[\int_0^{8\xi} dx x^6 e^{-x} \right] \approx 1.4 \times 10^{-4} \frac{H^4}{\xi^3} e^{2\pi\xi}. \quad (3.69)$$

Noticing that both integrals are constant in this regime, we can conclude that the gauge field production is balancing their dissipation due to the expansion of the universe.

Substituting (3.68) into the axion equation, we obtain

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = -\frac{\mathcal{I}\alpha}{f} \left(\frac{H}{\xi}\right)^4 e^{2\pi\xi}. \quad (3.70)$$

Focusing on inflationary scenarios in which the slow-roll regime is sustained by gauge friction we expect that both the acceleration $\ddot{\phi}$ and the Hubble friction $3H\dot{\phi}$ are negligible with respect to the slope V' , so that we can estimate ξ as follows:

$$\xi \approx \frac{1}{2\pi} \log \left[\frac{9}{\mathcal{I}\alpha} \frac{M_p^4 f |V'(\phi)|}{V^2(\phi)} \right] \quad (3.71)$$

where we have assumed $3M_p^2 H^2 \approx V(\phi)$. If ϕ is not rolling close to an extremum (and α is not exponentially large or small) we can approximate $|V'/V| \approx 1/f$ and $V \approx \Lambda^4$ in order to estimate

$$\xi \approx \frac{2}{\pi} \log \left[\frac{M_p}{\Lambda} \right]. \quad (3.72)$$

Due to the logarithmic dependence, we expect ξ to be $\mathcal{O}(10)$: for $\Lambda \approx 10^7 \text{ GeV}$, we obtain $\xi \approx 20$, while for $\Lambda \approx 10^{16} \text{ GeV}$ $\xi \approx 5$.

Let us now explore the constraints deriving from the slow-roll requirement. In order to approximate $H^2 \approx V/(3M_p^2)$ we must require the kinetic and the electromagnetic energies to be negligible with respect to the potential energy. Exploiting the approximate axion equation, $\mathcal{I}\alpha(H/\xi)^4 e^{2\pi\xi} \approx f|V'(\phi)|$, we estimate the electromagnetic energy with

$$\frac{\langle \mathbf{E}^2 + \mathbf{B}^2 \rangle}{2} \approx \frac{6! e^{2\pi\xi}}{2^{19} \pi^2} \frac{H^4}{\xi^3} \simeq \frac{4}{7} \frac{\xi}{\alpha} f V'(\phi), \quad (3.73)$$

hence as long as $\alpha \gg \xi$ this term is negligible, since $|fV'/V| \approx \mathcal{O}(1)$ far from the extremal points of the potential. Regarding the kinetic energy, we obtain

$$\frac{\dot{\phi}^2}{2V(\phi)} = 2 \frac{\xi^2 f^2 H^2}{\alpha^2 V(\phi)} \approx \frac{2}{3} \frac{\xi^2}{\alpha^2} \frac{f^2}{M_p^2}, \quad (3.74)$$

where we have employed $\dot{\phi} = 2fH\xi/\alpha$ and the approximate Friedmann equation. As previously stated, in order to neglect quantum gravity effects and to contemplate UV completions coming from string theory, we must ask that $f < M_p$, hence we notice that requiring $\alpha \gg \xi$ makes the kinetic energy negligible with respect to the potential one.

In order to have an inflationary epoch, the first slow-roll condition $\varepsilon < 1$ must be satisfied. Employing the equations of motion (3.26)-(4.27) we derive the following expression:

$$\varepsilon = \frac{1}{2M_p^2 H^2} \left[\dot{\phi}^2 + \frac{2}{3} \langle \mathbf{E}^2 + \mathbf{B}^2 \rangle + \frac{\langle \nabla \cdot (\mathbf{E} \times \mathbf{B}) \rangle}{3aH} \right], \quad (3.75)$$

where due to isotropy of the background $\langle \nabla \cdot (\mathbf{E} \times \mathbf{B}) \rangle = 0$. Then, using the values of $\dot{\phi}$, $\langle \mathbf{E}^2 + \mathbf{B}^2 \rangle$ and H^2 found earlier, we obtain

$$\varepsilon \approx \frac{2\xi^2}{\alpha^2} \frac{f^2}{M_p^2} + \frac{8}{7} \frac{\xi}{\alpha} \frac{f V'(\phi)}{V(\phi)}, \quad (3.76)$$

hence, thanks to the same condition required above the first slow-roll condition is satisfied.

As we observed earlier in the estimate of ξ , we expect $\ddot{\phi}$ and $3H\dot{\phi}$ to be negligible with respect to V' . This requirement translates into the following relations:

$$\frac{3H\dot{\phi}}{V'} \approx \frac{\xi}{2\alpha} \frac{f V/V'}{M_P^2} \ll 1, \quad (3.77)$$

$$\frac{\ddot{\phi}}{V'} \approx \frac{2\xi}{3\alpha} \left(-\varepsilon \frac{f V/V'}{M_P^2} + \frac{f^2}{\pi M_P^2} \frac{V V''/V'^2 - 2}{\alpha} \right) \ll 1. \quad (3.78)$$

Both these conditions are satisfied again if $\alpha \gg \xi$, since $|f V/V'| \approx f^2$ and $V V''/V'^2 \approx 1$.

The strongest requirement comes from asking inflation to last long enough. The number of e-folds can be estimated as follows:

$$N = \int_{\phi_i}^{\phi_f} \frac{H d\phi}{\dot{\phi}} = \frac{\alpha}{2f} \int_{\phi_i}^{\phi_f} \frac{d\phi}{\xi} \approx \frac{\alpha}{2\xi} \frac{\phi_f - \phi_i}{f}. \quad (3.79)$$

Because the largest field variation is bounded from above in this model, $|\phi_f - \phi_i| \lesssim \pi f$, eq. (3.79) implies

$$\alpha \gtrsim \frac{2}{\pi} \xi N, \quad (3.80)$$

hence for $\xi \approx 5$, we need $\alpha > 200$ to have at least 60 e-folds of inflation.

3.3 Gravitational waves production

One of the most relevant phenomenological features of this model is the generation of gravitational waves (GW) induced by gauge field production. In this section we outline the GW creation mechanism following the treatment in [10, 64, 65].

We work with the perturbed FLRW metric, keeping only the transverse and traceless metric perturbations, h_{ij}^{TT} , up to first order:

$$ds^2 = a^2(\tau) [-d^2\tau + (\delta_{ij} + h_{ij}^{TT}) dx^i dx^j]. \quad (3.81)$$

Expanding the full action, (3.8), up to second order, we obtain

$$S_{GW} = \int d^4x \left[\frac{M_p^2 a^2}{8} (h'_{ij} h'_{ij} - \partial_k h_{ij} \partial_k h_{ij}) - \frac{a^4}{2} h_{ij} (E_i E_j + B_i B_j)^{TT} \right], \quad (3.82)$$

where the superscript TT indicates the transverse and traceless projection of the tensor in brackets. Since the tensor perturbations display only two degrees of freedom, we can decompose h_{ij}^{TT} into two circular polarization modes in momentum space:

$$h_{ij}^{TT}(\tau, \mathbf{x}) = \int \frac{d^3k}{(2\pi)^{3/2}} e^{i\mathbf{k}\cdot\mathbf{x}} \sum_{\lambda=\pm} \Pi_{ij,\lambda}^*(\mathbf{k}) h_\lambda(\tau, \mathbf{k}) \quad \text{with } \Pi_{ij,\lambda}^*(\mathbf{k}) := \epsilon_i^\lambda(\mathbf{k}) \epsilon_j^\lambda(\mathbf{k}), \quad (3.83)$$

where the polarization operators $\epsilon^\lambda(\mathbf{k})$ are the same as those defined in (3.33). In order to canonically normalize the perturbed action we define the operator $\hat{v}_\lambda := \frac{M_p a}{2} \dot{h}_\lambda$, satisfying the following equation of motion:

$$\hat{v}_\lambda'' + \left[k^2 - \frac{a''}{a} \right] \hat{v}_\lambda = \hat{S}_\lambda(\tau, \mathbf{k}), \quad (3.84)$$

where the source term $\hat{S}_\lambda(\tau, \mathbf{k})$ takes the following form:

$$\hat{S}_\lambda(\tau, \mathbf{k}) := -\frac{a^3}{M_p} \Pi_{ij,\lambda}(\mathbf{k}) \int \frac{d^3x}{(2\pi)^{3/2}} e^{-i\mathbf{k}\cdot\mathbf{x}} \left[\hat{E}_i \hat{E}_j + \hat{B}_i \hat{B}_j \right]. \quad (3.85)$$

The solution of eq. (3.84) can be split into a homogeneous part $\hat{v}_\lambda^0$, obtained neglecting the source term, and a particular one $\hat{v}_\lambda^{ind}$, taking into account the gauge field production:

$$\hat{v}_\lambda = \hat{v}_\lambda^0 + \hat{v}_\lambda^{ind}, \quad (3.86)$$

where the superscript *ind* indicates the GW contribution induced by $U(1)$ gauge fields. Via canonical quantization we decompose the vacuum contribution as follows:

$$\hat{v}_\lambda^0(\tau, \mathbf{k}) = \hat{b}_{\lambda,\mathbf{k}} v_\lambda^0(\tau, k) + \hat{b}_{\lambda,-\mathbf{k}}^\dagger v_\lambda^0(\tau, k)^*, \quad (3.87)$$

where the creation and annihilation operators, $b_{\lambda,\mathbf{k}}^\dagger$ and $\hat{b}_{\lambda,\mathbf{k}}$, fulfill the canonical commutation relations:

$$[\hat{b}_{\lambda,\mathbf{k}}, \hat{b}_{\lambda',\mathbf{k}'}^\dagger] = \delta_{\lambda,\lambda'} \delta^{(3)}(\mathbf{k} - \mathbf{k}') \quad (3.88)$$

and the mode functions satisfy the following equation of motion:

$$v_\lambda^{0''} + \left[k^2 - \frac{a''}{a} \right] v_\lambda^0 = 0. \quad (3.89)$$

On the other hand, the solution for the induced contribution acquires the form:

$$\hat{v}_\lambda^{ind}(\tau, \mathbf{k}) = \int_{-\infty}^{+\infty} d\tau' G_k(\tau, \tau') \hat{S}_\lambda(\tau', \mathbf{k}) \quad (3.90)$$

where G_k satisfies

$$\left[\frac{\partial^2}{\partial \tau^2} + k^2 - \frac{a''}{a} \right] G_k(\tau, \tau') = \delta(\tau - \tau') \quad (3.91)$$

and is called the *retarded Green function*, because, due to causality, we decompose it as follows:

$$G_k(\tau, \tau') = \theta(\tau - \tau') \tilde{G}_k(\tau, \tau') \quad (3.92)$$

where $\theta(\tau - \tau')$ is the Heaviside step function. By imposing the following conditions

$$\tilde{G}_k(\tau, \tau) = 0, \quad \frac{\partial}{\partial \tau} \tilde{G}_k(\tau, \tau') \Big|_{\tau=\tau'} = 1, \quad (3.93)$$

that imply the relation

$$\frac{\partial^2}{\partial \tau^2} \left[\theta(\tau - \tau') \tilde{G}_k(\tau, \tau') \right] = \theta(\tau - \tau') \frac{\partial^2 \tilde{G}_k(\tau, \tau')}{\partial \tau^2} + \delta(\tau - \tau'), \quad (3.94)$$

we deduce the following equation for $\tilde{G}_k$:

$$\left[\frac{\partial^2}{\partial \tau^2} + k^2 - \frac{a''}{a} \right] \tilde{G}_k(\tau, \tau') = 0, \quad (3.95)$$

which has the same form as the equation of the vacuum mode functions.

The dimensionless power spectrum of GWs is defined as:

$$\begin{aligned}\Delta_{GW,\lambda}(\tau, k)\delta^{(3)}(\mathbf{k} + \mathbf{k}') &:= \frac{k^3}{2\pi^2} \langle \hat{h}_\lambda(\tau, \mathbf{k}) \hat{h}_\lambda(\tau, \mathbf{k}') \rangle \\ &= \frac{k^3}{2\pi^2} \frac{4}{M_p^2 a^2(\tau)} \langle \hat{v}_\lambda(\tau, \mathbf{k}) \hat{v}_\lambda(\tau, \mathbf{k}') \rangle.\end{aligned}\quad (3.96)$$

We notice that the vacuum contribution to the power spectrum is proportional to ladder operators $\hat{b}$, $\hat{b}^\dagger$, while in the expression of the induced contribution only the gauge field ladder operators $\hat{a}$, $\hat{a}^\dagger$ appear. The first set of operators commutes with the second one:

$$[a, b] = [a^\dagger, b] = [a, b^\dagger] = [a^\dagger, b^\dagger] = 0, \quad (3.97)$$

hence the two variables $\hat{v}_\lambda^0$ and $\hat{v}_\lambda^{ind}$ are not correlated and we can discern two separate contributions in the variance of $\hat{v}_\lambda$:

$$\langle \hat{v}_\lambda(\tau, \mathbf{k}) \hat{v}_\lambda(\tau, \mathbf{k}') \rangle = \langle \hat{v}_\lambda^0(\tau, \mathbf{k}) \hat{v}_\lambda^0(\tau, \mathbf{k}') \rangle + \langle \hat{v}_\lambda^{ind}(\tau, \mathbf{k}) \hat{v}_\lambda^{ind}(\tau, \mathbf{k}') \rangle. \quad (3.98)$$

Therefore, the GW power spectrum can be decomposed analogously:

$$\Delta_{GW,\lambda} = \Delta_{GW,\lambda}^0 + \Delta_{GW,\lambda}^{ind}. \quad (3.99)$$

Assuming the slow-roll regime, we recover the expression found in the previous chapter for tensor perturbations:

$$\Delta_{GW,\lambda}^0 = \left(\frac{H_*}{\pi M_p} \right)^2 \left(\frac{k}{k_*} \right)^{n_T} \quad (3.100)$$

where $n_T \approx -2\varepsilon$. If the potential becomes steeper, as we expect during the last stages of inflation, this expression is no longer valid, but we suppose that the induced contribution would be dominant during this phase. On the other hand, the variance of $\hat{h}_\lambda^{ind}$ acquires the form:

$$\langle \hat{h}_\lambda^{ind}(\tau, \mathbf{k}) \hat{h}_\lambda^{ind}(\tau, \mathbf{k}') \rangle = \frac{4}{M_p^2} \int d\tau' d\tau'' \frac{G_k(\tau, \tau')}{a(\tau)} \frac{G_k(\tau, \tau'')}{a(\tau)} \langle \hat{S}_\lambda(\tau', \mathbf{k}) \hat{S}_\lambda(\tau'', \mathbf{k}') \rangle, \quad (3.101)$$

then, by using the commutation relations and the following formula

$$\left| \boldsymbol{\epsilon}^\alpha(\mathbf{p}) \cdot \boldsymbol{\epsilon}^\beta(\mathbf{q}) \right| = \frac{1}{2} \left(1 - \alpha\beta \frac{\mathbf{p} \cdot \mathbf{q}}{pq} \right), \quad (3.102)$$

we express the two point correlation function of the source operator as

$$\begin{aligned}\langle \hat{S}_\lambda(\tau', \mathbf{k}) \hat{S}_\lambda(\tau'', \mathbf{k}') \rangle &= \frac{2\delta^{(3)}(\mathbf{k} + \mathbf{k}')}{a(\tau')a(\tau'')M_p^2} \int \frac{d^3p}{(2\pi)^3} \frac{1}{16} \times \\ &\times \sum_{\alpha, \beta = \pm} \left(1 + \lambda\alpha \frac{\mathbf{k} \cdot \mathbf{p}}{kp} \right)^2 \left(1 + \lambda\beta \frac{k^2 - \mathbf{k} \cdot \mathbf{p}}{kq} \right)^2 \times \\ &\times [A'_\alpha(\tau', p) A'_\beta(\tau', q) + \alpha\beta pq A_\alpha(\tau', p) A_\beta(\tau', q)] \times \\ &\times [A^{*'}_\alpha(\tau'', p) A^{*'}_\beta(\tau'', q) + \alpha\beta pq A^*_\alpha(\tau'', p) A^*_\beta(\tau'', q)]\end{aligned}\quad (3.103)$$

with $q := |\mathbf{k} - \mathbf{p}|$.

The full expression for the induced GW power spectrum is:

$$\Delta_{GW,\lambda}^{ind}(\tau, k) = \frac{k^3}{4\pi^2 M_p^4} \int \frac{d^3 p}{(2\pi)^3} \sum_{\alpha, \beta = \pm} \left(1 + \lambda \alpha \frac{\mathbf{k} \cdot \mathbf{p}}{kp} \right)^2 \left(1 + \lambda \beta \frac{k^2 - \mathbf{k} \cdot \mathbf{p}}{kq} \right)^2 \times \left| \int_{-\infty}^{k\tau} d(k\tau') \frac{\tilde{G}_k(\tau, \tau')}{k a(\tau) a(\tau')} [A'_\alpha(\tau', p) A'_\beta(\tau', q) + \alpha \beta p q A_\alpha(\tau', p) A_\beta(\tau', q)] \right|^2. \quad (3.104)$$

Eventually, we are interested in the GW energy-density power spectrum, i.e. the GW energy density per logarithmic frequency in units of the critical energy density ρ_c ,

$$\Omega_{GW}(f) := \frac{1}{\rho_c} \frac{\partial \rho_{GW}(f)}{\partial \ln f}. \quad (3.105)$$

We can relate Ω_{GW} straightforwardly to the power spectrum by taking into account two effects: cosmological redshift and the mode evolution after the source has switched off. The redshift from emission until today is taken care of by $f = k/(2\pi a_0)$, where a_0 is the value of the scale factor today. After a mode k is created, it will exit the horizon at a time τ_{out} during inflation, when $k \approx \mathcal{H}(\tau_{out})$ and then re-enter later at a time τ_{in} after inflation, when $k \approx \mathcal{H}(\tau_{in})$. Assuming that the source term has switched off before horizon re-entry [66] as it is the case in this context, since the gauge fields are no longer produced and they dilute away due to the expansion of the universe, the evolution of a mode after τ_{in} is described by the vacuum equation of motion (3.89). The solution acquires the form of a transfer function relating the initial amplitude set during inflation to the one we predict to measure today [67]. In the case of modes re-entering the horizon during the radiation domination epoch, it reads:

$$|\mathcal{T}_{GW}(f)|^2 \simeq \frac{H_0^2 \Omega_r g_*(T_f)}{8\pi^2 f^2 g_*(T_0)} \left(\frac{g_{*,S}(T_0)}{g_{*,S}(T_f)} \right)^{4/3}, \quad (3.106)$$

where H_0 is the Hubble rate measured today, Ω_r is the fractional energy density of radiation measured today, $g_*(T)$ and $g_{*,S}(T)$ count the effective relativistic degrees of freedom of the heat bath at temperature T for the energy density and the entropy density respectively, T_0 is the temperature of the universe today and T_f the heat bath temperature when the mode of frequency f re-entered the horizon. The conservation of entropy, $g_{*,S}(T) a^3 T^3 = \text{const}$, has also been employed in this computation.

Considering modes re-entering the horizon during an era of domination of radiation, the final result is

$$\Omega_{GW}(f) = \frac{\Omega_r g_*(T_f)}{24 g_*(T_0)} \left(\frac{g_{*,S}(T_0)}{g_{*,S}(T_f)} \right)^{4/3} \sum_{\lambda} \Delta_{GW,\lambda}(\tau_{out}(k_f), k_f), \quad k_f = 2\pi a_0 f. \quad (3.107)$$

We can estimate the maximum frequency of GWs generated during inflation by considering the last mode to leave the horizon, i.e. with $k = a_{end} H_{end}$:

$$f_{end} = \frac{H_{end}}{2\pi} \left(\frac{a_{end}}{a_{reh}} \right) \left(\frac{g_{*,S}(T_0)}{g_{*,S}(T_{reh})} \right)^{1/3} \frac{T_0}{T_{reh}} \stackrel{\text{inst. reh.}}{\approx} 1.83 \times 10^{11} \left(\frac{H_{end}}{M_P} \right)^{1/2} \text{ Hz} \quad (3.108)$$

where the evolution of the scale factor below the reheating temperature T_{reh} is deduced thanks to entropy conservation. The final estimate has been computed assuming instantaneous reheating, i.e. $T_{reh}^4 = 90 H_{end}^2 M_P^2 / (\pi^2 g_*(T_{reh}))$, and $g_*(T_{reh}) = g_{*,S}(T_{reh}) \approx 104.4$ for $T_{reh} \geq 100$ GeV [68].

3.3.1 Green functions

In this section we show how to compute the retarded Green functions for tensor perturbations, $G(\tau, \tau') := \theta(\tau - \tau') \tilde{G}(\tau, \tau')$, which satisfy the following equation and conditions:

$$\left[\frac{\partial^2}{\partial \tau^2} + k^2 - \frac{a''}{a} \right] \tilde{G}_k(\tau, \tau') = 0, \quad \tilde{G}_k(\tau, \tau) = 0, \quad \frac{\partial}{\partial \tau} \tilde{G}_k(\tau, \tau') \Big|_{\tau=\tau'} = 1. \quad (3.109)$$

Let us first consider two linearly independent solutions $F_{1,2}$ of the equation:

$$\left(\frac{\partial^2}{\partial \tau^2} + k^2 - \frac{a''}{a} \right) F_i(\tau, k) = 0, \quad i = 1, 2, \quad (3.110)$$

where we can choose $F_1^* = F_2$. A given mode of momentum k satisfies the equation of motion of a harmonic oscillator, $(\partial_\tau^2 + k^2)F_i = 0$, at asymptotically early times since the k^2 term dominates over a''/a . Hence, we fix the Bunch-Davies initial conditions:

$$\lim_{\tau \rightarrow -\infty} F_1(\tau, k) = \frac{e^{-ik\tau}}{\sqrt{2k}}. \quad (3.111)$$

At asymptotically early times the Wronskian normalization condition holds:

$$\lim_{\tau \rightarrow -\infty} F_1'(\tau, k) F_2(\tau, k) - F_2'(\tau, k) F_1(\tau, k) = -i \quad (3.112)$$

and, deriving the argument of the limit with respect to conformal time, we notice that it is constant in τ therefore the Wronskian normalization condition is satisfied at all times.

A generic solution for $\tilde{G}$ satisfying the boundary conditions in (3.109) is expressed by

$$\tilde{G}_k(\tau, \tau') = \frac{F_1(\tau, k) F_2(\tau', k) - F_2(\tau, k) F_1(\tau', k)}{F_1'(\tau', k) F_2(\tau', k) - F_2'(\tau', k) F_1(\tau', k)} = 2 \operatorname{Im} [F_1^*(\tau', k) F_1(\tau, k)], \quad (3.113)$$

and the retarded Green function acquires the form

$$G_k(\tau, \tau') = 2 \theta(\tau - \tau') \operatorname{Im} [F_1^*(\tau, k) F_1(\tau', k)]. \quad (3.114)$$

We proceed now to carry out the computation of $\tilde{G}_k$ in a simplified scenario, that is de Sitter spacetime. Under the assumption of constant H we have that $a = -1/(\tau H) \implies a''/a = 2/\tau^2$ and the analytical solution to eq. (3.110) takes the following expression:

$$F_{1,dS} = \frac{e^{-ik\tau}}{\sqrt{2k}} \left(1 - \frac{i}{k\tau} \right) \quad (3.115)$$

which recovers the Bunch-Davies initial conditions for $-k\tau \gg 1$. Following from (3.114) The full expression of $\tilde{G}_k$ in de Sitter spacetime is given by

$$\tilde{G}_{k,dS}(\tau, \tau') = \frac{(1 + k^2 \tau \tau') \sin(k(\tau - \tau')) - k(\tau - \tau') \cos(k(\tau - \tau'))}{k^3 \tau \tau'}. \quad (3.116)$$

Notably, in the super-horizon limit, $-k\tau, -k\tau' \rightarrow 0$, the last expression is independent of k and it reduces to the following:

$$\lim_{-k\tau, -k\tau' \rightarrow 0} \tilde{G}_{k,dS}(\tau, \tau') = \frac{\tau^3 - \tau'^3}{3\tau\tau'}. \quad (3.117)$$

This property, i.e. k independence, is actually more general and we can see it in the following.

In the super-horizon limit, the equation of motion (3.110) simplifies to

$$\left[\frac{\partial^2}{\partial \tau^2} - \frac{a''}{a} \right] F \approx 0, \quad (3.118)$$

since the k^2 term is negligible in comparison with a''/a . A solution to the last equation is clearly given by $a(\tau)$, then the other linearly independent solution is found exploiting Abel identity; hence the full solution on super-horizon scales acquires the form

$$\lim_{-k\tau \ll 1} F(\tau, k) = a(\tau) \left[c_1(k) + c_2(k) \int_{\tau_*}^{\tau} \frac{d\tau'}{a^2(\tau')} \right] \quad (3.119)$$

where $c_1(k)$ and $c_2(k)$ are momentum dependent integration constants and τ_* is some reference time from which the super-horizon approximation starts to be valid. In the $-k\tau \ll 1$ limit, the expression for $\tilde{G}_k$ is the following:

$$\begin{aligned} \lim_{-k\tau, -k\tau' \ll 1} \tilde{G}_k(\tau, \tau') &= \lim_{-k\tau, -k\tau' \ll 1} \frac{\text{Im} [F(\tau, k) F^*(\tau', k)]}{\text{Im} [F'(\tau, k) F^*(\tau', k)]} \\ &= a(\tau) a(\tau') \int_{\tau'}^{\tau} \frac{d\tau''}{a^2(\tau'')} . \end{aligned} \quad (3.120)$$

We observe that this last expression is independent of k as we expected and that it is also independent of the integration constants, c_1 and c_2 .

Chapter 4

Numerical techniques in axion inflation

As we derived in the previous chapter, the equation of motion for a homogeneous axion $\phi(t)$ coupled to a $U(1)$ gauge field in an FLRW spacetime and the related Friedmann equation read

$$\left\{ \begin{array}{l} H^2 = \frac{1}{3M_p^2} \left[\frac{1}{2} \dot{\phi}^2 + V(\phi) + \frac{1}{2} \langle \mathbf{E}^2 + \mathbf{B}^2 \rangle \right] \\ \quad = \frac{1}{3M_p^2} \left[\frac{1}{2} \dot{\phi}^2 + V(\phi) + \frac{1}{4\pi^2 a^4} \sum_{\lambda=\pm} \int_0^{k_{h,\lambda}(t)} dk \left(k^4 |A_\lambda|^2 + k^2 a^2 |\dot{A}_\lambda|^2 \right) \right], \\ \ddot{\phi} + 3H\dot{\phi} + V'(\phi) = \frac{1}{f} \langle \mathbf{E} \cdot \mathbf{B} \rangle \\ \quad = -\frac{1}{4\pi^2 f a^3} \sum_{\lambda=\pm} \int_0^{k_{h,\lambda}(t)} dk \lambda k^3 \frac{\partial}{\partial t} |A_\lambda(t, k)|^2, \end{array} \right. \quad (4.1)$$

where we remark that it is possible to write the electromagnetic bilinear correlators $\langle \mathbf{E}^2 + \mathbf{B}^2 \rangle$ and $\langle \mathbf{E} \cdot \mathbf{B} \rangle$ in terms of integrals in momentum space of the mode functions $A_\lambda(t, k)$, which need to be evolved by their own equation of motion.

The dynamics of the inflaton and of the scale factor can be evaluated numerically by closing the system of equation (4.1) in two different ways:

- the *Mode-by-mode approach* (MbM): this method employs the gauge mode equation

$$\ddot{A}_\lambda(t, k) + H \dot{A}_\lambda(t, k) + \left[\frac{k^2}{a^2} - 2\lambda \frac{k}{a} H \xi \right] A_\lambda(t, k) = 0, \quad (4.2)$$

and it relies on a discretization of momentum space so that we evolve only a finite number of gauge modes. We will focus on its implementation as presented in [10] in a code called *BackHom*;

- the *Gradient Expansion Formalism* (GEF): this technique is instead based on a tower of first order differential equations for electromagnetic bilinear correlators, such as $\langle \mathbf{E} \cdot [(\nabla \times)^n \mathbf{B}] \rangle$, in which an increasing number of spatial derivatives are inserted. We will present GEF as in [9] and the code implementation is called the *Gradient Expansion Formalism Factory* (GEFF) [69].

In this chapter, we will present these two numerical schemes and their realization in programming code and in the next one we will compare these methods in two different benchmark scenarios.

4.1 Mode-by-mode approach and *BackHom*

The MbM approach, as implemented in the *Mathematica* code *BackHom* [10], is built upon two approximations:

- *discretization of momentum space*: in order to solve numerically the dynamics of the system, we cannot deal with an infinite set of equations for the mode functions A_λ , one for each value of momentum, therefore we need to discretize momentum space and choose a specific set of momenta;
- *omission of the negative polarization mode A_-* : only the positive polarization mode A_+ is accounted for. As previously mentioned at the end of Sec. 3.1.2, starting from the slow-roll regime in which $\dot{\phi} \approx -V'(\phi)/(3H)$, the slope of the potential sets the sign of $\dot{\phi}$, which can briefly flip only in few instances. Consequently, also the sign of the instability variable ξ almost never flips and essentially only one of the two gauge polarizations will be tachyonically amplified. This implies that choosing to evolve only A_+ forces us to work always with potentials that have a negative slope.

In this code, the scale factor is normalized to 1 at t_0 , the initial time of the simulation, so that

$$a = e^N, \quad \tau = \tau_0 + \int_{t_0}^t \frac{dt'}{a(t')} = \tau_0 + \int_0^N \frac{dN'}{H(N') e^{N'}}, \quad (4.3)$$

where N is the number of e-folds.

4.1.1 Rescaling of the variables

Denoting the initial value of the inflaton potential with V_0 , we define the dimensionless quantities and fields employed in the code as follows

$$\begin{aligned} \tilde{k} &:= \frac{M_p}{\sqrt{V_0}} k, \quad \tilde{H} := \frac{M_p}{\sqrt{V_0}} H, \quad \tilde{f} := \frac{f}{M_p}, \quad \tilde{\phi} := \frac{\phi}{f}, \\ \tilde{V}(\tilde{\phi}) &:= \frac{V(\phi)}{V_0}, \quad \tilde{V}_0 := \frac{V_0}{M_p^4}, \quad \tilde{\tau} := \frac{\sqrt{V_0}}{M_p} \tau, \\ \bar{A}_\pm &:= \sqrt{2k} e^{ik(\tau-\tau_0)} A_\pm = \frac{\sqrt{2M_p \tilde{k}}}{V_0^{1/4}} e^{i\tilde{k} \int_0^N \frac{dN'}{\tilde{H}(N') e^{N'}}} A_\pm, \end{aligned} \quad (4.4)$$

where the gauge modes are rescaled so that their fast evolving time-dependent phase in the UV regime is eliminated (see eq. (4.8) below).

4.1.2 Equations of motion in the code

Adopting this set of rescaled variables and employing the number of e-folds N as time variable in place of the cosmic time t , our system of equations reads

$$\begin{cases} \frac{d^2 \tilde{\phi}}{dN^2} = -\frac{d\tilde{\phi}}{dN} + \frac{\tilde{f}^2}{6} \left(\frac{d\tilde{\phi}}{dN} \right)^3 - \frac{\tilde{V}'(\tilde{\phi})}{\tilde{f}^2 \tilde{H}^2} - \frac{2\tilde{V}(\tilde{\phi})}{3\tilde{H}^2} \frac{d\tilde{\phi}}{dN} + \frac{\langle \mathbf{E} \cdot \mathbf{B} \rangle}{\tilde{f}^2 V_0 \tilde{H}^2} \\ \frac{d\tilde{H}}{dN} = -2\tilde{H} - \frac{\tilde{f}^2 \tilde{H}}{6} \left(\frac{d\tilde{\phi}}{dN} \right)^2 + \frac{2\tilde{V}(\tilde{\phi})}{3\tilde{H}} \\ \frac{d^2 \bar{A}_\lambda}{dN^2} = \left[1 + \frac{\tilde{f}^2}{6} \left(\frac{d\tilde{\phi}}{dN} \right)^2 + \frac{2i\tilde{k}}{\tilde{H} e^N} - \frac{2\tilde{V}(\tilde{\phi})}{3\tilde{H}^2} \right] \frac{d\bar{A}_\lambda}{dN} + \lambda \frac{\tilde{k} \bar{A}_\lambda}{\tilde{H} e^N} \frac{d\tilde{\phi}}{dN} \end{cases} \quad (4.5)$$

where prime on the rescaled inflaton potential denotes derivative with respect to the rescaled inflaton and, as stated earlier, we fix $\lambda = +$. Instead of the Friedmann equation, in (4.5) we employ the equation for the second derivative of the scale factor:

$$\begin{aligned}\frac{\ddot{a}}{a} &= -\frac{1}{6M_p^2} \sum_i (\rho_i + 3p_i) = -\frac{1}{6M_p^2} \left[2\dot{\phi}^2 - 2V(\phi) + \langle \mathbf{E}^2 + \mathbf{B}^2 \rangle \right] \\ &= -\frac{1}{3M_p^2} \left[\frac{1}{2}\dot{\phi}^2 - 2V(\phi) \right] - H^2 ,\end{aligned}\tag{4.6}$$

where, to obtain the expression in the second row, we used the Friedmann equation, then we get the second equation in (4.5) by remembering that $\ddot{a}/a = \dot{H} + H^2$. Hence, the code computes only one integral of the gauge modes (namely, one backreaction term) instead of two.

In dimensionless variables, the electromagnetic expectation values acquire the form

$$\begin{aligned}\langle \mathbf{E} \cdot \mathbf{B} \rangle &= -\frac{V_0^2}{8\pi^2 M_p^4} \frac{\tilde{H}}{e^{3N}} \int d\tilde{k} \tilde{k}^2 \sum_{\lambda} \lambda \frac{d}{dN} |\bar{A}_{\lambda}|^2 , \\ \frac{\langle \mathbf{E}^2 + \mathbf{B}^2 \rangle}{2} &= \frac{V_0^2}{8\pi^2 M_p^4 e^{4N}} \int d\tilde{k} \tilde{k} \sum_{\lambda} \left[e^{2N} \tilde{H}^2 \left| \frac{d\bar{A}_{\lambda}}{dN} - i \frac{\tilde{k} \bar{A}_{\lambda}}{\tilde{H} e^N} \right|^2 + \tilde{k}^2 |\bar{A}_{\lambda}|^2 \right] .\end{aligned}\tag{4.7}$$

As mentioned before, the negative polarization modes are neglected, hence the code does not include a sum over polarizations $\sum_{\lambda}$, contrary to what is explicitly shown in (4.7).

4.1.3 Initial conditions

Concerning the gauge modes, we assume they satisfy the Bunch-Davies boundary condition at asymptotically early times:

$$\lim_{\tau \rightarrow -\infty} A_+(\tau, k) = \frac{e^{-ik\tau}}{\sqrt{2k}} .\tag{4.8}$$

In the code, the gauge modes are all initialized deep inside the horizon in the Bunch-Davies vacuum. The initialization procedure of gauge modes is specified in greater detail in the next two sections, 4.1.4 and 4.1.5. By initializing the system on a portion of the inflaton potential which is flat enough to allow for standard slow-roll inflation, we neglect the contribution of the gauge fields at $N = 0$, so that we can set

$$H(N=0) = \frac{V(\phi_0)^{1/2}}{\sqrt{3}M_p} , \quad \dot{\phi}(N=0) = -\frac{V'(\phi)}{3H} \Big|_{N=0} .\tag{4.9}$$

where $\phi_0 = \phi(N=0)$ is the initial value of the inflaton field. In code variables, the initial conditions read

$$\tilde{\phi}(0) = \tilde{\phi}_0 , \quad \tilde{H}(0) = \frac{1}{\sqrt{3}} , \quad \frac{d\tilde{\phi}}{dN} \Big|_{N=0} = -\frac{1}{\tilde{f}^2} \frac{d\tilde{V}}{d\tilde{\phi}} \Big|_{\tilde{\phi}=\tilde{\phi}_0} , \quad \bar{A}_{\lambda}(0) = 1 , \quad \frac{d\bar{A}_{\lambda}}{dN} \Big|_{N=0} = 0 .\tag{4.10}$$

4.1.4 Definition of the grid in momentum space

It is not possible to numerically solve the system (4.5) taking into account an infinite set of equations for the polarization mode $A_+(\tau, k)$, hence we need to discretize momentum space. In the homogeneous inflaton approximation, the mode equation of motion (4.2) involves only the magnitude of the momentum and not its angular components, therefore we do not have to discretize the three dimensional space in which $\mathbf{k}$ lives, but rather just the half-line to which the modulus k belongs. We need to choose a finite interval in $(0, +\infty)$ and then fraction it appropriately.

We select a set of momenta $\tilde{k}_i$ that are spaced in a logarithmically even manner:

$$\tilde{k}_i = \tilde{k}_{\min} \left(\frac{\tilde{k}_{\max}}{\tilde{k}_{\min}} \right)^{\frac{i-1}{i_{\max}-1}}, \quad i \in [1, i_{\max}], \quad (4.11)$$

where $\tilde{k}_{\min}$ is the minimum momentum we consider in code units, $\tilde{k}_{\max}$ the maximum one and $i_{\max}$ is the total number of modes. In order to exploit Bunch-Davies initial conditions, we need to fix $\tilde{k}_{\min}$ so that all of the modes taken into account are deep inside the horizon at the beginning of the simulation, while we estimate $\tilde{k}_{\max}$ to be the mode exiting the Hubble horizon at the end of the simulation, i.e. $k_{\max} = a_f H_f$. We remark that the interval in momentum space must be chosen preferring larger momenta in order to accurately portray the back-reaction of the gauge fields onto the inflaton dynamics, since the effect of the IR modes of momentum below $\tilde{k}_{\min}$ is negligible. In code units, these requirements read

$$\tilde{k}_{\min} \gg \tilde{k}_{\text{thr}}(0) \quad \text{and} \quad \tilde{k}_{\max} \lesssim e^{N_{\text{end}}} \tilde{H}(0). \quad (4.12)$$

where we assumed that $H \approx \text{const}$, N_{end} is the total number of e-folds we are evolving the system for, and the rescaled momentum threshold for the tachyonic instability is

$$k_{\text{thr}}(t) := 2a(t)H(t)\xi(t) \xrightarrow{\text{in code variables}} \tilde{k}_{\text{thr}}(N) := e^N \tilde{H}(N) \frac{d\tilde{\phi}}{dN}. \quad (4.13)$$

For practical applications, the authors of [10] fixed $i_{\max} = 400$, $\tilde{k}_{\max} = e^{N_{\text{end}}} \tilde{H}(0)$ and $\tilde{k}_{\min} = 10^4 \tilde{k}_{\text{thr}}(0)$.

The backreaction integral in the first line of (4.7) is computed via the trapezoid rule:

$$\begin{aligned} \int_{\tilde{k}_{\min}}^{\tilde{k}_{\max}} \mathcal{I}(\tilde{k}) d\tilde{k} &\Rightarrow \sum_{i=1}^{i_{\max}-1} \Delta \ln \tilde{k}_i \frac{\tilde{k}_i \mathcal{I}_{\tilde{k}_i} + \tilde{k}_{i+1} \mathcal{I}_{\tilde{k}_{i+1}}}{2} = \\ &= \frac{\Delta \ln \tilde{k}}{2} \left(\tilde{k}_{\min} \mathcal{I}_{\tilde{k}_{\min}} + \tilde{k}_{\max} \mathcal{I}_{\tilde{k}_{\max}} + 2 \sum_{i=2}^{i_{\max}-1} \tilde{k}_i \mathcal{I}_{\tilde{k}_i} \right) \\ &\approx \Delta \ln \tilde{k} \sum_{i=1}^{i_{\max}} \tilde{k}_i \mathcal{I}_{\tilde{k}_i}, \end{aligned} \quad (4.14)$$

where, according to the even logarithmic spacing in (4.12), the integration step $\Delta \ln \tilde{k}$ is the same $\forall i$. In the last step we are making a mistake in the first and last terms, since the integrand is negligible at the boundaries of the integration domain; if it was not, the result would depend on the choice of the upper and lower ends of the momentum interval.

4.1.5 The cutoffs

The structure of *BackHom* relies on two cutoffs:

- the first one acts as a boundary separating physical modes that have experienced the tachyonic instability during their evolution from oscillatory modes contained in the adiabatic vacuum. It is the same cutoff described in the previous chapter in (3.62) and in terms of dimensionless variables it takes the form

$$\tilde{k}_{\text{reg}}(N) := \max_{N' \leq N} \left[\tilde{k}_{\text{thr}}(N') \right]; \quad (4.15)$$

- the purpose of the other cutoff is to set the moment at which a gauge mode starts its evolution from the Bunch-Davies vacuum. A mode is frozen in its initial conditions until the ratio between its momentum and the physical cutoff (4.15) reaches a value α , then at a later time the mode will also be included in the gauge friction integral, contributing to the back-reaction. We define it as follows:

$$\tilde{k}_{\text{vac}}(N) := \alpha \tilde{k}_{\text{reg}}(N) , \quad (4.16)$$

where α is set to $10^{5/2}$ in [10], while in [70], work in which *BackHom* is also employed, the value of α is relaxed to 10 since there is no significant loss of accuracy, according to the authors.

We observe that, for self-consistency reasons, the following relation must be satisfied

$$\tilde{k}_{\text{vac}}(0) \leq \tilde{k}_{\text{min}} , \quad (4.17)$$

because, otherwise, we would be evolving modes that should have been initialized before $N = 0$. Moreover, we remark that since at early times $\tilde{k}_{\text{reg}}(N) < \tilde{k}_{\text{min}}$, no gauge modes at all are included in the gauge friction integral, therefore the computation of backreaction will become trustworthy only when $\tilde{k}_{\text{reg}}(N) \gg \tilde{k}_{\text{min}}$. This is not an issue for simulating the system accurately since in the models we are working with the backreaction is negligible at the first stages, hence there is no issue in the overall evolution of the system even though at early stages the gauge friction integral will include only few modes.

Numerical implementation of the cutoffs

In *BackHom*, the cutoffs are implemented via two regulating functions that are smooth realizations of the Heaviside step function:

$$R(N, \tilde{k}) := \frac{1}{2} \left\{ \tanh \left[\mathcal{K} \ln \left(\frac{\tilde{k}_{\text{reg}}(N)}{\tilde{k}} \right) \right] + 1 \right\} , \quad (4.18)$$

and

$$R_{\text{vac}}(N, \tilde{k}) := \frac{1}{2} \left\{ \tanh \left[\mathcal{K} \ln \left(\frac{\tilde{k}_{\text{vac}}(N)}{\tilde{k}} \right) \right] + 1 \right\} , \quad (4.19)$$

where the parameter $\mathcal{K}$ modulates the sharpness of the regulators, recovering the Heaviside step function in the limit $\mathcal{K} \rightarrow +\infty$. For practical applications, the code uses $\mathcal{K} = 10^3$. The first one is used for the gauge friction integral in which the integrand is substituted in the code by its regularized counterpart:

$$\mathcal{I}(\tilde{k}) \rightarrow \mathcal{I}_{\text{reg}}(\tilde{k}) := R(N, \tilde{k}) \mathcal{I}(\tilde{k}) . \quad (4.20)$$

On the other hand, the second regulator must appear in the mode equations for the gauge field, so that we recover the full equation only if $\tilde{k} \ll \tilde{k}_{\text{vac}}(N)$. Hence, the mode equation is implemented in the code as:

$$\frac{d^2 \bar{A}_+}{dN^2} = R_{\text{vac}}(N, \tilde{k}) \left\{ \left[1 + \frac{\tilde{f}^2}{6} \left(\frac{d\tilde{\phi}}{dN} \right)^2 + \frac{2i\tilde{k}}{\tilde{H}e^N} - \frac{2\tilde{V}(\tilde{\phi})}{3\tilde{H}^2} \right] \frac{d\bar{A}_+}{dN} + \frac{\tilde{k}\bar{A}_+}{\tilde{H}e^N} \frac{d\tilde{\phi}}{dN} \right\} . \quad (4.21)$$

Let us now focus on the numerical realization of the momentum cutoff $\tilde{k}_{\text{reg}}(N)$. Until $\tilde{k}_{\text{thr}}(N)$ is increasing monotonically, as we expect at early times when backreaction is negligible, we have that $\tilde{k}_{\text{reg}}(N) = \tilde{k}_{\text{thr}}(N)$. But, when the back-reaction becomes strong, the inflaton may slow down significantly and potentially $\dot{\phi}$ could even change sign implying that the inflaton is momentarily climbing up the potential. Hence, $\tilde{k}_{\text{thr}}$ could diminish. In this kind of instances we demand $\tilde{k}_{\text{reg}}(N)$ to stay equal to the last maximum $\tilde{k}_{\text{thr}}(N)$ in the past evolution of the system, until $\tilde{k}_{\text{thr}}(N)$ grows up to a value greater than the previous local maximum, when we will have again that $\tilde{k}_{\text{reg}}(N) = \tilde{k}_{\text{thr}}(N)$. In *Mathematica*, this feature can be encoded via a **WhenEvent** trigger inside the differential equation solver **NDSolve**. The event we focus on is

$$\frac{d\tilde{k}_{\text{thr}}}{dN} = e^N \tilde{H} \frac{d\tilde{\phi}}{dN} + e^N \frac{d\tilde{H}}{dN} \frac{d\tilde{\phi}}{dN} + e^N \tilde{H} \frac{d^2\tilde{\phi}}{dN^2} < 0, \quad (4.22)$$

marking the change in sign of the first derivative of $\tilde{k}_{\text{thr}}$. When it is activated the value $\tilde{k}_{\text{thr}}$ acquired at that time is appended to a list of local maxima and a discrete variable $\tilde{k}_{\text{loc,max}}(N)$ is set to the maximum value present in this list. The option **DiscreteVariables** is employed to describe the evolution of $\tilde{k}_{\text{loc,max}}(N)$ since it enters the **NDSolve** with an initial condition, but is not associated to any differential equation.

Ultimately, the first momentum cutoff is realized as

$$\begin{aligned} \tilde{k}_{\text{reg}}(N) = & \frac{1}{2} \left\{ \tanh \left[10^3 \ln \left(\frac{\tilde{k}_{\text{thr}}(N)}{\tilde{k}_{\text{loc,max}}(N)} \right) \right] + 1 \right\} \tilde{k}_{\text{thr}}(N) \\ & + \frac{1}{2} \left\{ \tanh \left[-10^3 \ln \left(\frac{\tilde{k}_{\text{thr}}(N)}{\tilde{k}_{\text{loc,max}}(N)} \right) \right] + 1 \right\} \tilde{k}_{\text{loc,max}}(N). \end{aligned} \quad (4.23)$$

In the following figure, borrowed from [10], we can see the evolution of $\tilde{k}_{\text{reg}}$ against $\tilde{k}_{\text{thr}}$ for a scenario thoroughly examined in the cited work.

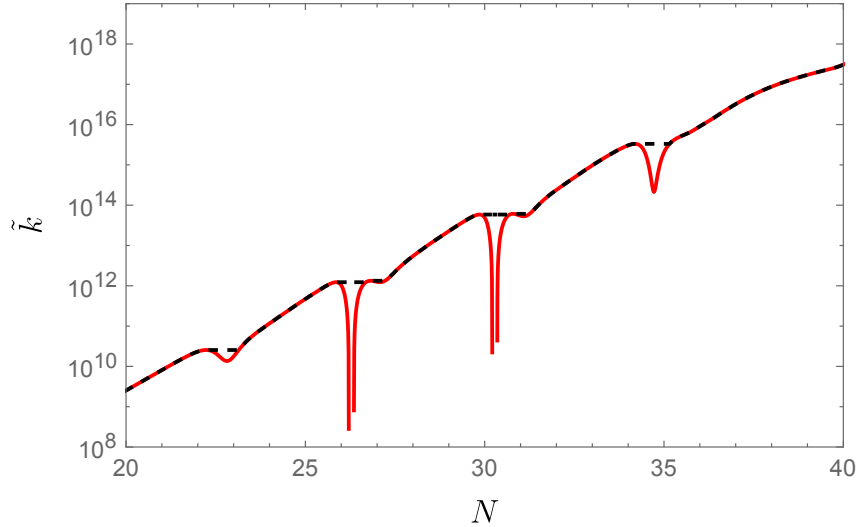


Figure 4.1: Comparison between $\tilde{k}_{\text{thr}}(N)$, the red solid line, and $\tilde{k}_{\text{reg}}(N)$, the black dashed line. The red lines shortly disappear when the ξ changes sign, because there is not enough resolution in the plot. This figure is borrowed from [10].

4.1.6 Comments on the computational strategy

The total system of equations consists of one first order differential equation for the Hubble rate and $1 + i_{\text{max}}$ second order differential equations, one for the inflaton and i_{max} for the gauge mode functions.

To enhance numerical stability, we distinguish between the real and the imaginary part of the mode functions so that the code has to deal only with real variables and no complex numbers:

$$\bar{A}_+(N, \tilde{k}_i) = \bar{A}_{+,RE}(N, \tilde{k}_i) + i \bar{A}_{+,IM}(N, \tilde{k}_i), \quad (4.24)$$

where the subscripts *RE* and *IM* refer to the real and imaginary part, respectively. Hence, we are doubling the number of equations for the mode functions. Moreover, for a given variable $y(N)$ obeying a second order differential equation, we define an auxiliary variable $y_{aux}(N)$ so that

$$y''(N) = f(y, N, \dots) \quad \text{is substituted by} \quad \{y'(N) = y_{aux}(N), y'_{aux}(N) = f(y, N, \dots)\}, \quad (4.25)$$

where $f(y, N, \dots)$ is some functional of the variables involved in the system of equations and the prime denotes derivative with respect to the argument N . Eventually, the system is constituted by $3 + 4 i_{\max}$ first order differential equations and, taking into account the discrete variable $k_{\text{loc},\max}(N)$, *BackHom* has to solve in total $4 + 4 i_{\max}$ equations, 1604 for $i_{\max} = 400$.

In the *Mathematica* solver **NDSolve**, we adopt the following method,

Method $\rightarrow$ {"StiffnessSwitching", **Method** $\rightarrow$ {"ExplicitRungeKutta", **Automatic**}},

thanks to which the code switches automatically between stiff and non-stiff solvers depending if an internal criterion for stiffness is met. The solver in the case of weak back-reaction employs the Runge-Kutta method, which is fast and efficient, and permits the use of *autoparallelization*, a process by which a program can run on multiple processor cores at the same time.

4.2 Gradient expansion formalism and GEFF

The gradient expansion formalism is rooted in the idea that, to study the back-reaction of the gauge fields onto the dynamics of the inflaton $\phi(t)$ and of the scale factor $a(t)$, we can rely on the time evolution of just the expectation values, $\langle \mathbf{E} \cdot \mathbf{B} \rangle$ and $\langle \mathbf{E}^2 + \mathbf{B}^2 \rangle / 2$, without solving the gauge mode equation directly.

Starting from the equations of motion for the electric and magnetic fields

$$\dot{\mathbf{E}} + 2H\mathbf{E} - \frac{1}{a}\nabla \times \mathbf{B} + \frac{\dot{\phi}}{f}\mathbf{B} = 0, \quad (4.26)$$

$$\dot{\mathbf{B}} + 2H\mathbf{B} + \frac{1}{a}\nabla \times \mathbf{E} = 0, \quad (4.27)$$

$$\nabla \cdot \mathbf{E} = 0, \quad \nabla \cdot \mathbf{B} = 0, \quad (4.28)$$

we can construct first order differential equations for the three expectation values, $\langle \mathbf{E}^2 \rangle$, $\langle \mathbf{B}^2 \rangle$ and $\langle \mathbf{E} \cdot \mathbf{B} \rangle$, by manipulating (4.26) and (4.27) as follows:

$$\begin{aligned} \langle \mathbf{E} \cdot [\text{L.H.S of (4.26)}] + [\text{L.H.S of (4.26)}] \cdot \mathbf{E} \rangle = \\ = \partial_t \langle \mathbf{E}^2 \rangle + 4H \langle \mathbf{E}^2 \rangle - \frac{2}{a} \langle \mathbf{E} \cdot (\nabla \times \mathbf{B}) \rangle_{sym} + 2 \frac{\dot{\phi}}{f} \langle \mathbf{E} \cdot \mathbf{B} \rangle = 0, \end{aligned} \quad (4.29)$$

$$\begin{aligned} \langle \mathbf{E} \cdot [\text{L.H.S of (4.27)}] + [\text{L.H.S of (4.26)}] \cdot \mathbf{B} \rangle = \\ = \partial_t \langle \mathbf{E} \cdot \mathbf{B} \rangle + 4H \langle \mathbf{E} \cdot \mathbf{B} \rangle + \frac{1}{a} [\langle \mathbf{E} \cdot (\nabla \times \mathbf{E}) \rangle - \langle \mathbf{B} \cdot (\nabla \times \mathbf{B}) \rangle] + \frac{\dot{\phi}}{f} \langle \mathbf{B}^2 \rangle = 0, \end{aligned} \quad (4.30)$$

$$\begin{aligned} \langle \mathbf{B} \cdot [\text{L.H.S of (4.27)}] + [\text{L.H.S of (4.27)}] \cdot \mathbf{B} \rangle = \\ = \partial_t \langle \mathbf{B}^2 \rangle + 4H \langle \mathbf{B}^2 \rangle + \frac{2}{a} \langle \mathbf{E} \cdot (\nabla \times \mathbf{B}) \rangle_{sym} = 0, \end{aligned} \quad (4.31)$$

where for simplicity we are not accounting for the time-dependent cutoff in momentum space, the subscript *sym* indicates that the operators contained in the expectation value are symmetrized and we exploited the fact that it does not matter whether the curl acts on the first or the second operator in the expectation value thanks to the identity

$$\langle (\nabla \times \mathbf{f}) \cdot \mathbf{g} \rangle - \langle \mathbf{f} \cdot (\nabla \times \mathbf{g}) \rangle = \nabla \cdot \langle (\mathbf{f} \times \mathbf{g}) \rangle = 0, \quad (4.32)$$

which is valid because expectation values of vector-valued quantities, such as $\langle (\mathbf{f} \times \mathbf{g}) \rangle$, vanish in an isotropic universe. Crucially, we notice that the equations (4.29) - (4.31) do not form a closed system due to the insertion of the curl in some expectation values.

More in general we define the following functions accounting for n curl insertions:

$$\mathcal{E}^{(n)} := \frac{1}{a^n} \langle \mathbf{E} \cdot [(\nabla \times)^n \mathbf{E}] \rangle, \quad (4.33)$$

$$\mathcal{G}^{(n)} := -\frac{1}{2a^n} \langle \mathbf{E} \cdot [(\nabla \times)^n \mathbf{B}] + [(\nabla \times)^n \mathbf{B}] \cdot \mathbf{E} \rangle, \quad (4.34)$$

$$\mathcal{B}^{(n)} := \frac{1}{a^n} \langle \mathbf{B} \cdot [(\nabla \times)^n \mathbf{B}] \rangle. \quad (4.35)$$

We note that the only expectation value that must be explicitly symmetrized is (4.34) when n is odd, since all of the other correlators are already symmetric. In terms of the gauge modes A_λ , we can express them as

$$\mathcal{E}^{(n)} = \sum_{\lambda=\pm} \int_0^{k_h} \frac{dk}{k} \lambda^n \frac{k^{n+3}}{2\pi^2 a^{n+2}} |\dot{A}_\lambda(t, k)|^2 = \sum_{\lambda=\pm} \int_0^{k_h} \frac{dk}{k} \lambda^n \frac{k^{n+5}}{2\pi^2 a^{n+4}} |D_\lambda(t, k)|^2, \quad (4.36)$$

$$\mathcal{G}^{(n)} = \sum_{\lambda=\pm} \int_0^{k_h} \frac{dk}{k} \lambda^{n+1} \frac{k^{n+4}}{4\pi^2 a^{n+3}} \frac{d}{dt} |A_\lambda(t, k)|^2 = \sum_{\lambda=\pm} \int_0^{k_h} \frac{dk}{k} \lambda^{n+1} \frac{k^{n+5}}{2\pi^2 a^{n+4}} \text{Re} [A_\lambda^* D_\lambda], \quad (4.37)$$

$$\mathcal{B}^{(n)} = \sum_{\lambda=\pm} \int_0^{k_h} \frac{dk}{k} \lambda^n \frac{k^{n+5}}{2\pi^2 a^{n+4}} |A_\lambda(t, k)|^2, \quad (4.38)$$

where $\text{Re}[\dots]$ stands for the real part and the function D_λ is defined as

$$D_\lambda(t, k) := \frac{a}{k} \frac{\partial A_\lambda}{\partial t} = \frac{1}{k} \frac{\partial A_\lambda}{\partial \tau}. \quad (4.39)$$

The equations of motion for these variables can be derived analogously to how we obtained (4.29) - (4.31) and they acquire the form

$$\dot{\mathcal{E}}^{(n)} + (n+4)H \mathcal{E}^{(n)} - 2\frac{\dot{\phi}}{f} \mathcal{G}^{(n)} + 2\mathcal{G}^{(n+1)} = [\dot{\mathcal{E}}^{(n)}]_b, \quad (4.40)$$

$$\dot{\mathcal{G}}^{(n)} + (n+4)H \mathcal{G}^{(n)} - \mathcal{E}^{(n+1)} + \mathcal{B}^{(n+1)} - \frac{\dot{\phi}}{f} \mathcal{B}^{(n)} = [\dot{\mathcal{G}}^{(n)}]_b, \quad (4.41)$$

$$\dot{\mathcal{B}}^{(n)} + (n+4)H \mathcal{B}^{(n)} - 2\mathcal{G}^{(n+1)} = [\dot{\mathcal{B}}^{(n)}]_b, \quad (4.42)$$

where the R.H.S. of these equations are called *boundary terms* that arise due to the time dependence of the upper extremum of integration $k_h(t)$ in eqs. (4.36) - (4.38), as we discuss in details in the next section.

We critically notice that for any n the three variables (4.33) - (4.35) satisfy an equation of motion that features a term with $n+1$ curl insertions. Therefore, we observe that with respect to the MbM approach we have effectively traded an infinite number of second order gauge mode equations for an infinite system of coupled first order differential equations.

Remark on the symmetrization of the expectation values

In principle, the order of the operators appearing in an expectation value must be always symmetrized in order to get a physical result. Clearly, (4.33) and (4.35) are already symmetric due to the identity (4.32). On the other hand, by exploiting the properties of the polarization operators $\epsilon^\lambda(\mathbf{k})$ and the canonical commutation relation, we see that

$$\begin{aligned} \langle \mathbf{E} \cdot [(\nabla \times)^n \mathbf{B}] \rangle &= \int \frac{d^3 k}{a^3 (2\pi)^3} k^{n+1} \sum_\lambda \lambda^{n+1} \dot{A}_\lambda A_\lambda^*, \\ \langle [(\nabla \times)^n \mathbf{B}] \cdot \mathbf{E} \rangle &= \int \frac{d^3 k}{a^3 (2\pi)^3} k^{n+1} \sum_\lambda \lambda^{n+1} A_\lambda \dot{A}_\lambda^*. \end{aligned} \quad (4.43)$$

Hence, due to the fact that

$$\dot{A}_+ A_+^* - \dot{A}_- A_-^* = A_+ \dot{A}_+^* - A_- \dot{A}_-^* \quad (4.44)$$

where we exploited the Wronskian renormalization condition, $A_\lambda(t, k) \dot{A}_\lambda^*(t, k) = A_\lambda^*(t, k) \dot{A}_\lambda(t, k) + \frac{i}{a}$, we conclude that (4.34) needs to be explicitly symmetrized only if n is odd.

4.2.1 The boundary terms

We let X represent any of the electromagnetic bilinears, $\mathcal{E}^{(n)}$, $\mathcal{G}^{(n)}$, $\mathcal{B}^{(n)}$, then their spectral decomposition reads

$$X = \int_0^{k_h(t)} \frac{dk}{k} \frac{dX}{d \ln k}, \quad (4.45)$$

where $k_h(t)$ is the momentum cutoff defined as

$$k_h(t) := \max_{t' \leq t} \{2a(t')H(t')|\xi(t')|\}. \quad (4.46)$$

In general, a bilinear X evolves in time and its evolution comes from the time dependence of both the integrand and the upper integration boundary. In the equations (4.40) - (4.42) the L.H.S. accounts

for the evolution of the electromagnetic fields, while the boundary terms on the R.H.S. track the time dependence of the cutoff. Therefore, a boundary term is expressed by

$$(\dot{X})_b := \left. \frac{dX}{d \ln k} \right|_{k=k_h} \cdot \frac{d \ln k_h}{dt} . \quad (4.47)$$

In order to compute the boundary terms, we need to know the analytical form of the gauge modes about the time when they exit the adiabatic vacuum. Hence, assuming that the instability variable ξ evolves adiabatically slowly and that the Hubble rate is almost constant, we approximate the mode function when $k = k_h(t)$ with the Anber-Sorbo solution:

$$A_\lambda(\tau, k) = \frac{1}{\sqrt{2k}} e^{\lambda \pi \xi / 2} W_{i\lambda \xi, \frac{1}{2}}(-2ik\tau) . \quad (4.48)$$

Using the decompositions in terms of the mode functions (4.36-4.38) and substituting the AS solution, the boundary terms acquire the following form

$$[\dot{\mathcal{E}}^{(n)}]_b := \frac{d \ln k_h(t)}{dt} \frac{1}{4\pi^2} \left(\frac{k_h(t)}{a(t)} \right)^{n+4} \sum_{\lambda=\pm} \lambda^n E_\lambda(\xi(t)), \quad (4.49)$$

$$[\dot{\mathcal{G}}^{(n)}]_b := \frac{d \ln k_h(t)}{dt} \frac{1}{4\pi^2} \left(\frac{k_h(t)}{a(t)} \right)^{n+4} \sum_{\lambda=\pm} \lambda^{n+1} G_\lambda(\xi(t)), \quad (4.50)$$

$$[\dot{\mathcal{B}}^{(n)}]_b := \frac{d \ln k_h(t)}{dt} \frac{1}{4\pi^2} \left(\frac{k_h(t)}{a(t)} \right)^{n+4} \sum_{\lambda=\pm} \lambda^n B_\lambda(\xi(t)), \quad (4.51)$$

where

$$E_\lambda(\xi) := \frac{e^{\pi \lambda \xi}}{4\xi^2} \left| (i2|\xi| - i\lambda\xi) W_{-i\lambda\xi, \frac{1}{2}}(-4i|\xi|) + W_{1-i\lambda\xi, \frac{1}{2}}(-4i|\xi|) \right|^2, \quad (4.52)$$

$$G_\lambda(\xi) := \frac{e^{\pi \lambda \xi}}{2|\xi|} \left\{ \text{Re} \left[W_{i\lambda\xi, \frac{1}{2}}(4i|\xi|) W_{1-i\lambda\xi, \frac{1}{2}}(-4i|\xi|) \right] \right\}, \quad (4.53)$$

$$B_\lambda(\xi) := e^{\pi \lambda \xi} \left| W_{-i\lambda\xi, \frac{1}{2}}(-4i|\xi|) \right|^2 . \quad (4.54)$$

Since the direct computation of these expressions is numerically heavy, for large ξ we perform a series expansion in terms of inverse powers of $|\xi|$, facilitating their computation. We will express in more detail this procedure in Section (4.2.2).

Remark on the choice of physical modes

Contrary to the definition of physical gauge modes we gave in the previous chapter, i.e. modes that have experienced the tachyonic instability, in the GEF a mode is labeled as physical when its momentum goes below the monotonic cutoff $k_h(t)$, no matter if it is stable or not. According to this definition, we just need a single cutoff for both polarizations, unlike what we stated in (3.62).

We observe that the GEF actually takes into account both polarizations thanks to the boundary terms in which we explicitly sum over both of them. The inclusion of the two polarizations and the specific implementation of the cutoff is fully controlled by the boundary terms since in the equations (4.40-4.42) they are the only terms explicitly depending on λ and $k_h(t)$.

4.2.2 Details on the numerical implementation of the boundary terms

In this section we carefully describe how the boundary terms are computed.

In the approximation of constant ξ and constant H , the solution to the mode equation can be expressed in terms of the Coulomb functions:

$$A_\lambda(\tau, k) = \frac{1}{\sqrt{2k}} [G_0(\lambda\xi, -k\tau) + iF_0(\lambda\xi, -k\tau)]. \quad (4.55)$$

This expression is equivalent to (4.48). The boundary terms are computed at $k = k_h(t) = 2aH|\xi|$, hence the second argument of the Coulomb functions becomes $-k\tau \rightarrow -k_h(t)/aH = 2|\xi|$, where we used $\tau = -1/(aH)$ in de Sitter. Thus, we can express the boundary terms in terms of the following functions:

$$E_\lambda(\xi) = G_0'^2 + F_0'^2, \quad (4.56)$$

$$G_\lambda(\xi) = -[G_0 G_0' + F_0 F_0'], \quad (4.57)$$

$$B_\lambda(\xi) = G_0^2 + F_0^2, \quad (4.58)$$

where $F_0 = F_0(\lambda\xi, 2|\xi|)$ and the prime denotes the derivative with respect to the second argument, i.e. $F_0' = \partial F_0(\lambda\xi, z)/\partial z|_{z=2|\xi|}$.

The functions E_λ , G_λ , and B_λ can be expressed in terms of either the Whittaker functions, as in (4.52 - 4.54), or the Coulomb functions as we just pointed out. Being $W_{i\lambda\xi, 1/2}(-k\tau)$, $G_0(\lambda\xi, -k\tau)$ and $F_0(\lambda\xi, -k\tau)$ confluent hypergeometric functions, their full expression is poorly suitable for numerical computations, especially for large values of ξ since the working precision must increase significantly. Hence, an approximate expression for these functions is needed in order to facilitate their evaluation and we can exploit power series for the Whittaker functions, or alternatively the Coulomb wave functions, in terms of inverse powers of ξ .

In the case $\lambda = \text{sign } \xi$, the Coulomb wave functions are more suitable for our purposes. The expansions for $F_0(|\xi|, 2|\xi|)$, $G_0(|\xi|, 2|\xi|)$, and their derivatives for $|\xi| \gg 1$ can be found from the integral representations of the Coulomb wave functions as presented in Ref. [71]; the first terms of these expansions can be found in equations (14.5.10) and (14.5.11) of [62], as well as in [72, 73].

In this way, the series expansions for E_λ , G_λ , and B_λ with a relative error less than $\mathcal{O}(10^{-6})$ for $|\xi| \geq 3$ acquires the following form:

$$\begin{aligned} E_{\text{sign } \xi}(\xi) = & \frac{(\frac{3}{2})^{1/3} \Gamma^2(\frac{2}{3})}{\pi |\xi|^{1/3}} - \frac{\sqrt{3}}{15 |\xi|} + \frac{(\frac{2}{3})^{1/3} \Gamma^2(\frac{1}{3})}{100 \pi |\xi|^{5/3}} + \frac{(\frac{3}{2})^{1/3} \Gamma^2(\frac{2}{3})}{1575 \pi |\xi|^{7/3}} - \frac{27\sqrt{3}}{19250 |\xi|^3} + \\ & + \frac{359 (\frac{2}{3})^{1/3} \Gamma^2(\frac{1}{3})}{866250 \pi |\xi|^{11/3}} + \frac{8209 (\frac{3}{2})^{1/3} \Gamma^2(\frac{2}{3})}{13162500 \pi |\xi|^{13/3}} - \frac{690978\sqrt{3}}{1861234375 |\xi|^5} + \\ & + \frac{13943074 (\frac{2}{3})^{1/3} \Gamma^2(\frac{1}{3})}{127566140625 \pi |\xi|^{17/3}} + \mathcal{O}(|\xi|^{-19/3}), \end{aligned} \quad (4.59)$$

$$\begin{aligned}
 G_{\text{sign } \xi}(\xi) = & \frac{1}{\sqrt{3}} - \frac{(\frac{2}{3})^{1/3} \Gamma^2(\frac{1}{3})}{10 \pi |\xi|^{2/3}} + \frac{3(\frac{3}{2})^{1/3} \Gamma^2(\frac{2}{3})}{35 \pi |\xi|^{4/3}} - \frac{\sqrt{3}}{175 |\xi|^2} - \frac{41(\frac{2}{3})^{1/3} \Gamma^2(\frac{1}{3})}{34650 \pi |\xi|^{8/3}} + \\
 & + \frac{10201(\frac{3}{2})^{1/3} \Gamma^2(\frac{2}{3})}{2388750 \pi |\xi|^{10/3}} - \frac{8787\sqrt{3}}{21896875 |\xi|^4} - \frac{1927529(\frac{2}{3})^{1/3} \Gamma^2(\frac{1}{3})}{4638768750 \pi |\xi|^{14/3}} + \\
 & + \frac{585443081(\frac{3}{2})^{1/3} \Gamma^2(\frac{2}{3})}{393158390625 \pi |\xi|^{16/3}} - \frac{65977497\sqrt{3}}{495088343750 |\xi|^6} + O(|\xi|^{-20/3}),
 \end{aligned} \tag{4.60}$$

$$\begin{aligned}
 B_{\text{sign } \xi}(\xi) = & \frac{(\frac{2}{3})^{1/3} \Gamma^2(\frac{1}{3}) |\xi|^{1/3}}{\pi} + \frac{2\sqrt{3}}{35 |\xi|} - \frac{4(\frac{2}{3})^{1/3} \Gamma^2(\frac{1}{3})}{225 \pi |\xi|^{5/3}} + \frac{9(\frac{3}{2})^{1/3} \Gamma^2(\frac{2}{3})}{1225 \pi |\xi|^{7/3}} + \frac{132\sqrt{3}}{56875 |\xi|^3} + \\
 & - \frac{9511(\frac{2}{3})^{1/3} \Gamma^2(\frac{1}{3})}{5457375 \pi |\xi|^{11/3}} + \frac{1448(\frac{3}{2})^{1/3} \Gamma^2(\frac{2}{3})}{1990625 \pi |\xi|^{13/3}} + \frac{1187163\sqrt{3}}{1323765625 |\xi|^5} + \\
 & - \frac{22862986(\frac{2}{3})^{1/3} \Gamma^2(\frac{1}{3})}{28465171875 \pi |\xi|^{17/3}} + O(|\xi|^{-19/3}).
 \end{aligned} \tag{4.61}$$

Instead in the case $\lambda = -\text{sign } \xi$, the Whittaker functions are more favorable and we can find the asymptotic expansion through the method presented in Sec. 6.13.3 of [74] and Sec. 10 of [75]. In this way, E_λ , G_λ , and B_λ can be expanded in a power series with an error less than $\mathcal{O}(10^{-6})$ for $|\xi| \geq 3$ as follows:

$$E_{-\text{sign } \xi}(\xi) = \sqrt{2} \left[1 - \frac{9}{2^{10} \xi^2} + \frac{2059}{2^{21} \xi^4} - \frac{448157}{2^{31} \xi^6} + O(|\xi|^{-8}) \right], \tag{4.62}$$

$$G_{-\text{sign } \xi}(\xi) = -\frac{\sqrt{2}}{32 |\xi|} \left[1 - \frac{67}{2^{10} \xi^2} + \frac{21543}{2^{21} \xi^4} - \frac{6003491}{2^{31} \xi^6} + O(|\xi|^{-8}) \right], \tag{4.63}$$

$$B_{-\text{sign } \xi}(\xi) = \frac{1}{\sqrt{2}} \left[1 + \frac{11}{2^{10} \xi^2} - \frac{2397}{2^{21} \xi^4} + \frac{508063}{2^{31} \xi^6} + O(|\xi|^{-8}) \right]. \tag{4.64}$$

4.2.3 Truncation of the chain of equations

In principle, the system of equations (4.40) - (4.42) is infinite; hence, in order to solve it, we must truncate it at some order $n \gg 1$. The goal of this subsection is to present a physically sensible argument for the closure of the system.

It is useful to rewrite an electromagnetic bilinear $X^{(n)}$ ($X \in \{\mathcal{E}, \mathcal{G}, \mathcal{B}\}$) in terms of a spectral density function $\mathcal{X}_\lambda(t, k)$:

$$X^{(n)} = \sum_{\lambda=\pm} \lambda^n \int_0^{k_h} \left(\frac{k}{a}\right)^{n+4} \mathcal{X}_\lambda(t, k) dk, \tag{4.65}$$

where we notice that $\mathcal{X}_\lambda(t, k)$ is independent of n . We expect that for $n \gg 1$ the integral is dominated by the contribution close to the upper bound of integration. In this region, where $k \lesssim k_h(t)$, we approximate

$$\mathcal{X}_\lambda(t, k) \approx \frac{Q_\lambda(t)}{a} \left(\frac{k}{a}\right)^p, \tag{4.66}$$

where the factor $1/a$ was added for convenience and p is a spectral index we expect to be of $\mathcal{O}(1)$. We assume the index p to be the same for both polarizations, because, even though just one of the two is tachyonically enhanced, when $k = k_h(t)$ both of them are very close to the vacuum solution. Therefore, a bilinear quantity $X^{(n)}$ can be approximated as follows

$$X^{(n)} \approx \frac{(k_h/a)^{n+p+5}}{n+p+5} \sum_{\lambda=\pm} \lambda^n Q_\lambda(t). \quad (4.67)$$

We can trivially extrapolate this expression to $n+2$ since the sum in this last expression is not affected by this change. Hence, we obtain

$$X^{(n+2)} \approx \frac{n+p+5}{n+p+7} \left(\frac{k_h}{a}\right)^2 X^{(n)} \approx \left(\frac{k_h}{a}\right)^2 X^{(n)}, \quad (4.68)$$

where in the last step we used $n \gg p = \mathcal{O}(1)$. In this way we can approximate higher-order quantities in terms of the lower-order ones in order to truncate the chain. Truncating the system of equations at $n_{\max}$, the equations of motion for the electromagnetic bilinears are the same as in (4.40 - 4.42) until $n = n_{\max} - 1$, while we use (4.68) in the last three equations for $n = n_{\max}$, so that they acquire the form:

$$\dot{\mathcal{E}}^{(n_{\max})} + (n_{\max} + 4)H \mathcal{E}^{(n_{\max})} - 2\frac{\dot{\phi}}{f} \mathcal{G}^{(n_{\max})} + 2\left(\frac{k_h}{a}\right)^2 \mathcal{G}^{(n_{\max}-1)} = [\dot{\mathcal{E}}^{(n_{\max})}]_b, \quad (4.69)$$

$$\dot{\mathcal{G}}^{(n_{\max})} + (n_{\max} + 4)H \mathcal{G}^{(n_{\max})} - \left(\frac{k_h}{a}\right)^2 [\mathcal{E}^{(n_{\max}-1)} - \mathcal{B}^{(n_{\max}-1)}] - \frac{\dot{\phi}}{f} \mathcal{B}^{(n_{\max})} = [\dot{\mathcal{G}}^{(n_{\max})}]_b, \quad (4.70)$$

$$\dot{\mathcal{B}}^{(n_{\max})} + (n_{\max} + 4)H \mathcal{B}^{(n_{\max})} - 2\left(\frac{k_h}{a}\right)^2 \mathcal{G}^{(n_{\max}-1)} = [\dot{\mathcal{B}}^{(n_{\max})}]_b. \quad (4.71)$$

In order to initialize the system in slow-roll, we assume that all of the electromagnetic bilinears are initially vanishing and that the initial conditions of the other quantities are fixed by the slow-roll conditions (4.9).

4.2.4 Details on the code implementation

Let us now focus on the code itself in which the GEF is implemented. We refer to the numerical strategy as GEF and to the code as GEFF.

The Gradient Expansion Formalism Factory is a *Python* package mainly developed by R. von Eckardstein, see [69], and it can be downloaded from a GitHub repository,¹ where it is also possible to find a detailed manual.² The *Python* package features two models of axion inflation: pure axion inflation, which is the model we are dealing with in this thesis, and fermionic axion inflation, in which the Standard Model hypercharge field is coupled to the inflaton and fermions are produced thanks to the Schwinger effect. Fermionic axion inflation is not treated in this work, but it is discussed within the gradient expansion formalism in [9, 76, 77].

¹See <https://github.com/richard-von-eckardstein/GEFF>

²See <https://richard-von-eckardstein.github.io/GEFF/geff.html>

Rescaling of the variables

In numerical computations using dimensionless variables is necessary. In GEFF, physical variables are rescaled with respect to a frequency ω and an energy scale μ . For example, since a scalar field has mass dimension 1, we rescale it as $\phi = \mu\bar{\phi}$, where we are referring to the dimensionless counterpart of a quantity X as $\bar{X}$; instead, for the gauge fields we have that $A_\mu = \omega\bar{A}_\mu$, since they appear in the covariant derivative and they must have the same dimension as $[\partial_t] = [\omega] = [t^{-1}]$, where here the square brackets indicate that we are referring to the mass dimension of what they contain.

In our model the reference quantities adopted are the initial Hubble rate for the frequency, $\omega \equiv H(0)$, and the Planck mass for the energy scale, $\mu \equiv M_p$.

Each cosmological quantity the code is dealing with is characteristically rescaled with respect to a frequency or an energy scale as follows:

- cosmic time: $t = \bar{t}/\omega$,
- comoving momentum: $k = \omega\bar{k}$,
- Hubble rate: $H = \omega\bar{H}$,
- axion: $\phi = \mu\bar{\phi}$,
- inflaton potential: $V(\phi) = \omega^2\mu^2\bar{V}(\mu\bar{\phi})$
- gauge modes: $A_\mu = \omega\bar{A}_\mu$.

Structure of the code

GEFF is structured in two main modules, **GEFSolver** and **ModeSolver**, and a self-correction algorithm. Internally, to solve all of the differential equations GEFF exploits **scipy.integrate.solve_ivp**.

In the **GEFSolver** module, we fully determine the dynamics of our system through the GEF from its initial conditions until the end of inflation, when the equation of state reads $w = -1/3$. The system is composed by the first order differential equations for the functions $\mathcal{E}^{(n)}$, $\mathcal{G}^{(n)}$, $\mathcal{B}^{(n)}$ with $n \in [0, n_{max}]$, as previously stated, in addition to the following system:

$$\left\{ \begin{array}{l} \frac{dN}{dt} = H = \frac{1}{\sqrt{3}M_p} \left[\frac{1}{2}\pi_\phi^2 + V(\phi) + \frac{1}{2} \left(\mathcal{E}^{(0)} + \mathcal{B}^{(0)} \right) \right]^{1/2} \\ \frac{d\pi_\phi}{dt} = -3H\pi_\phi - V'(\phi) - \frac{\beta}{M_p}\mathcal{G}^{(0)} \\ \frac{d\phi}{dt} = \pi_\phi \\ \frac{d \ln k_h}{dt} = \theta \left(H + \frac{\dot{\pi}_\phi}{\pi_\phi} \right) \theta \left(\ln k_{thr}(t) - (1 - 10^{-5}) \ln k_h(t) \right) \left[H + \frac{\dot{\pi}_\phi}{\pi_\phi} \right] \end{array} \right. \quad (4.72)$$

where $N = \ln a$ since the scale factor is set to 1 at the beginning of the simulation, $\pi_\phi = d\phi/dt$ is as an auxiliary variable introduced to work with just first order differential equations. We are denoting the coupling strength as β/M_p instead of $1/f$, as it is fixed in the code. In the last equation, $k_{thr}(t) := 2aH|\xi|$, and the two Heaviside step functions $\theta(\dots)$ are introduced to enforce monotonicity of k_h : the first one imposes positivity of the R.H.S. of the last equation marking when $\partial_t k_{thr}(t) > 0$, while the second one makes sure that $k_h(t)$ increases only when $k_{thr}(t)$ grows larger than its previous local maximum. Also, the last equation does not account for the variation of H , i.e. terms proportional to $\dot{H}$ are neglected. In total, the number of first order differential equations GEFF has to solve amounts to $4 + 3(n_{max} + 1)$.

Regarding the initial conditions, the number of e-folds N is set to zero, $\phi(t=0)$ and $\dot{\phi}(t=0)$ can be set freely and $k_h(0) = 2a(0)H(0)\xi(0) = 2\beta\dot{\phi}(0)/M_p$. Since the Hubble rate is not a dynamical variable directly evolved by the solver, it is always set according to the Friedmann equation. Hence, if we are choosing slow-roll initial conditions for ϕ and $\dot{\phi}$, then $H(t=0)$ is computed accounting for both the kinetic and potential energies and not just the potential one, as we did for *BackHom*. Obviously, the electromagnetic energy does not appear in the Friedmann equation at $t=0$, since all of the electromagnetic bilinears are initially vanishing.

Throughout the whole run of **GEFSolver**, the code checks at each time step for the positivity of $\langle \mathbf{E}^2 \rangle$ and $\langle \mathbf{B}^2 \rangle$. If this criterion is not met, an error is raised and the run is rejected. Another run is automatically attempted, after increasing n_{max} by some amount set by default to 10.

A solid crosscheck for the validity of the run is given by the **ModeSolver** module. After the equations of motion are solved via the GEF, we solve the gauge mode equation on the top of the background dynamics of a and ϕ that has just been computed. This is done via the MbM approach in a way strictly analogous to *BackHom*: a grid in momentum space is fixed with lower bound $k_{min} = 10^4 k_h(0)$ and upper bound $k_{max} = k_h(t_{end})$, where t_{end} is the time at which inflation ends, then a mode is evolved from the Bunch-Davies boundary condition when its momentum k is $10^{5/2}$ times greater than k_h .

We remark that in the version of GEF currently available on GitHub (**v 0.1.1**), the bounds of the interval in momentum space are first fixed as just stated, but then, in the method `create_k_array` of the Class **BaseModeSolver**, where the momentum array is actually built, a further factor of 10 is applied to the bounds of this interval making it $[10^5 k_h(0), 10k_h(t_{end})]$. We think this factor was applied for visualization purposes only, since modes of momentum such that $k_h(t_{end}) \leq k \leq 10k_h(t_{end})$ will never be tachyonically amplified during the simulation and they allow us to have a better picture of the evolution of the peak of the physical modes in the spectrum. Obviously, the reasonable choice for this interval would be $[10^4 k_h(0), 10k_h(t_{end})]$ so that we are consistent with the premises and we can visualize the evolution of the spectrum better.

Noticing this mismatch is part of the work behind this thesis and it will be fixed in the next version of GEF. We do not expect this fact to make a noticeable difference in the results of papers that used this code in the past since we expect the backreaction of lower momentum modes to be negligible.

The variables for the mode function and its derivative are rescaled as follows in the code:

$$A_\lambda \rightarrow \sqrt{2k} A_\lambda, \quad \dot{A}_\lambda \rightarrow \sqrt{\frac{2}{k}} a(t) \dot{A}_\lambda \quad (4.73)$$

in order to deal with dimensionless quantities. The solver adopted to solve the mode equations is `scipy.integrate.solve_ivp` with an explicit Runge–Kutta method of order 5(4).

The crosscheck consists in computing the following errors at each time step:

$$\varepsilon_{\mathcal{X}} := \left| 1 - \frac{\mathcal{X}_{MbM}^{(0)}}{\mathcal{X}_{GEF}^{(0)}} \right|, \quad (4.74)$$

where $\mathcal{X} \in \{\mathcal{E}, \mathcal{G}, \mathcal{B}\}$, $\mathcal{X}_{MbM}^{(0)}$ refers to the quantity computed mode-by-mode and $\mathcal{X}_{GEF}^{(0)}$ to the same one obtained via the GEF. Since at early times, in the stage when $k_h(t) < k_{min}$, no modes are evolved and later in the evolution, only when $k_h(t) \gg k_{min}$, the computation of the expectation values becomes reliable, we expect that initially the MbM quantity will be systematically different from the GEF one. Hence, starting from $t=0$ all of the errors such that $\varepsilon_{\mathcal{X}} > \varepsilon_{thr}$ are excluded from the crosscheck until $\varepsilon_{\mathcal{X}} < \varepsilon_{thr}$, where ε_{thr} is a threshold value set to 0.025. Then, an error is raised when $\varepsilon_{\mathcal{X}} > \varepsilon_{thr}$ after this discarding process has taken place.

A self-correction algorithm is also implemented. In the case of an error $\varepsilon_{\mathcal{X}}$ going above the threshold, GEF will find a time t_{reinit} a bit earlier than the moment of detection and reject the run for $t > t_{reinit}$, then it will use $\mathcal{X}_{MbM}^{(n)}(t_{reinit})$ as new initial conditions for a GEF run starting from t_{reinit} .

The following figure synthesizes how the code operates.

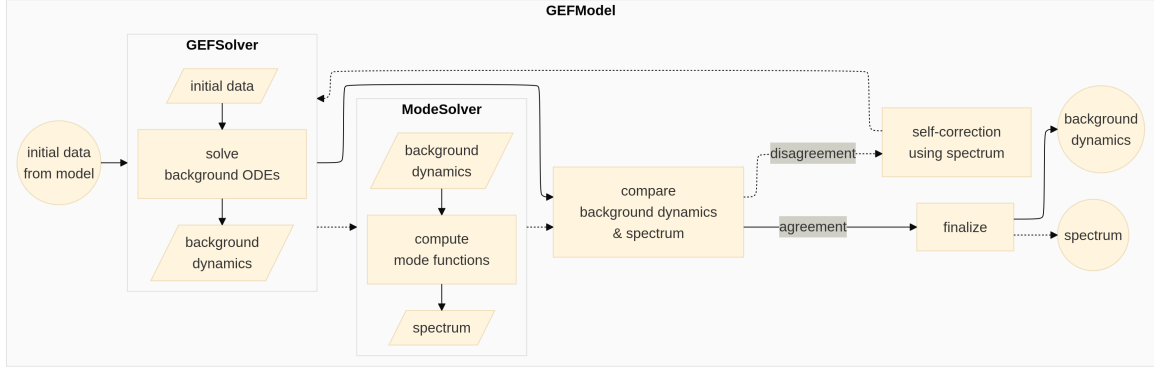


Figure 4.2: Sketch of the algorithm behind the solving method. This figure is taken from [69].

Chapter 5

BackHom vs GEF: the comparison

This chapter is fully dedicated to a systematic comparison between *BackHom* and GEF. We have chosen two benchmark potentials: a chaotic inflation quadratic potential and a composite potential defined piecewise in [78]. We will analyze the convergence behavior of the two codes changing the order of approximation, the number of gauge modes, $i_{\max}$, on the MbM side and the order, $n_{\max}$, at which the infinite chain of equations of GEF is truncated.

5.1 Benchmark scenario no. 1

The model adopted in this subsection is described by the potential

$$V(\phi) = \frac{1}{2}m^2\phi^2 \quad \text{with} \quad m = 6.16 \times 10^{-6}M_p, \quad (5.1)$$

and the following coupling strength

$$\beta = 15 \longleftrightarrow f = \frac{M_p}{15}. \quad (5.2)$$

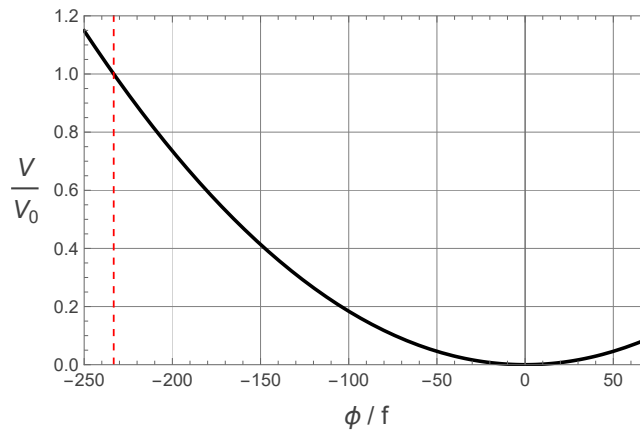


Figure 5.1: The potential in terms of the rescaled variables adopted in *BackHom*. The vertical red dashed line marks the initial value for the inflaton.

The initial conditions for the inflaton and for its velocity are set in the slow-roll approximation:

$$\phi(0) = -15.55M_p, \quad \dot{\phi}(0) = -\sqrt{\frac{2}{3}}mM_p, \quad (5.3)$$

where specifically the initial field value $\phi(0)$ is computed as follows

$$N = \int H dt = \int \frac{H}{\dot{\phi}} d\phi \stackrel{SR}{\approx} -\frac{1}{M_p^2} \int \frac{V(\phi)}{V'(\phi)} d\phi = \frac{\phi_0^2 - \phi_f^2}{4M_p^2} \quad (5.4)$$

$$\Rightarrow \phi_0 = -\sqrt{4M_p^2 + \phi_f^2} = -\sqrt{4N + 2} M_p \approx -15.55M_p,$$

where we fixed $N=60$ and ϕ_f is the field value at which we estimate inflation to end by setting $\varepsilon_V := M_p^2 V'(\phi)^2 / (2V(\phi)^2)$ equal to 1, yielding $\phi_f = \sqrt{2}M_p$. Instead, the initial value of the Hubble rate is fixed according to the Friedmann equation: $H(0) = \sqrt{[\dot{\phi}(0)^2/2 + V(\phi(0))]/(3M_p^2)}$, as explained in the previous chapter. The initial conditions are set ignoring the backreaction of gauge fields and, as we will see, backreaction will increase the duration of inflation.

Chaotic inflation

The quadratic potential we selected in eq. (5.1) was originally adopted by Andrei Linde in [41]. In his work *chaotic inflation* is proposed: in that paper, local regions in the spacetime manifold inflate stochastically and our universe is contained in one of these regions. The inflaton is lying in equilibrium in the minimum of the potential, but quantum fluctuations drive it out of equilibrium in different locations of spacetime in a stochastic manner, hence the name *chaotic*. Thus, the regions that happen to have fluctuated out of equilibrium start inflating and blow up.

In general, this theory predicts large values of the tensor-to-scalar ratio r that are excluded by experimental probes such as Planck [2]. In any case, we still consider this potential as a suitable case study since we use it to analyze the production of gauge fields in the late stages of inflation: because the backreaction becomes strong only during the last e-folds, as we will see in the following, the phenomenologically interesting effects become relevant closer to the bottom of the potential and not in the early stages of inflation when the backreaction is negligible. Therefore, we expect that during the last e-folds of inflation the phenomenology of potentials that are quadratic with similar concavity around their global minimum should be analogous.

5.1.1 Phenomenological remarks on the first benchmark scenario

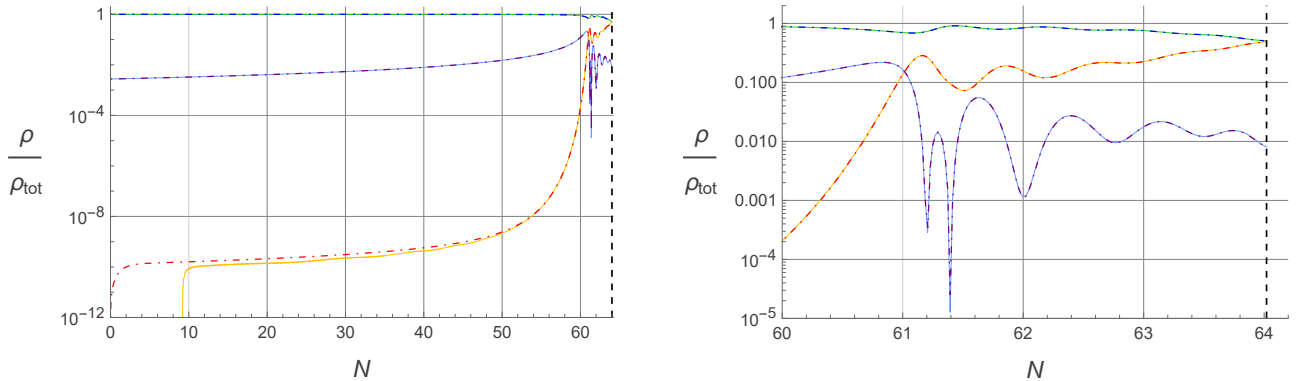


Figure 5.2: In the left panel we show the whole evolution of the kinetic energy ρ_K (solid light blue/dot-dashed purple), the potential energy ρ_V (solid green/dot-dashed blue) and the electromagnetic energy ρ_{EM} (solid yellow/dot-dashed red), all normalized with respect to the total energy $\rho_{tot} = 3H^2 M_p^2$. The right panel shows a close up of the last 4 e-folds of inflation. The solid lines correspond to *BackHom* and the dot-dashed lines to *GEFF*. The black dashed vertical line marks the end of inflation, happening at $N \approx 64$, few e-folds later than the case with no backreaction as we estimated in (5.4).

In this section we comment on the phenomenological predictions of the model just described by employing both *BackHom* and *GEFF*. In the following figures we can see the evolution of the kinetic energy $\rho_K = \dot{\phi}^2/2$ (solid light blue/dot-dashed purple), the potential energy $\rho_V = V(\phi)$ (solid green/dot-dashed blue) and the electromagnetic energy $\rho_{EM} = \langle \mathbf{E}^2 + \mathbf{B}^2 \rangle/2$ (solid yellow/dot-dashed

red) normalized by the total energy $\rho_{tot} = 3H^2 M_p^2$. The results of *BackHom* (solid) and GEFF (dot-dashed) are both shown and these runs are obtained with $i_{\max} = 1500$ and $n_{\max} = 240$, respectively, where we recall that $i_{\max}$ is the number of gauge modes evolved by *BackHom* and $n_{\max}$ is the truncation order of the chain of differential equation for electromagnetic bilinears in GEFF.

In Fig. 5.2, we notice that ρ_K and ρ_V are in agreement always, while ρ_{EM} is in disagreement for $N \lesssim 54$ in a region however where it is negligible. Also ρ_{EM} is in agreement in the region where it becomes important. In *BackHom*, the electromagnetic contribution appears at around 9 e-folds, because the gauge fields are not included at all in the backreaction until $k_{thr}(t) = k_{min} := 10^4 k_{thr}(0)$. But this does not affect the evolution of the system since the backreaction can be neglected initially. Instead, GEFF tracks the evolution of ρ_{EM} right from the beginning of the simulation. Moreover, after this first phase, there is a clear discrepancy between the computation of the electromagnetic energy from 10 to about 54 e-folds. This difference lies in the different definition of physical modes adopted in the two codes: *BackHom* accounts just for the plus polarization of modes, while GEFF includes in the backreaction also the polarization modes that have not undergone the tachyonic instability. Qualitatively, this difference is present only during the weak backreaction regime when it is approximately $\mathcal{O}(10^{-10})$ and then it essentially vanishes. We also observe that in this stage $\rho_{EM, BackHom}$ slightly oscillates. These oscillations are just a numerical artifact and appear only when the electromagnetic contribution is negligible. We will discuss the appearance of “numerical noise” in the next subsection.

During this first stage of weak backreaction, the potential energy is dominating over the kinetic energy density by 2-3 orders of magnitude and the electromagnetic energy density lies approximately 7 orders of magnitude below the kinetic energy. In this phase, the standard slow-roll solution is essentially recovered: not only does the Friedmann equation reduce to $H^2 \approx V(\phi)/(3M_p^2)$ as we can see in Fig. 5.2, but also the Hubble friction $3H\dot{\phi}$ and the slope $V'(\phi)$ terms of the axion equation are coupled. During this phase, the other terms of the axion equation are negligible, as we can see from Fig. 5.3. The evolution of the Hubble friction $3H\dot{\phi}$ (solid green/dot-dashed blue), the gauge friction $\beta\langle\mathbf{E} \cdot \mathbf{B}\rangle/M_p$ (solid yellow/dot-dashed red), the acceleration $\ddot{\phi}$ (solid light blue/dot-dashed purple) terms is shown in the following figures, while the opposite of the slope of the potential $-V'(\phi)$ is used as a reference and it is marked with a horizontal dashed gray line.

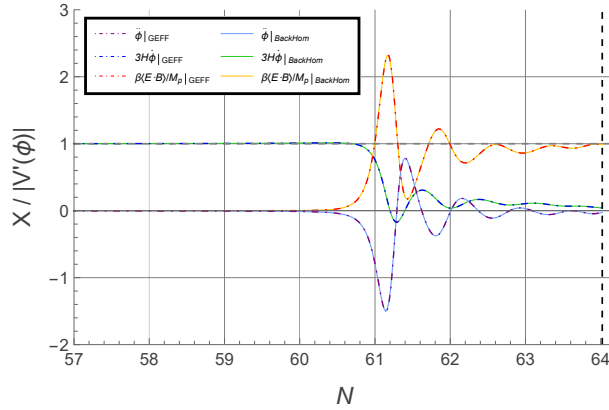


Figure 5.3: The figure shows the evolution of the terms of the axion equation marked by X and normalized by $|V'(\phi)|$ from 57 e-folds until inflation ends: the acceleration $\ddot{\phi}$ (solid light blue/dot-dashed purple), the Hubble friction $3H\dot{\phi}$ (solid green/dot-dashed blue) and the gauge friction $\beta\langle\mathbf{E} \cdot \mathbf{B}\rangle/M_p$ (solid yellow/dot-dashed red). The solid lines correspond to *BackHom* and the dashed lines to GEFF. The black dot-dashed vertical line marks the end of inflation and the gray horizontal line the value of 1.

During the last stages of inflation, the electromagnetic energy density grows exponentially and at around 60.5-61 e-folds the gauge field backreaction begins to impact the inflaton field dynamics: as we can see in the right panel of Fig. 5.2 oscillations in the evolution of all three energy densities settle in. The oscillations in the electromagnetic energy density are almost in opposite phase with the ones in the kinetic energy. This is due to the fact that the faster the inflaton is rolling down the potential, the larger the instability variable ξ is, thus promoting gauge field production. But, when backreaction

becomes significant, gauge friction slows down the axion, effectively reducing the amount of gauge fields. The effect of gauge friction can be so strong, that $\dot{\phi}$ flips sign, as it happens in correspondence to the sharp drops at 61.1 – 61.2 e-folds. On the other hand, in the right panel of Fig. 5.3 we can see that the terms of the axion equation start oscillating from the moment the backreaction criterion

$$\delta_{BR} := \left| \frac{\beta \langle \mathbf{E} \cdot \mathbf{B} \rangle / M_p}{3H\dot{\phi}} \right| \geq 1 \quad (5.5)$$

is met for the first time. We notice that the oscillations, both in these terms of the axion equation and in the energy densities, are damped and that, when inflation ends, the electromagnetic energy density is at a crossing point with the potential energy density. Let us show the equation of state at the end of inflation:

$$w = \frac{\dot{\phi}^2/2 - V(\phi) + \langle \mathbf{E}^2 + \mathbf{B}^2 \rangle / 6}{\dot{\phi}^2/2 + V(\phi) + \langle \mathbf{E}^2 + \mathbf{B}^2 \rangle / 2} \approx \frac{-V(\phi) + \langle \mathbf{E}^2 + \mathbf{B}^2 \rangle / 6}{V(\phi) + \langle \mathbf{E}^2 + \mathbf{B}^2 \rangle / 2} \quad (5.6)$$

where in the first step we neglected the kinetic energy. The end of inflation occurs when $w = -1/3$, namely when $\langle \mathbf{E}^2 + \mathbf{B}^2 \rangle / 6$ becomes equal to $V(\phi)/3$ or, more physically, when $\rho_{EM} = \langle \mathbf{E}^2 + \mathbf{B}^2 \rangle / 2$ becomes equal to $V(\phi)$. This is clearly visible in the right panel of Fig. 5.2. We also observe that the gauge friction seems to stabilize around the value of 1 in the right panel of Fig. 5.3 before the simulation is cut off. As we commented in Sec. 3.2, the coupling of the gauge friction term to the slope is a marking feature of the Anber-Sorbo regime, but it is not achieved due to the presence of oscillations.

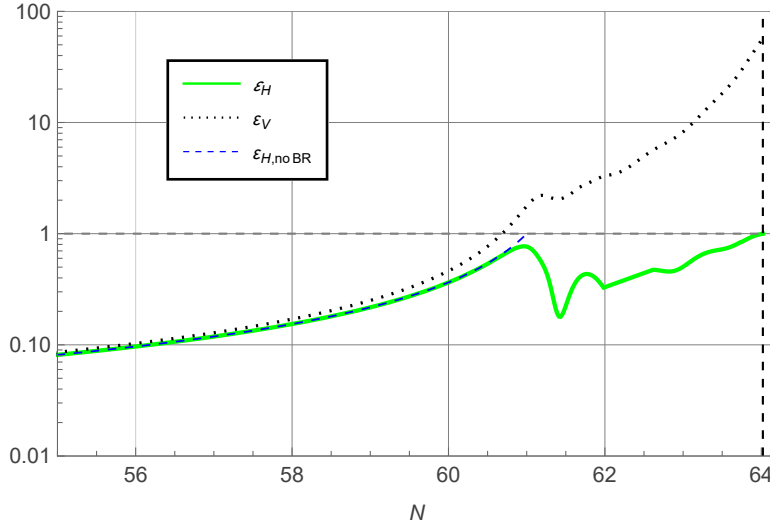


Figure 5.4: Close up of the evolution of the potential-related slow-roll parameter ε_V (black dotted), the Hubble slow-roll parameter ε_H (solid green) and Hubble slow-roll parameter neglecting backreaction $\varepsilon_{H,\text{no BR}}$. We notice that in absence of gauge fields inflation ends at 61.0 e-folds, while accounting for backreaction it is extended to 64.02 e-folds.

In Fig. 5.4, we show the evolution of the first slow-roll parameters: $\varepsilon_V := M_p^2 V'(\phi)^2 / (2V(\phi)^2)$ (dotted) and $\varepsilon_H := -\dot{H}/H^2$ (solid green when including the backreaction, dashed blue when neglecting it). We observe that the end of inflation is postponed by about 3 e-folds due to the effect of gauge friction slowing down the inflaton and blocking it from rolling too quickly. Recalling that accelerated expansion is sustained if $\varepsilon_H < 1$, we see that this condition is true for longer than in the case with no gauge fields, meaning that backreaction is actually helping to maintain the inflationary epoch when the potential becomes too steep to sustain standard slow-roll inflation ($\varepsilon_V \approx \mathcal{O}(10 - 100)$). In absence of backreaction and neglecting corrections that are of second order in slow-roll, we remark that $\varepsilon_V = \varepsilon_H$ and this is seen earlier ($N \lesssim 56$) in the evolution shown. We can also observe that the effect of gauge fields is affecting the dynamics noticeably starting from ≈ 61 e-folds.

5.1.2 The comparison

This subsection has the main goal to determine the most efficient values of the internal code parameters $i_{\max}$ and $n_{\max}$. Setting these values is a matter of compromise between the accuracy and the speed of the codes. The most efficient implementation consists in choosing the minimum values compatible with any desired accuracy. We divide our study in three separate steps. Step 1: we look at different *BackHom* runs increasing the number of gauge modes $i_{\max}$; step 2: we do the same for GEFF increasing the truncation order of the infinite system of coupled differential equations $n_{\max}$; step 3, we compare against each other the solutions obtained with the two different codes, so to determine threshold values for $i_{\max}$ and $n_{\max}$ above which the accuracy of the obtained solution does no longer improve appreciably.

We compare the runs within a single method via relative errors with respect to a reference quantity defined as follows:

$$\varepsilon_{j,\alpha}(x) := \left| 1 - \frac{\rho_{j,\alpha}(x)}{\bar{\rho}_{j,\alpha}} \right| \quad (5.7)$$

where $j \in \{K, V, EM\}$, $\alpha \in \{BackHom, GEFF\}$, x stands for either the number of gauge modes $i_{\max}$ or the GEF truncation order $n_{\max}$ and $\bar{\rho}_{j,\alpha}$ denotes the j -th energy density obtained for a reference run with $i_{\max} = 1500$ for *BackHom* and $n_{\max} = 240$ for GEFF. Instead, to compare one numerical technique with the other, we fix a run obtained with one method with a high order of approximation and we compute the relative errors with respect to runs done with the other technique varying the order of the approximation in the following way:

$$\tilde{\varepsilon}_{j,\alpha}(x) := \left| 1 - \frac{\rho_{j,\alpha}(x)}{\bar{\rho}_{j,-\alpha}} \right|, \quad \text{for } j = K, V, \quad \text{and} \quad \tilde{\varepsilon}_{EM,\alpha}(x) := 1 - \frac{\rho_{EM,\alpha}(x)}{\bar{\rho}_{EM,-\alpha}} \quad (5.8)$$

where differently from the definition of $\varepsilon_{j,\alpha}(x)$ we are using $\bar{\rho}_{j,-\alpha}$ instead of $\bar{\rho}_{j,\alpha}$ with $-\alpha = BackHom$ if $\alpha = GEFF$ and viceversa for $\alpha = BackHom$. For $j = EM$, we want to keep track of the sign of $\tilde{\varepsilon}_{EM,\alpha}(x)$ to check if one of the two methods systematically overestimates the gauge field contribution, hence the absence of the absolute value.

It is worth emphasizing the added value of the third step with respect to the previous two. If a single code is used, one can increase the corresponding internal parameter until solutions obtained with different values do not show appreciable difference. This convergence guarantees precision, but, at least in principle, not accuracy, since intrinsic numerical instabilities of an algorithm (or mistakes in a code) might result in a convergence on a spurious solution. Comparing different codes, based on independent numerical schemes, is a much stronger probe of accuracy, as it is extremely unlikely that they would converge to the same incorrect solution.

Step 1: convergence of *BackHom*

We now examine the convergence behavior of *BackHom* increasing $i_{\max}$, the number of modes we are evolving. In [10, 70, 78], works in which *BackHom* was used, the choices of $i_{\max}$ were 400, 350 and 600, respectively. Here, we want to consider cases in which $i_{\max}$ is both lower and greater than these values in order to assess how the result of the code changes: first we take a run with a high number of modes $i_{\max} = 1500$, then we consider the relative errors of runs with lower values of $i_{\max}$ so that we can define a threshold value i_{thr} over which the relative errors stop decreasing. In this work we consider the following set of numbers of modes: $\{50, 100, 150, 200, 300, 400, 600, 800, 1200\}$.

In Fig. 5.5, we can see the relative errors of the following numbers of gauge modes: 50 (light blue), 100 (yellow), 150 (green) and 200 (orange). In all three panels of Fig. 5.5 we see a converging behavior, since by increasing the number of gauge modes the relative errors diminish, except initially. In these three panels we can distinguish three different stages: the first 10 e-folds, from 10 to 60 e-folds and the last part of the evolution. Firstly, we see that, independently of the number of gauge modes,

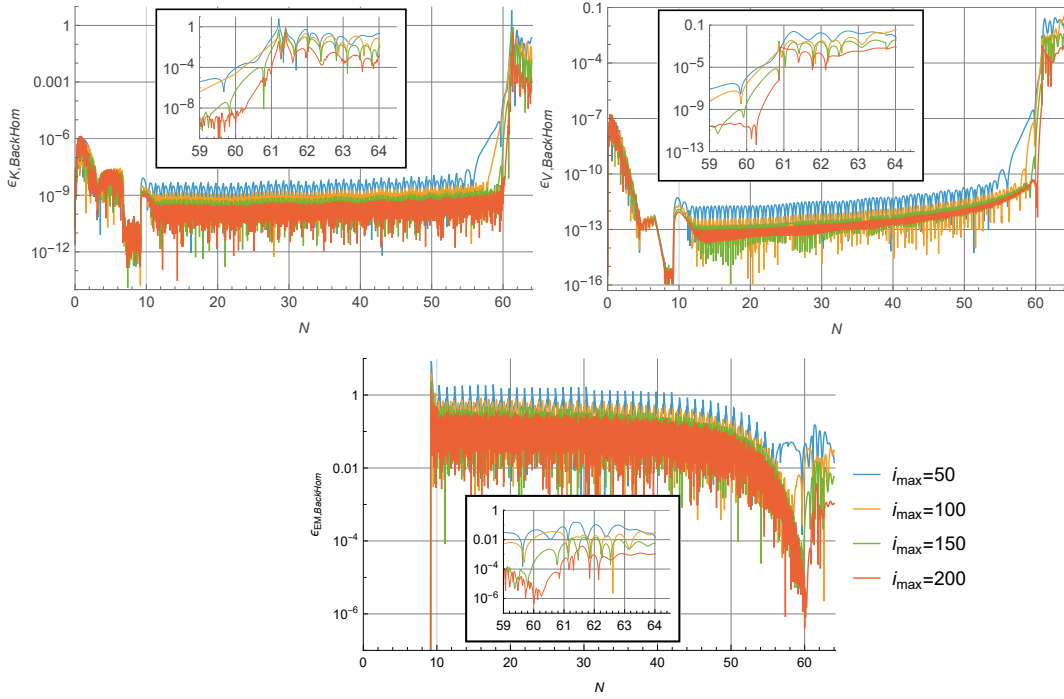


Figure 5.5: Full evolution of the relative errors of *BackHom* runs with close-ups for kinetic energy (top left panel), potential energy (top right panel) and electromagnetic energy (bottom panel) are shown. The displayed values of $i_{\max}$ are 50 (light blue), 100 (yellow), 150 (green) and 200 (orange). The inset shows the last 5 e-folds of inflation.

the relative errors of the kinetic and potential energies follow the same decreasing pattern. Then, at $N \approx 10$, the relative errors shortly spike up. From this time until $N \approx 60$, we notice that the relative errors for the kinetic energy oscillate quickly, but they are all $\lesssim \mathcal{O}(10^{-9})$, while for the potential energy they all remain roughly below $\mathcal{O}(10^{-10})$. During this same stage, we notice that the relative errors in ρ_{EM} are large, reaching values very close to 1.

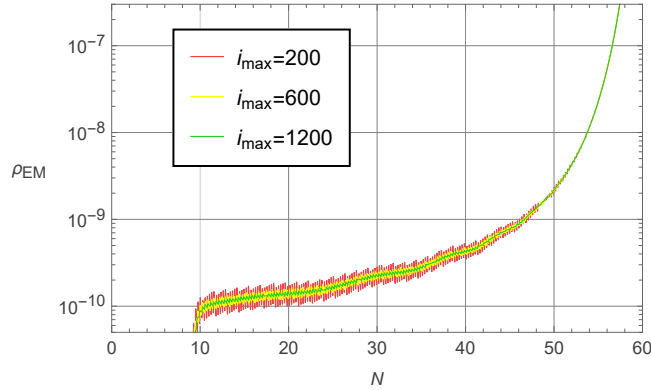


Figure 5.6: Evolution up to ≈ 58 e-folds of the electromagnetic energy density computed with *BackHom* for $i_{\max}=200$ (red), 600 (yellow), 1200 (green). Notice how the “numerical noise” is greatly reduced, even though some wide oscillations are still present. This “numerical noise” does not affect the dynamics since backreaction is negligible.

The results just described can be attributed to the following reasons: the initial pattern of the errors of kinetic and potential energy can be attributed to the system adjusting to the slow-roll attractor solution from the initial conditions; the spike in the errors at $N \approx 10$ corresponds to the moment when *BackHom* starts accounting for backreaction, making the computations more demanding for the code; the large errors in ρ_{EM} in the central section are due to “numerical noise”, since at this stage very few gauge modes are included in the backreaction and we are dealing with small numbers, that are $\mathcal{O}(10^{-10})$ (see Fig. 5.2). But, this noise gets severely reduced by enhancing the number of gauge

modes as we can see in Fig. 5.6. As we already remarked, we stress that a “noisy behavior” of the electromagnetic energy density at this stage does not affect the dynamics, since the backreaction is negligible. In Fig. 5.6, we can also clearly see that already at about 50 e-folds, when gauge fields production increases exponentially, the noise essentially disappears and, in fact, also in the bottom panel of Fig. 5.5 we observe the relative errors dropping.

There is a sharp peak in the relative errors for the kinetic and potential energies at approximately 61 e-folds, while for the electromagnetic one the peak is more shallow. As noted before, at 61 e-folds the backreaction of the gauge modes becomes relevant, hence we attribute this sudden increase to the non-linearities of the strong backreaction regime settling in: from this moment an oscillatory stage starts and the errors get larger, since a fast-moving system is computationally harder to keep track of. The sharp drops ending in a cusp in this last stage are due to the oscillations of the energy densities and they correspond to the crossings with the reference run.

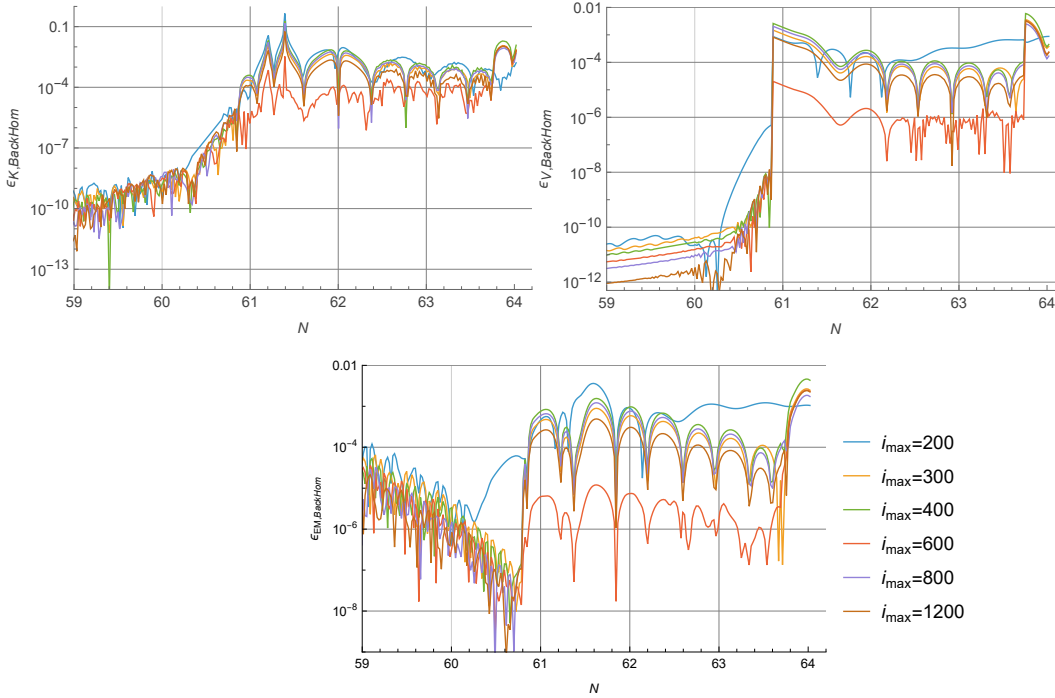


Figure 5.7: Evolution from 59 e-folds of the relative errors of *BackHom* runs for kinetic energy (top left panel), potential energy (top right panel) and electromagnetic energy (bottom panel) are shown. The displayed values of $i_{\max}$ are 200 (light blue), 300 (yellow), 400 (green), 600 (orange), 800 (purple) and 1200 (brown).

In Fig. 5.7, we show the last 5 e-folds of inflation for higher numbers of gauge modes, i.e. 200, 300, 400, 600, 800, 1200. On general grounds, we expect that precision should improve with increasing $i_{\max}$, until one reaches a threshold that can be regarded as the intrinsic limit of a given numerical scheme. The results show that no significant improvement is obtained if $i_{\max}$ is increased beyond about 200, with the noticeable exception of $i_{\max} = 600$. We are unable to explain why 600 results in a closer evolution to the reference $i_{\max} = 1,500$ case than, say, 800 or 1200, and we therefore believe that this better agreement is incidental (namely, the results suggest that runs with different $i_{\max}$ start to ‘randomly’ deviate from each other for $i_{\max} \gtrsim 200$, and the agreement between $i_{\max} = 600$ and the reference $i_{\max} = 1,500$ case is accidental). We observe that the relative errors for all three energy densities are all $\lesssim \mathcal{O}(10^{-3})$ for the runs in Fig. 5.7. There are two spikes crossing the value of 10^{-3} : they happen in correspondence of the kinetic energy crossing zero when $\dot{\phi}$ flips its sign, hence we do not consider them as deviations of one run from the other. In all three panels, we can clearly notice a jump in the relative errors right before the end of inflation. We ascribe this occurrence to the fact that ρ_{EM} gets very close to crossing ρ_V at very late times: the pattern we are noticing is that when the electromagnetic energy is crossing one of the other two energy densities, evolving the system becomes more numerically challenging, increasing the relative errors. We observe this first at about 61 e-folds when ρ_{EM} overcomes ρ_K marking the beginning of the oscillatory regime and then towards the end of

the simulation. It is very clear from the top right panel of Fig. 5.7 that the errors increase in these two instances.

Step 2: convergence of GEFF

This subsection is dedicated to analyze how GEFF converges increasing the truncation order $n_{\max}$ of the system of equations for the electromagnetic bilinears. The values of $n_{\max}$ employed in [9] are 80 for $\beta = 20$ and 115 for $\beta = 25$, in order to keep the relative errors below 0.6%. More generically, in Appendix A.1 of [64], the authors suggest to use $n_{\max} = \mathcal{O}(100)$. The lowest value of $n_{\max}$ we consider is 60, because it is the minimum value giving a successful GEFF run according to the internal crosschecks implemented in the code (see Sec. 4.2.4). Then, after fixing $n_{\max} = 240$ as the reference truncation order, we took runs with increasing values of $n_{\max}$: $\{60, 80, 100, 200, 210, 220, 230, 235\}$. The convergence analysis is done as in the previous sub-subsection by computing the relative errors for the same three observables: ρ_K , ρ_V and ρ_{EM} .

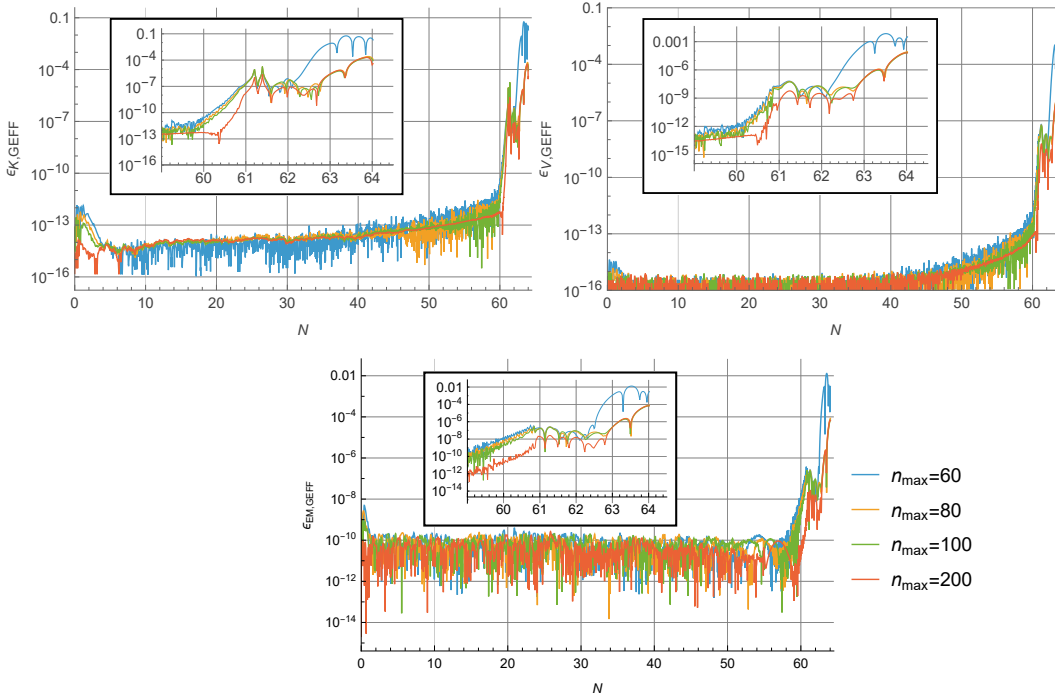


Figure 5.8: Full evolution with close-ups of the relative errors obtained with GEFF for kinetic energy (top left panel), potential energy (top right panel) and electromagnetic energy (bottom panel) are shown. The truncation orders of the displayed runs are 60 (light blue), 80 (yellow), 100 (green) and 200 (orange). The inset shows the last 5 e-folds of inflation.

In Fig. 5.8, we show the evolution of the relative errors for GEFF runs with truncation order of 60 (light blue), 80 (yellow), 100 (green) and 200 (orange). Overall, the silhouette of the relative errors is very similar to the one obtained with *BackHom*. During the first 5 e-folds of the evolution, we notice a bump which is more clearly visible in the relative errors for the kinetic energy and we see that it disappears for $n_{\max} = 200$. Then, until 60 e-folds the relative errors stay almost constant, increasing slightly closer to $N = 60$. During this stage, we do not observe a clear converging behavior, but the relative errors are very small for all of the three plots in Fig. 5.8 for the kinetic and potential energies, relative errors do not exceed $\mathcal{O}(10^{-11})$ before $N = 60$, while relative errors for the electromagnetic energy remain at $\mathcal{O}(10^{-10})$.

Moving towards the end of inflation, we notice that, as for *BackHom*, there is an increase in the relative errors of all three energy densities at 61 e-folds, but then at 63 e-folds we notice an additional increment in the relative errors. Moreover, in Fig. 5.9 we can see the improvements in the relative errors with values of $n_{\max}$ from 100 up to 235: we observe that while the bump at 60 e-folds is smoothed out by increasing the truncation order, the trend of the relative errors for the last 1.5 e-folds does not change.

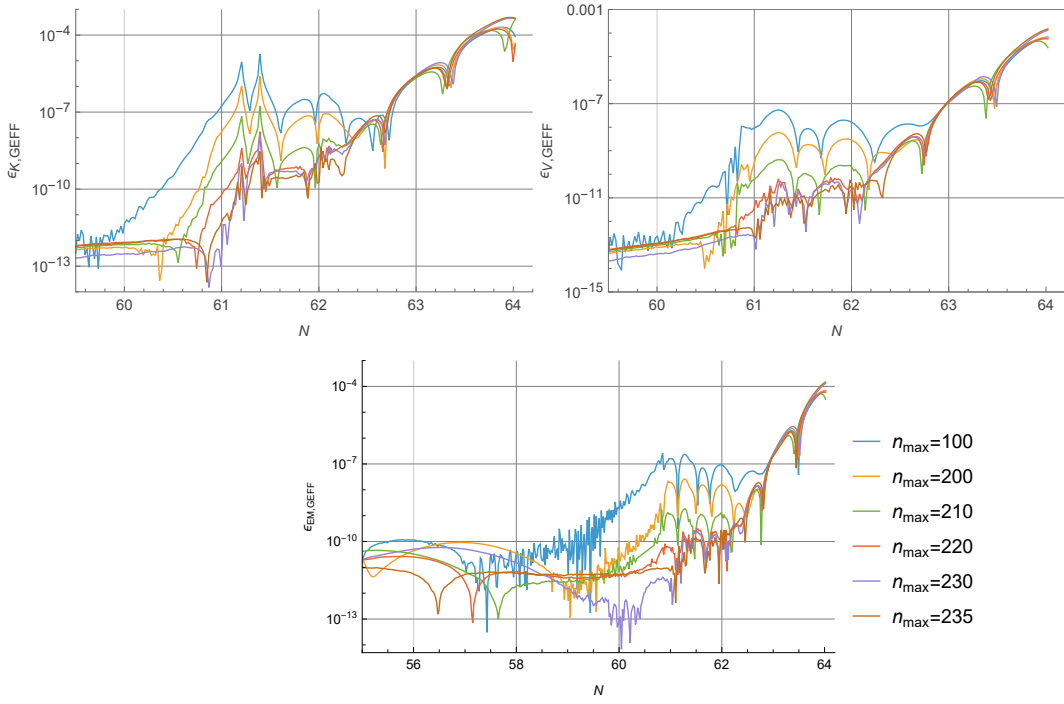


Figure 5.9: Evolutions for the last e-folds of inflation of relative errors of GEFF for kinetic energy (top left panel), potential energy (top right panel) and electromagnetic energy (bottom panel) are shown. The truncation orders are 100 (light blue), 200 (yellow), 210 (green), 220 (orange), 230 (purple) and 235 (brown).

We do not see an improvement in precision for values of $n_{\max}$ greater than 220. Hence, we consider it as a threshold for this model. Already for $n_{\max} = 80$, the maximum relative error is $\lesssim \mathcal{O}(10^{-4})$. But for $n_{\max} = 220$, considering the evolution before the last 2 e-folds, they do not exceed $\mathcal{O}(10^{-9})$ for the kinetic energy and $\mathcal{O}(10^{-10})$ for the potential and electromagnetic ones.

Step 3: matching *BackHom* and GEFF

In Fig. 5.10, we show the evolution of the relative errors, computed as in (5.8), of *BackHom* runs computed with 100 (light blue), 200 (yellow), 400 (green), 800 (orange) and 1500 (purple) gauge modes and compared with a GEFF run with $n_{\max} = 240$. We observe that in the top panels (relative errors related to kinetic and potential energies, respectively), first there is some “numerical noise” probably related to the gauge fields settling in, then, until 55 e-folds, the relative errors remain very low for each value of $i_{\max}$ we considered, $\lesssim \mathcal{O}(10^{-8} - 10^{-9})$. During the last 7-8 e-folds, the relative errors increase, as we expected due to the oscillations in the energy. We notice that there is no significant improvement for $i_{\max} \geq 200$ and that the relative errors for the 200 gauge modes case are $\lesssim \mathcal{O}(10^{-2})$ for the kinetic energy (in the top left panel, we do not consider the two cusps at around 61 e-folds, since they correspond to vanishing kinetic energy) and $\lesssim \mathcal{O}(10^{-3})$ for the potential energy. Regarding ρ_{EM} , we notice that “numerical noise” is greatly reduced by increasing the number of gauge modes and that the GEFF contribution is clearly greater than the *BackHom* one. The reason behind this discrepancy lies in the different definitions of physical gauge modes the two techniques rely on: in the GEF, a mode is deemed physical and accounted for in the backreaction if its momentum lies below the UV cutoff $k_h(t) := \max_{t' < t} \{2a(t')H(t')|\xi(t')|\}$, while, in the version of the MbM approach we presented, backreaction is computed only for modes that have undergone the tachyonic instability. In practice, GEFF includes both polarizations in the backreaction always, while *BackHom* selects just the one that gets tachyonically enhanced predominantly. Therefore, GEFF accounts also for modes that are never affected by the tachyonic instability, unlike *BackHom*. This difference is noticeable only during the first stage, in which backreaction is negligible and does not impact the dynamics, because then, when the axion speeds up, only the plus polarization is amplified and the other one becomes negligible with respect to the latter. Indeed, towards the end of inflation, the relative errors satisfy $|\tilde{\epsilon}_{\text{EM}, \text{BackHom}}| \lesssim \mathcal{O}(10^{-2})$ for $i_{\max} \geq 200$. As confirmation for this difference in the two codes, Fig. 5.3

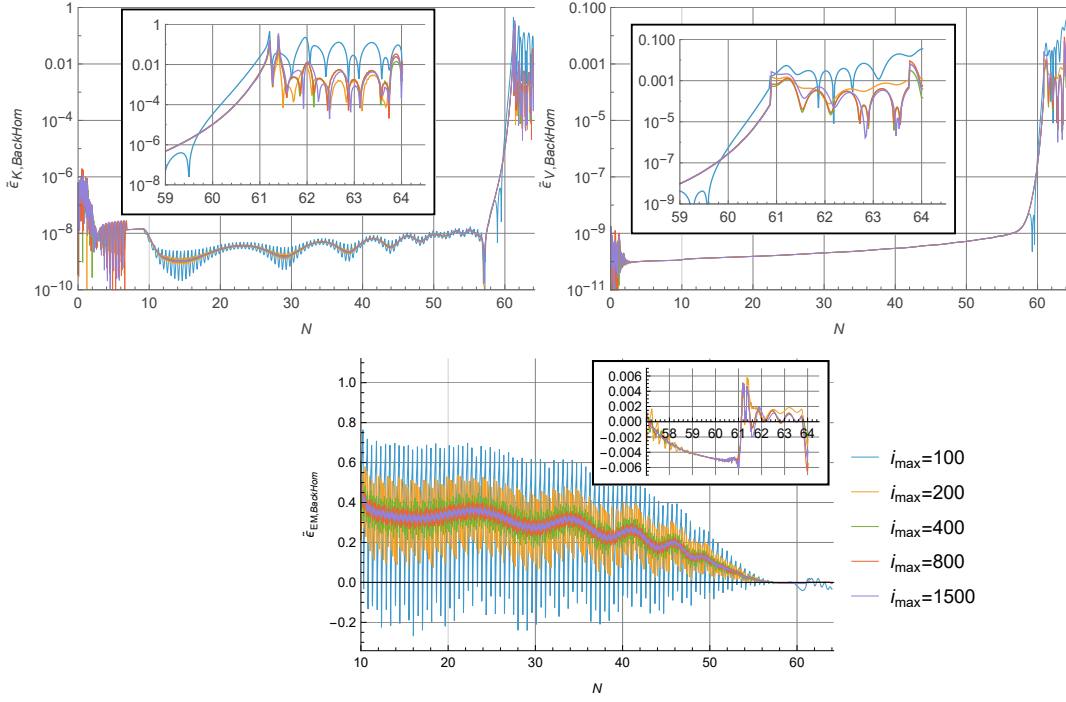


Figure 5.10: Full evolution with close-ups of the relative errors of *BackHom* runs computed with 100 (light blue), 200 (yellow), 400 (green), 800 (orange) and 1500 (purple) gauge modes and compared with a GEFF run with $n_{\max} = 240$. In the top left panel there are the relative errors for ρ_K and in the top right panel the ones for ρ_V . In the bottom panel, we are displaying the relative errors of the ρ_{EM} in linear scale, while the other plots are in log scale. We remark that in this panel the definition of $\tilde{\epsilon}$ (5.8) does not include the absolute value. Also in its inset we are excluding the case $i_{\max} = 100$ for visualization purposes.

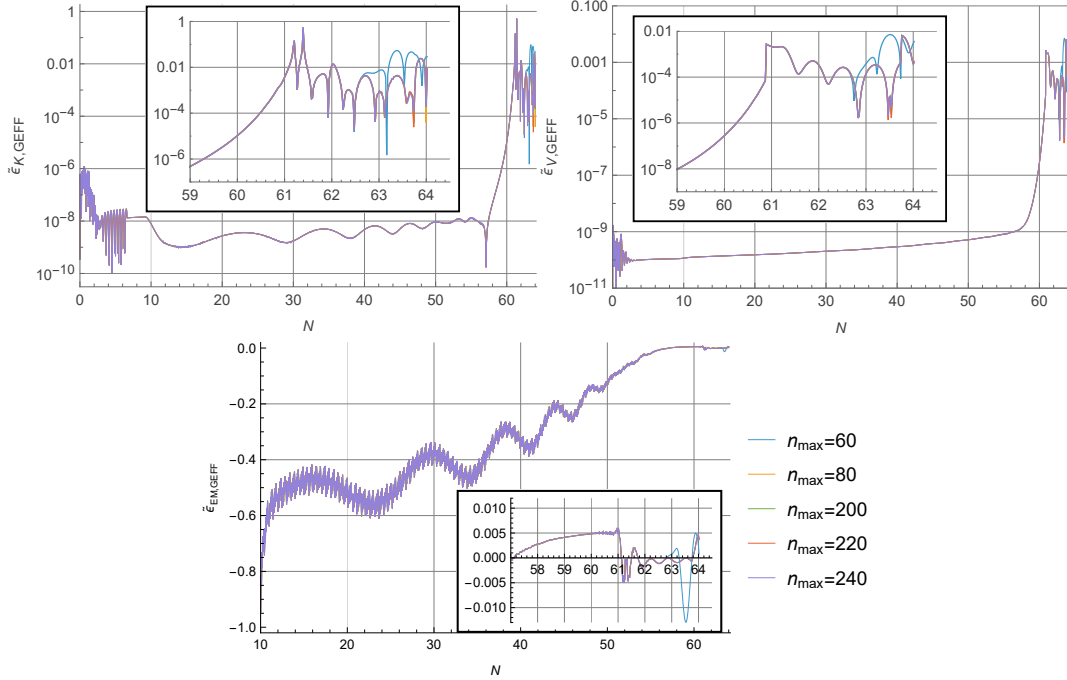


Figure 5.11: Full evolution with close-ups of the relative errors of GEFF runs computed with truncation orders of 60 (light blue), 80 (yellow), 200 (green), 220 (orange) and 240 (purple) that we compare with a *BackHom* run with $i_{\max} = 1500$. In the top left panel there are the relative errors for the kinetic energy and in the top right panel the ones for the potential energy. In the bottom panel, we are displaying the relative errors of the electromagnetic energy in linear scale, while the other plots are in log scale. Note that all of the lines overlap almost perfectly for $n_{\max} \geq 80$. The inset shows the last e-folds of inflation.

shows that the Hubble friction term is almost always positive and it flips sign only for $N = 61.2 - 61.4$, implying that also the instability variable ξ turns negative in this instance. Since the sign of ξ controls which of the two polarization modes is affected by the tachyonic instability, the minus mode A_- is tachyonically amplified just for a very brief moment. In the bottom panel of Fig. 5.10, we see a spike in the relative error of ρ_{EM} precisely in correspondence to this instance, meaning that GEFF is actually sensitive to this little contribution coming from the negative polarization, while *BackHom* ignores it. In any case the relative difference towards the end of inflation is small, $\lesssim \mathcal{O}(10^{-2})$ as previously mentioned.

In Fig. 5.11, where we are showing relative errors of GEFF runs with respect to a *BackHom* run with $i_{\text{max}} = 1500$, we notice very analogous patterns to the ones we just described in the previous paragraph. In this case, we observe that starting from $n_{\text{max}} = 80$ there is not any detectable improvement in the relative errors by increasing the truncation order and, in the cases of the kinetic and potential energies, they remain $\lesssim \mathcal{O}(10^{-2})$, except for the cusps in the top left panel at 61 e-folds. Regarding the electromagnetic relative errors, we still notice the same discrepancy in the gauge field contribution as before and that it is reduced towards the end of inflation, when $|\tilde{\varepsilon}_{\text{EM},\text{GEFF}}| \lesssim \mathcal{O}(10^{-2})$.

Conclusion

To conclude, this study shows that for the benchmark scenario defined by (5.1) and (5.2), the choices $i_{\text{max}} \approx 200$ and $n_{\text{max}} = 80$ guarantee a 10^{-3} accuracy.

5.2 Benchmark scenario no. 2

The second potential we selected is employed in [78] in the context of production of Primordial Black Holes by $U(1)$ gauge fields coupled to an axion-inflaton via a Chern–Simons-type interaction term, as in our model. The expression of this potential reads

$$V(\phi) = V_0 \times \begin{cases} 1 - \frac{\sqrt{r}}{2\sqrt{2}} \hat{\phi} - \frac{1}{4} \left(1 - n_s - \frac{3r}{8}\right) \hat{\phi}^2 & , \quad 0 \leq \hat{\phi} \leq \hat{\phi}_1 \\ p_1 \hat{\phi}^2 + p_2 \hat{\phi} + p_3 & , \quad \hat{\phi}_1 \leq \hat{\phi} \leq \hat{\phi}_2 \\ -v \hat{\phi} + p_4 & , \quad \hat{\phi}_2 \leq \hat{\phi} \leq \hat{\phi}_3 \\ p_5 \hat{\phi}^3 + p_6 \hat{\phi}^2 + p_7 \hat{\phi} + p_8 & , \quad \hat{\phi}_3 \leq \hat{\phi} \leq \hat{\phi}_4 \\ \frac{1}{2} m^2 \frac{M_p^2}{V_0} (\hat{\phi} - \hat{\phi}_5)^2 & , \quad \hat{\phi} \geq \hat{\phi}_4 \end{cases} \quad (5.9)$$

where $\hat{\phi} := \phi/M_p$ is the dimensionless code variable for the inflaton, differently from what is stated in the previous chapter, where we rescaled the scalar field by the axion decay constant f . The inflaton values are shifted so that $\phi = 0$ corresponds to the time when the Planck pivot scale $k_p = 0.05 \text{ Mpc}^{-1} a_0$ left the horizon, with a_0 the present value of the scale factor. The potential acquires the value V_0 in this instance. The initial conditions are given in the slow-roll regime for ϕ and $\dot{\phi}$, while H is set according to the Friedmann equation, analogously to the first scenario.

5.2.1 The sections of the potential

Let us go through the five branches of the potential: it is composed by three main sections (given, respectively, by the first, third, and fifth lines of (5.9)) and by two intermediate ones (the second and fourth line, respectively), that join the main sections in a way that the potential and its first derivative are continuous.

The first branch of the potential, comprised in $[0, \hat{\phi}_1]$, accounts for the CMB scales. Fixing a coupling strength such that there is no significant backreaction in this first branch, we can match the parameters

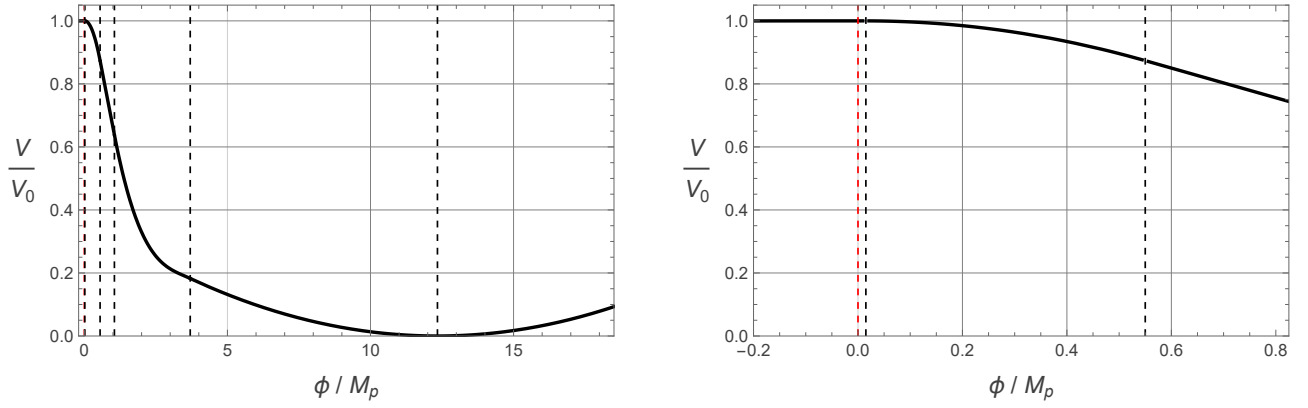


Figure 5.12: The potential proposed in [78] (left panel) with a close-up of the initial branches (right panel). The vertical dashed lines correspond to the different values of $\hat{\phi}_i$ with $i \in [1, 5]$. The red line marks the value $\hat{\phi} = 0$.

r and n_s with experimental measurements to first order in the slow-roll parameters: the tensor-to-scalar ratio $r_{obs} := \Delta_t^2/\Delta_s^2$, where Δ_t^2 is the tensor power spectrum and Δ_s^2 the scalar one, and the scalar spectral tilt $n_{s,obs} := d \ln \Delta_s^2 / d \ln k + 1$, respectively. Hence, from Planck [2] we fix $n_s = 0.965$ and we relate the ratio between V_0 and r to the measured scalar power spectrum as follows:

$$\Delta_s^2 \approx \frac{H_0^2}{8\pi^2 M_p^2 \varepsilon} \approx \frac{2V_0}{3\pi^2 M_p^4 r} \approx 2.1 \times 10^{-9}. \quad (5.10)$$

Since r has not been measured, there are only experimental upper bounds: combining Planck and BICEP/Keck data sets [79], we have $r < 0.034$ at 95% CL. For our analysis we fix $r = 3.50 \times 10^{-6}$, consistently with [78].

We notice that in this section of the potential the coefficients of the linear and quadratic terms are chosen so that

$$\begin{aligned} \partial_\phi V(\phi)|_{\phi=0} &= -\frac{V_0}{M_p} \sqrt{\frac{r}{8}}, & \partial_\phi^2 V(\phi)|_{\phi=0} &= \frac{V_0}{2M_p^2} \left(n_s - 1 + \frac{3r}{8} \right) \\ \implies \varepsilon_V|_{\phi=0} &= \frac{r}{16}, & \eta_V|_{\phi=0} &= \frac{n_s - 1}{2} + \frac{3r}{16}. \end{aligned} \quad (5.11)$$

In this way, $n_s - 1 = 2\eta_V - 3\varepsilon_V$ is automatically satisfied.

The third branch of (5.9), for $\hat{\phi}_2 \leq \hat{\phi} \leq \hat{\phi}_3$, is linear and we choose the value of the slope v so that gauge field production is sufficiently large so to lead to observable phenomenology.

The last branch, corresponding to $\hat{\phi} \geq \hat{\phi}_4$, is the bottom of the potential and is chosen to be quadratic for simplicity. The value of the inflaton mass parameter is bounded from above in order to avoid GW overproduction at this stage [64, 80, 81]. In Fig. 1 of [64], the upper bounds on m are plotted as a function of the coupling strength β by imposing that the energy density of GWs sourced by gauge fields at reheating is less than the one of 0.5 effective neutrino species. The result of [64] can be fitted by [78]:

$$\log_{10} \left(\frac{m}{M_p} \right) \lesssim 0.48 - 0.406 \frac{M_p}{f} = 0.48 - 0.406 \beta, \quad (5.12)$$

and, as done in [78], we saturate this upper bound on the inflaton mass.

5.2.2 Parameter choice

For definiteness, we consider the same parameter choice made in [78]:

$$\begin{aligned} f = M_p/\beta = M_p/20, \quad m = 2.31 \times 10^{-8} M_p, \quad r = 3.50 \times 10^{-6}, \quad n_s = 0.965, \\ v = 0.471, \quad \hat{\phi}_1 = 0.0151, \quad \hat{\phi}_2 = 0.55, \quad \hat{\phi}_3 = 1.05, \quad \hat{\phi}_4 = 3.70, \quad \hat{\phi}_5 = 12.34. \end{aligned} \quad (5.13)$$

The parameters p_i , $i \in [1, 8]$, of the joining sections are determined by imposing continuity of the potential and of its first derivative throughout the whole range of inflaton values.

In [78], the momentum of the selected gauge modes ranges from $k_{min} = 2 \times 10^5 (2aH\xi|_{N=0})$ to $k_{max} = 2aH\xi|_{N=N_{end}-5}$, where $2aH\xi$ is estimated by doing a run without backreaction, i.e. neglecting the contribution of gauge fields. They consider an inflationary epoch lasting about 55 e-folds, but they choose to consider gauge modes entering the horizon until 5 e-folds before the end of inflation. Modes that are excited during this last stage of inflation are excluded from their analysis. As stated before, we expect that modes of momenta below k_{min} to have negligible impact on the dynamics of the system since the inflaton velocity is small in the first branch of the potential. On the other hand, modes of momenta greater than k_{max} will be strongly excited during the last stages of inflation, resulting in a regime of strong backreaction, but we expect that these modes will also source inflaton gradients leading the system outside the homogeneous inflaton approximation. We will comment more on the breaking of the homogeneous inflaton approximation in Sec. 5.4 dedicated to a comparison of GEFF with lattice simulations. Since these high-momentum modes are not excited until the last e-folds of inflation, they do not affect the evolution of the system before they get amplified and the phenomenological predictions we are making are reliable.

Due to limitations of GEFF, we actually need to slightly change the setup of the simulations in order to do the comparison. We found that GEFF simulations are not able to evolve the system from values of the instability variable ξ below $1/2$ to values larger than that: by initializing the system at $\hat{\phi} = 0$ the simulation crashes after few e-folds, but by choosing $\hat{\phi} = 0.0698$, corresponding to $\xi(N=0) \approx 0.5$, we were able to achieve successful runs. On the other hand, *BackHom* has no issues in crossing the value $\xi = 1/2$ and we were able to recover the full results of [78]. From the complete runs, we are able to deduce that the inflaton value $\hat{\phi} = 0.0698$ is reached at $N \approx 24.8$, therefore we decide to do the comparison neglecting a first part of the field evolution, thus initializing the inflaton at $\hat{\phi} = 0.0698$ and ending the runs after 30 e-folds.

This issue of GEFF not being able to cross $\xi = 1/2$ is likely due to numerical difficulties in the computation of the boundary terms which involve hypergeometric functions. We observed that this problem is strongly tied to the specific value $1/2$, because, even by employing different values of β such as 10, 20, 30, 40, we were able to achieve successful runs only starting from field values corresponding to $\xi = 1/2$ with a precision of $\mathcal{O}(1\%)$. When initializing the system at $\hat{\phi} = 0$, GEFF would actually evolve the system for a few e-folds before crashing at some time before $\xi = 1/2$ was reached, hence it is not just an issue of dealing with small ξ values. More specifically, we notice how the value $1/2$ is special since it marks the crossing of two relevant scales characterizing the system: the inverse of the comoving Hubble radius aH and the threshold for the tachyonic instability $k_{thr}|_{\xi=1/2} = 2aH|\xi|_{\xi=1/2} = aH$.

As we will see in the next subsection, starting from $\hat{\phi} = 0.0698$, gauge field production is active almost immediately, hence we cannot choose as k_{min} the same value of [78], since backreaction becomes relevant before *BackHom* starts to keep track of the evolution of gauge fields. Therefore, in this section we set $k_{min} = 10 \times (2aH\xi|_{N=0})$ to $k_{max} = 2aH\xi|_{N=30}$, which is the minimum value we can choose since a polarization mode of momentum k is evolved from the Bunch-Davies boundary conditions when $k = 10 \times k_{thr}(t)$.

5.2.3 Phenomenological remarks on the second benchmark scenario

In Fig. 5.13, we show the evolution of the energy densities normalized by the total energy. Since the system, as explained in the previous subsection, is initialized closer to the linear branch of the

potential (included between $\hat{\phi}_2$ and $\hat{\phi}_3$), gauge field production is efficient right from the beginning.

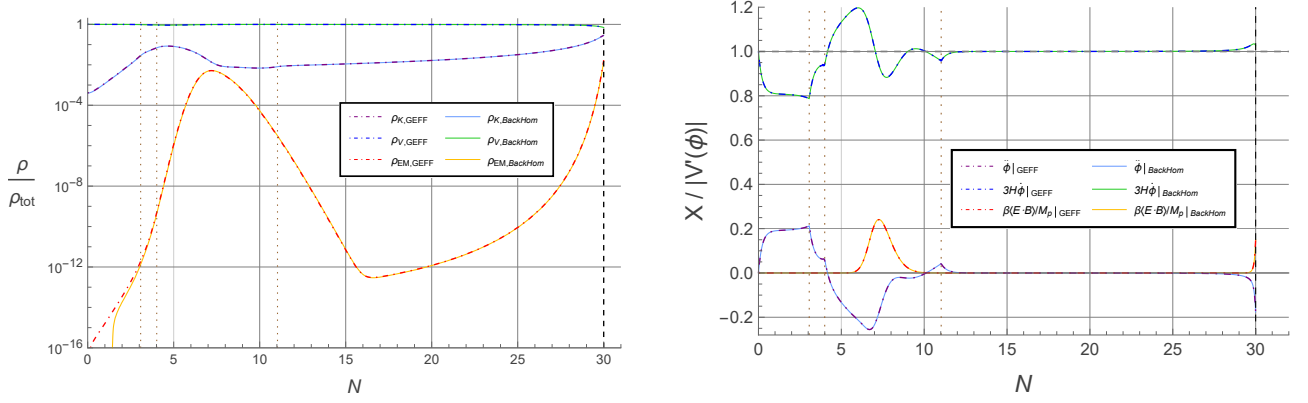


Figure 5.13: The left panel shows the evolution of the three energy densities normalized by the total energy $\rho_{\text{tot}} = 3H^2 M_p^2$: ρ_K (solid light blue/dot-dashed purple), ρ_V (solid green/dot-dashed blue) and ρ_{EM} (solid yellow/dot-dashed red). In the right panel we show the evolution of the terms of the axion equation marked by X and normalized by $|V'(\phi)|$: the acceleration $\ddot{\phi}$ (solid light blue/dot-dashed purple), the Hubble friction $3H\dot{\phi}$ (solid green/dot-dashed blue) and the gauge friction $\beta\langle\mathbf{E} \cdot \mathbf{B}\rangle/M_p$ (solid yellow/dot-dashed red). The GEFF run ($n_{\text{max}} = 240$) is marked by the dot-dashed lines, while the *BackHom* one ($i_{\text{max}} = 1500$) by the solid lines. The three dotted vertical brown lines mark the e-folds at which the inflaton values $\hat{\phi}_2$, $\hat{\phi}_3$ and $\hat{\phi}_4$ are crossed.

We observe that the kinetic energy increases, reaching a local maximum at 5 e-folds, and then, 2-3 e-folds later, the electromagnetic energy reaches a peak. As noted for the first benchmark scenario, we recognize the same delayed pattern of ρ_{EM} increasing a few e-folds after ρ_K has done so and has already diminished. Starting from 11 e-folds, the inflaton enters the quadratic branch of the potential and we observe essentially the same pattern as for the previous benchmark case, but the simulation is cut off before reaching the strong backreaction regime. This is because lattice simulations, that are able to evolve the axion keeping its spatial dependence, show that the gradient energy $\rho_G := \langle(\nabla\phi)^2\rangle/(2a^2)$ becomes relevant in the Friedmann equation when backreaction becomes strong, hence marking the breaking of the homogeneous backreaction approximation. The regime of validity of the homogeneous approximation is investigated in [82]. We will discuss more about lattice simulations at the end of this chapter.

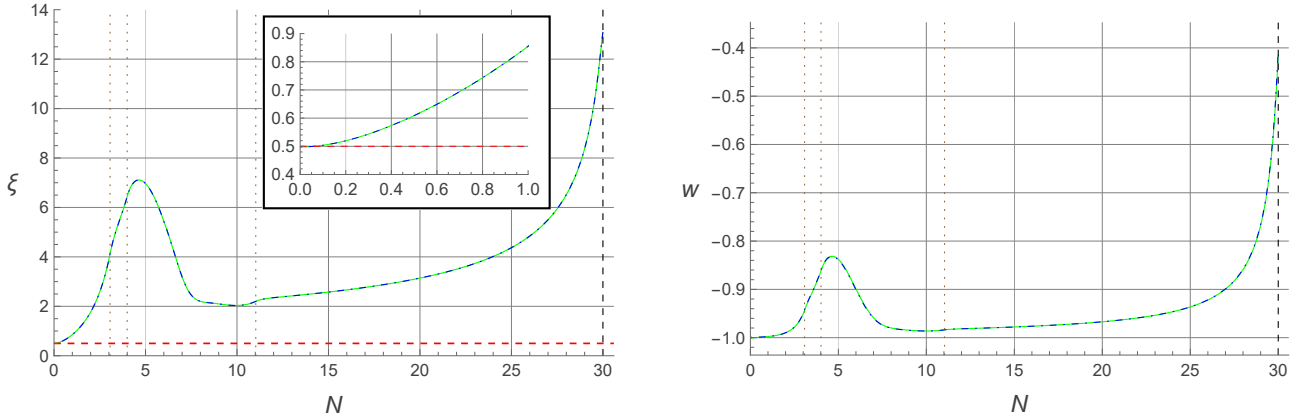


Figure 5.14: In the left panel we show the evolution of the instability variable ξ , while the right panel shows the evolution of the equation of state. The GEFF run ($n_{\text{max}} = 240$) is marked by the dot-dashed blue lines, while the *BackHom* one ($i_{\text{max}} = 1500$) by the solid green lines. The horizontal dashed red line in the left panel marks the value $1/2$ of ξ . The three dotted vertical brown lines mark the e-folds at which the inflaton values $\hat{\phi}_2$, $\hat{\phi}_3$ and $\hat{\phi}_4$ are crossed. The inset in the left panel shows the first e-folds of the evolution of ξ .

It is interesting to observe in the right panel of Fig. 5.13 the behavior of the terms of the axion equation during the first 12 e-folds of inflation. Firstly, we see the inflaton accelerating and getting faster, making the system drift from standard slow-roll in which the Hubble friction is almost exactly

coupled to the slope, as we can also see from the right panel of Fig. 5.14 where the evolution of the equation of state shows a bump for $N = 5$. Then, the inflaton decelerates and the gauge friction spikes briefly for approximately 3 e-folds.

Notably, we observe that in correspondence of the dotted brown lines marking the boundaries of the branches of the potential, the evolution of the acceleration and of the Hubble friction terms encounter sharp turning points, that will lead to spikes in the relative errors, as we will see in the next subsection. From the evolution of the axion terms, we can see how the system stabilizes in the standard slow-roll regime from 12 e-folds until the last e-fold of the simulation, when we see them briefly deviating from the constant behavior they were following.

In the left panel of Fig. 5.14, where the evolution of the instability variable is shown, we remark that the system is really initialized at values of ξ very close to $1/2$. Specifically in this case, the initial value of ξ is a little below $1/2$, but after the first time step is computed, the $\xi = 1/2$ threshold is actually overcome.

5.2.4 The comparison

The comparison of the second benchmark scenario is done in the exact same way as for the first one.

Step 1: convergence of *BackHom*

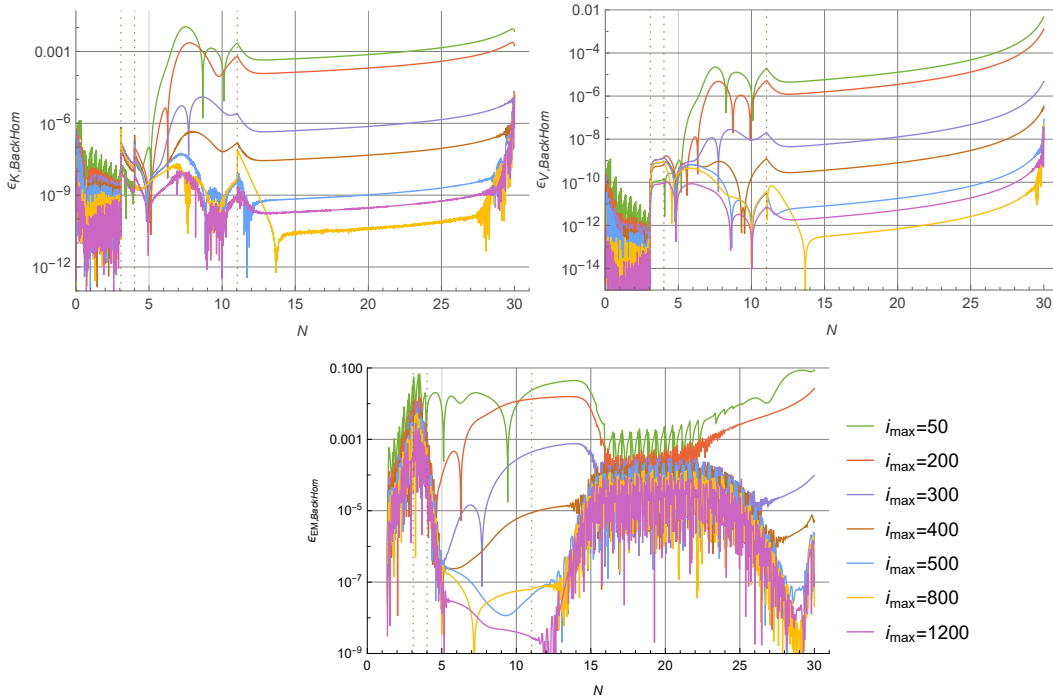


Figure 5.15: Full evolution of the relative errors of *BackHom* runs with varying values of $i_{\max}$ for the kinetic energy (top left panel), the potential energy (top right panel) and for the electromagnetic energy (bottom panel). The dotted vertical brown lines mark the e-folds at which the inflaton values $\hat{\phi}_2$, $\hat{\phi}_3$ and $\hat{\phi}_4$ are crossed.

The set of numbers of gauge modes considered is $\{50, 200, 300, 400, 500, 800, 1200\}$ and the reference run is taken with $i_{\max} = 1500$. In all three panels of Fig. 5.15, we can clearly see a convergent trend: until $i_{\max} = 500$ the relative errors are monotonically decreasing, except for the first 4-5 e-folds during which the system is adjusting to the gauge field production. For the cases with 800 and 1200 gauge modes, we do not see a clear convergent behavior anymore, hence we can consider 500 as a threshold value over which precision is not improving. Overall, for $i_{\max} = 500$, in the kinetic case (top left panel) the relative error remains below $\mathcal{O}(10^{-7})$, for the relative errors of the potential energy (top right panel) we do not exceed $\mathcal{O}(10^{-8})$ and in the electromagnetic case the relative errors stay below $\mathcal{O}(10^{-6})$, if we exclude the regions affected by noise (before 5 e-folds and in between 12 and 28 e-folds)

when backreaction is weak and the dynamics is not affected significantly.

The relative errors are the largest in two instances: between 5 and 10 e-folds when there is a local maximum in ρ_{EM} and towards the end of the simulation when the inflaton rolls more quickly and gauge modes start to proliferate. They get severely reduced in both instances by increasing the number of gauge modes. For the kinetic and potential relative errors, we observe little spikes corresponding to the boundaries of the branches of the potential. But we see that they never exceed significantly other maxima of the relative errors that instead happen due to gauge field production.

Step 2: convergence of GEFF

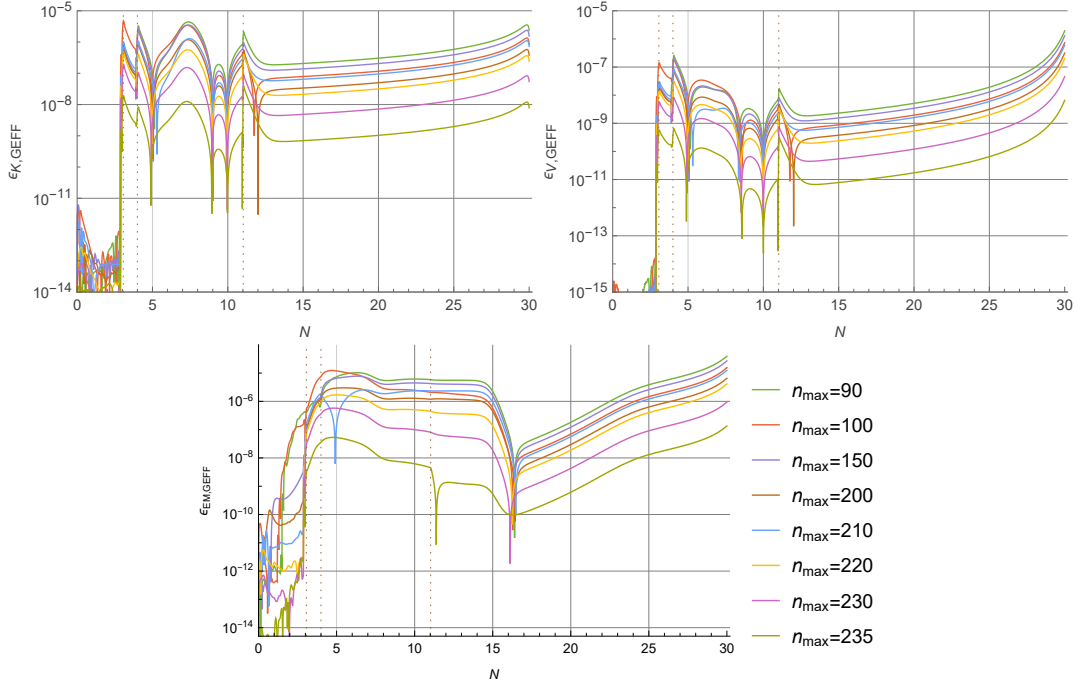


Figure 5.16: Full evolution of the relative errors of GEFF runs with varying values of n_{max} for the kinetic energy (top left panel), the potential energy (top right panel) and for the electromagnetic energy (bottom panel). The dotted vertical brown lines mark the e-folds at which the inflaton values $\hat{\phi}_2$, $\hat{\phi}_3$ and $\hat{\phi}_4$ are crossed.

To study the convergence of GEFF, we take into consideration the following set of truncation orders: $\{90, 100, 150, 200, 210, 220, 230, 235\}$. As we noted when analyzing the convergence trend of *BackHom*, there are clear spikes at the boundaries of the branches of the potential in the top panels of Fig. 5.16. The convergence pattern is not monotonic, meaning that in more than one instance increasing the truncation order increases the relative error, e.g. in all three panels of Fig. 5.16 the purple line corresponding to $n_{\text{max}} = 150$ stands above the orange line corresponding to $n_{\text{max}} = 100$ and the same happens for truncation orders of 210 and 200, respectively. We were not able to find a limit value for the truncation order over which there is no improvement in precision due to the fact that runs with $n_{\text{max}} = 250$ give a run time error after about 8 hours of running.

We notice that the pattern in the evolution of the errors is similar to the *BackHom* case, reaching a maximum when gauge field production is active. Instead, when backreaction is weak we do not see any increment associated to “numerical noise” (see bottom panel of Fig. 5.16). Already for $n_{\text{max}} = 90$ we observe that the relative errors are $\lesssim \mathcal{O}(10^{-5})$ for all three energy densities.

Step 3: matching *BackHom* and GEFF

In this sub-subsection we discuss the evolution of the relative errors presented in Fig. 5.17, in which we compare *BackHom* runs with increasing numbers of gauge modes to a GEFF run with $n_{\text{max}} = 240$, and in Fig. 5.18, in which we do the opposite, comparing GEFF runs with increasing truncation orders to a *BackHom* run with $i_{\text{max}} = 1500$.

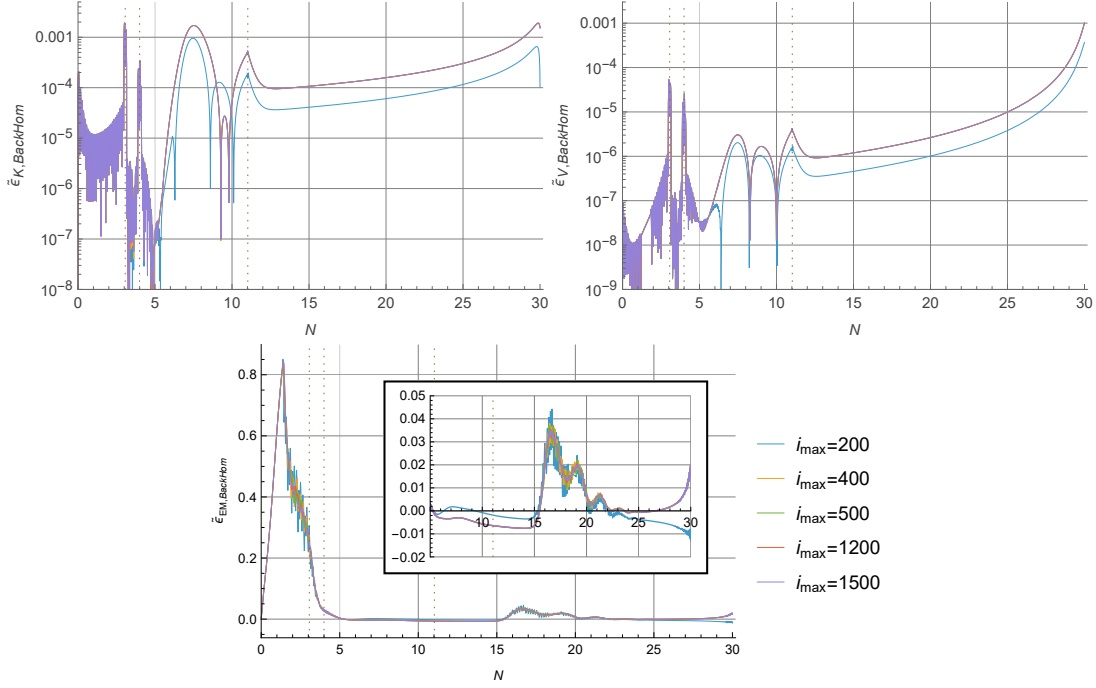


Figure 5.17: Full evolution of the relative errors of *BackHom* runs computed with 200 (light blue), 400 (yellow), 500 (green), 1200 (orange) and 1500 (purple) gauge modes and compared with a GEFF run with $n_{\max} = 240$. In the top row there are the relative errors for ρ_K (left) and ρ_V (right). In the bottom panel, we are displaying the relative errors of the ρ_{EM} in linear scale, while the other plots are in log scale. In this panel the definition of $\tilde{\epsilon}$ (5.8) does not include the absolute value. Note that all of the lines overlap almost perfectly.

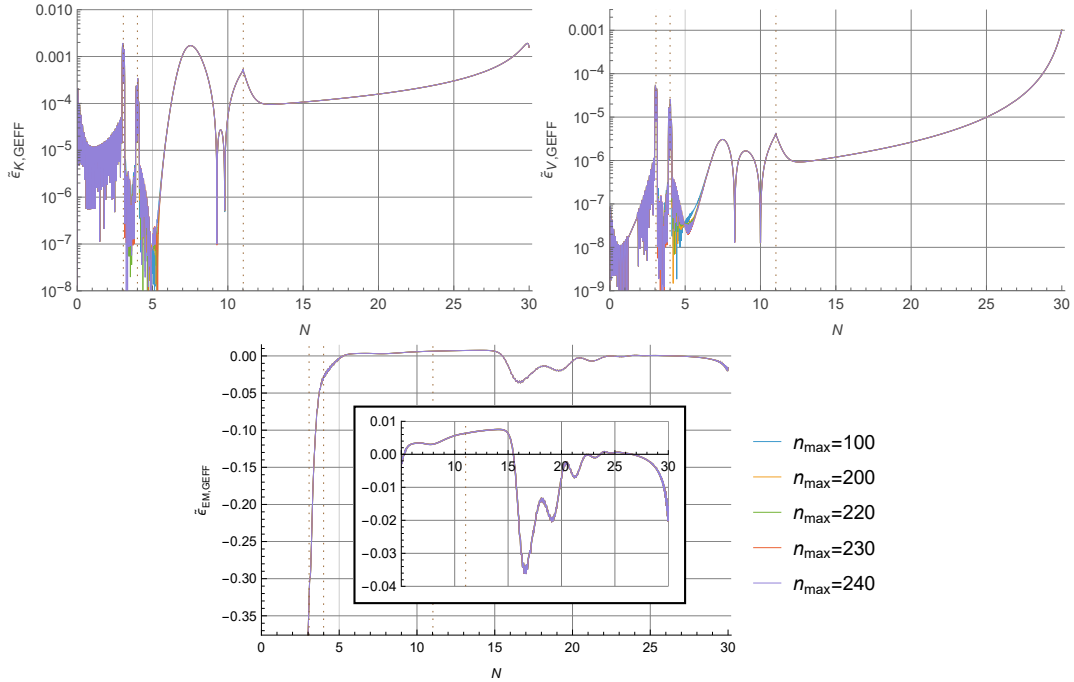


Figure 5.18: Full evolution of the relative errors of GEFF runs computed with $n_{\max} = 100$ (light blue), 200 (yellow), 220 (green), 230 (orange) and 240 (purple) that we compare with a *BackHom* run with $i_{\max} = 1500$. The panels are disposed as in the previous figure. Note that all of the lines overlap almost perfectly.

Similarly to what we observed in Fig. 5.11 for the first benchmark scenario, we notice that in both figures the relative errors do not vary by changing the level of the approximation, except for $i_{\max} = 200$ in Fig. 5.17. For the kinetic and the potential energies the relative error is always $\lesssim \mathcal{O}(10^{-3})$, while the electromagnetic relative error spikes initially, but then it remains $\lesssim \mathcal{O}(10^{-2})$. We can relate the

first big spike in $\tilde{\varepsilon}_{\text{EM}, \text{BackHom}}$ to the delay in the *BackHom* initialization time of gauge modes, but also to the contribution of the minus polarization mode A_- which is noticeable only when backreaction is negligible. Further confirmation of the presence of the A_- contribution only in GEFF comes from the spike in $\tilde{\varepsilon}_{\text{EM}, \text{BackHom}}$ between 15 and 22 e-folds: Fig. 5.13 shows that this instance corresponds to a local minimum in $\rho_{\text{EM}}/\rho_{\text{tot}}$ at a value of $\lesssim \mathcal{O}(10^{-12})$, hence we reach again the negligible backreaction regime in which the A_- contribution is comparable to the A_+ one, explaining the peak.

Conclusions

In conclusion, this study shows that for the benchmark scenario defined by (5.9) and (5.13), the choices $i_{\text{max}} \approx 200$ and $n_{\text{max}} = 100$ guarantee a 10^{-3} accuracy.

In both benchmark scenarios, we observed that the silhouettes of the $\tilde{\varepsilon}$ relative errors do not vary anymore by increasing the degree of approximation, either i_{max} or n_{max} , over a certain level. Since all of the external parameters fixing the potential, the coupling strength and the initial conditions are exactly the same, the two codes are converging to slightly different solutions due to intrinsic differences in the two methods. The main causes could be the delay in the initialization of the gauge modes in the two codes or, also, the exclusion of the A_- contribution in *BackHom*. Both of these features affect the simulations minimally since the runs are in great agreement, but they could in principle introduce little deviations affecting the whole evolution.

5.3 Comments on the run time

Let us make a few remarks on the run times of the two codes. In the following table, we show the run times of different sections of the two codes:¹ we display first the time that it took to fully solve for the dynamics (solver run time), then how long it took to compute the evolution of the energy density and gauge friction integrals for *BackHom* (vev run time) and to derive the gauge mode spectrum for GEFF (MbM run time). We remark that the last step is not necessary to solve for the dynamics, but it is done for comparing the two methods. The electromagnetic energy density and gauge friction integrals are computed every 0.05 e-folds for a total of 1281 times, while the mode-by-mode crosscheck of GEFF has been done with 200 gauge modes. Of course, the time step for the computation of the two integrals for *BackHom* and the number of gauge modes for GEFF heavily affect the vev and MbM run times, but we still include them in this section to give the reader an idea of how long it takes for the computation.

<i>BackHom</i>			GEFF		
i_{max}	solver run time	vev run time	n_{max}	solver run time	MbM run time
100	0 min 16 sec	0 min 46 sec	80	6 min 0 sec	16 sec
200	0 min 37 sec	1 min 49 sec	160	10 min 4 sec	17 sec
400	8 min 37 sec	6 min 8 sec	200	17 min 32 sec	20 sec
600	6 min 26 sec	21 min 11 sec	220	34 min 49 sec	20 sec
1500	38 min 46 sec	84 min 51 sec	240	104 min 25 sec	16 sec

Table 5.1: In this table we show the run times for varying levels of approximation, i.e. i_{max} and n_{max} , for the first benchmark scenario (quadratic potential). For each level of approximation we show how long it took to solve for the dynamics (Solver run time). We also display the run time for computing the evolution of the electromagnetic energy density and gauge friction integrals for *BackHom* (Vev run time) and for computing the gauge spectrum for GEFF (MbM run time), so that we get the same output from both codes. The integrals for *BackHom* are computed every 0.05 e-folds, thus for a total of 1281 times. The mode-by-mode crosscheck of GEFF is done with 200 gauge modes.

We notice that the vev run time increases greatly with the number of gauge modes, while the mode-

¹All of the runs were taken on the same device. The specifications are the following. Processor: AMD PRO A10-8730B R5, 10 Compute Cores 4C+6G, 2.40 GHz. Installed RAM: 8.00 GB (6.93 GB usable). Storage: 238 GB MTFDDAV2 56TBN-1AR15ABHA (SSD). Graphics card: AMD Radeon(TM) R5 Graphics (1006 MB).

by-mode crosscheck always takes about the same time and is quicker than the *BackHom* run for $i_{\max} = 200$, since we are just evolving the gauge modes on top of a background that has already been computed. In Sec. [5.1.2](#), we noticed that the GEF runs have relative errors (with respect to the $i_{\max} = 1500$ case) below $\mathcal{O}(10^{-4})$ already at $n_{\max} = 80$, while a *BackHom* run reaches relative errors of $\mathcal{O}(10^{-3})$ at $i_{\max} = 200$, if we neglect the noise in the weak backreaction regime. The first took 6 min, while the former took 1 min 49 sec. Hence, both codes take very little time to solve for the dynamics with a reasonable amount of internal precision. But, if we want to estimate accurately the evolution of the electromagnetic energy density and gauge friction integrals we need to increase the number of gauge modes greatly in order to reduce the “numerical noise” and for this purpose GEF is more convenient.

5.4 GEF and its comparison with lattice simulations

A lattice simulation is a computational technique capable of keeping track of the inhomogeneities of the inflaton field and the contribution of the axion spatial gradients in the equations of motion. The full non-linear physics of axion inflation with a Chern–Simons interaction to $U(1)$ gauge fields was treated for the first time in [\[83\]](#) via lattice simulations.

Accounting for the inhomogeneities of the inflaton, the equations of motion describing the dynamics in the temporal gauge, $A_0 = 0$, read

$$\begin{cases} \ddot{\phi} = -3H\dot{\phi} + \frac{1}{a^2}\nabla^2\phi - V'(\phi) + \frac{\beta}{M_p}\mathbf{E} \cdot \mathbf{B} \\ \dot{\mathbf{E}} = -2H\mathbf{E} - \frac{1}{a}\nabla \times \mathbf{B} - \frac{\beta}{M_p}(\dot{\phi}\mathbf{B} - \nabla\phi \times \mathbf{E}) \\ \ddot{a} = -\frac{a}{3M_p^2}(2\rho_K - \rho_V + \rho_{EM}) \end{cases} \quad (5.14)$$

while the following constraint equations must be also satisfied

$$\begin{cases} \nabla \cdot \mathbf{E} = -\frac{\beta}{M_p}\nabla\phi \cdot \mathbf{B} \\ H^2 = \frac{1}{3M_p^2}(\rho_K + \rho_G + \rho_V + \rho_{EM}) \end{cases} \quad (5.15)$$

where $\rho_K = \langle \dot{\phi}^2 \rangle / 2$ is the kinetic energy, $\rho_G = \langle (\nabla\phi)^2 \rangle / (2a^2)$ the gradient energy, $\rho_{EM} = \langle \mathbf{E}^2 + \mathbf{B}^2 \rangle / 2$ the electromagnetic energy and $\rho_V = \langle V(\phi) \rangle$ the potential energy. To solve this system of equations via lattice simulations, a cubic lattice of N^3 points with comoving volume L^3 is defined and field values ϕ_{n_1, n_2, n_3} and $\mathbf{A}_{n_1, n_2, n_3}$ are associated to each point on the grid. By fixing a discretization scheme for spatial derivatives, eqs. [\(5.14\)](#) becomes a system of coupled differential equations that can be solved numerically. A simulation is typically initialized with $L \lesssim 1/(aH)$ so that Bunch-Davies initial conditions can be assumed.

A lattice simulation is a powerful tool, but the total amount of variables we need to keep track of is extremely high: in [\[83\]](#) the number of lattice sites is $N^3 = 256^3 = \mathcal{O}(10^7)$, while in [\[84\]](#) they use up to $N^3 = 2340^3 = \mathcal{O}(10^{10})$ lattice points. In comparison, the numerical schemes for homogeneous backreaction presented in the previous chapter need to solve $\mathcal{O}(10^3)$ equations. Hence, simulations on the lattice are very expensive computationally and the system can be usually evolved for about 6 e-folds.

A study capturing the inhomogeneity and full dynamical range during strong backreaction until the end of inflation was done for the first time in [\[84\]](#), by distinguishing a linear regime of weak backreaction, in which the gauge friction term is neglected, from the non-linear regime during which the full equations are used. Considering a set of momenta included in the interval $[k_{IR}, k_{UV}]$, the simulation starts with all of the modes well inside the horizon in order to exploit the Bunch-Davies vacuum conditions. The

system is evolved in the linear regime up to a given cutoff k_{BD} , then we switch to the whole non-linear equations (5.14) exciting all of the modes in the selected range. The choice of k_{BD} is made so that the system stays in the linear regime for some time before the backreaction becomes strong. A detailed and thorough discussion on the lattice formulation is given in [85].

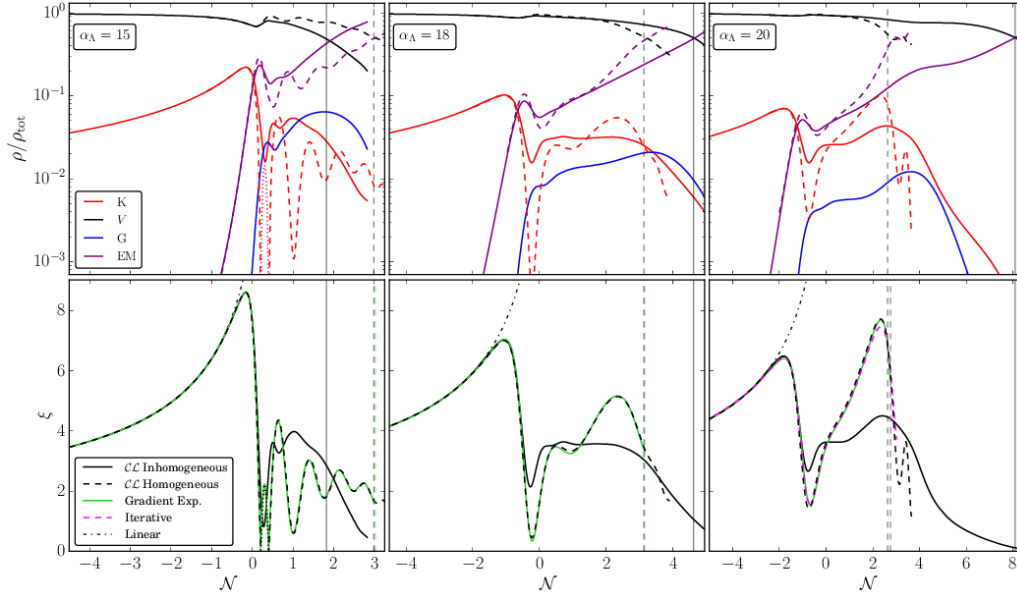


Figure 5.19: *Top Row:* Evolution of the electromagnetic (purple), inflaton potential (black), kinetic (red) and gradient (blue) energy densities, all normalized to the total energy density of the system ($3H^2 M_p^2$), for $\alpha_\Lambda = \beta = 15, 18, 20$. Lattice simulations of inhomogeneous and homogeneous backreaction are indicated with solid and dashed lines respectively. *Bottom Row:* Evolution of ξ for the same coupling constants, corresponding to simulations with inhomogeneous (black solid) and homogeneous (black dashed) backreaction, and to GEF (green solid) and iterative method [11] (magenta dashed). Solid and dashed vertical lines signal the end of inflation in each case. Evolution in the linear regime (black dash-dotted) is also shown for completeness. This figure is taken from [84].

Homogeneous backreaction is also considered in [84]. We can enforce this approximation by neglecting the terms $\propto \nabla^2 \phi, \nabla \phi \times \mathbf{E}, \nabla \phi \cdot \mathbf{B}$ in (5.14) and ρ_G in (5.15), while also considering $\mathbf{E} \cdot \mathbf{B} \rightarrow \langle \mathbf{E} \cdot \mathbf{B} \rangle$ in the equation of motion of the inflaton for consistency. In *CosmoLattice* ($\mathcal{CL}$), i.e. the code implementation [60, 86] adopted in [84], lattice simulations of both the inhomogeneous and the homogeneous regimes can be realized. In their work, the authors exploit GEF and an iterative method [11] as a mean of comparison to their lattice implementation of the homogeneous regime with the chaotic inflation potential, $V(\phi) = m^2 \phi^2/2$, the very same one we considered in the first benchmark scenario with same value of $m = 6.16 \times 10^{-6} M_p$. We report the following figure synthesizing this comparison for three different values of the coupling strength β , denoted with α_Λ in their work. Here, they show the evolution of the system during the last e-folds of inflation where the number of e-folds is denoted with $\mathcal{N}$ and $\mathcal{N} = 0$ corresponds to the end of inflation in absence of $U(1)$ gauge fields.

We observe that the GEF and the $\mathcal{CL}$ lattice simulations agree with remarkable precision in Figure 5.19, thus giving stronger validation for the GEF and consequently also for *BackHom* in light of the results found in this chapter, since the first benchmark scenario we considered is the exact same as this one.

Let us comment on the impact of inflaton gradients on the dynamics of the system. We notice that in all three cases the system stays in the linear regime for the first 2-3 e-folds before backreaction becomes important: during this initial phase the electromagnetic energy density grows exponentially until its contribution becomes a significant fraction of the kinetic energy ρ_K . At this point we see that ρ_K , that was monotonically increasing, starts to diminish, hence departing from the linear regime. Then, a stage of homogeneous backreaction follows during which GEF approximates well the full lattice simulation, but after around one e-fold the gradient energy ρ_G becomes relevant. We observe

that, while in the homogeneous axion approximation ρ_{EM} and ρ_K undergo large oscillations, these are heavily damped by the inflaton gradients. In the homogeneous case, the ξ variable even flips sign in the case of $\alpha_\Lambda = \beta = 15$ implying that the inflaton climbs back up the potential in this instance, but this does not happen when gradients are taken into account.

Another feature of these simulations worth noting is that accounting for inhomogeneities prolongs inflation monotonically with the coupling strength with respect to the linear case. For $\alpha_\Lambda = \beta = 15$ inflation ends later in the homogeneous case than in the full non-linear case, but enhancing the coupling strength the opposite is true. The trend we see here is clear: the stronger the coupling the earlier ρ_{EM} crosses ρ_K and the longer inflation lasts.

As argued in [70, 82, 87], the approximation of homogeneous backreaction, on which *BackHom* and GEFF are based, is still accurate if the potential is chosen so that $\rho_G/\rho_K \lesssim$ few percent. Hence, simulations obtained with these two codes can still produce phenomenologically accurate predictions.

Chapter 6

Numerical exploration of the Anber-Sorbo stability region

The Anber-Sorbo solution (AS) [8] is based on the analytical solution to the gauge mode equation of motion in the approximation of exact de Sitter spacetime, i.e. $H = \text{const}$, and of constant inflaton velocity, i.e. $\dot{\phi} = \text{const}$. In this regime, the solution is:

$$A_\lambda(\tau, k) = \frac{1}{\sqrt{2k}} e^{\lambda\pi\xi/2} W_{i\lambda\xi, \frac{1}{2}}(-2ik\tau) \quad (6.1)$$

where $\xi := \beta\dot{\phi}/(2M_p H)$ is the *instability variable* and $W_{i\lambda\xi, \frac{1}{2}}(-2ik\tau)$ is the Whittaker function [62]. It was initially proposed as a way to realize natural inflation on a potential too steep to satisfy the conditions of standard slow-roll. The authors of [15] showed analytically that the AS solution is unstable in the strong backreaction regime, meaning that perturbations about this solution grow exponentially, driving the solution away from (6.1) in a way that $\dot{\phi}$ oscillates. The study of the stability of the AS solution was then further investigated in [16, 88]. In [15, 88], the AS solution is found to be stable only in the case of weak backreaction when gauge friction is subdominant, not when gauge field production is highly effective. But, in the recent work [16], a new region of parameter space, in which the AS solution is stable in the strong backreaction regime, is found. The analysis of [16] is done at the linear level and within the approximations of the AS original paper [8], i.e. constant Hubble rate and constant inflaton velocity. In this chapter we will first review the techniques through which the stability study of [16] was done, then we will explore numerically this new region of parameter space and the transition of the system from the weak backreaction regime to the strong one.

6.1 The stability analysis

This section is dedicated to recovering the main results of [16]. First we introduce the toy model the authors adopted, then we describe how the perturbation study is set up in order to present their results.

6.1.1 The toy model

Let us start our analysis from the simplified scenario of an inflaton rolling in de Sitter spacetime at constant velocity down a linear potential of the form

$$V(\phi) = V_0 - \kappa(\phi - \phi_0), \quad (6.2)$$

where V_0 is the value of the potential at $\phi = \phi_0$, $\kappa = \text{const}$ is the slope and we assume that $\kappa\Delta\phi \ll V_0$ with $\Delta\phi$ being the typical inflaton excursion interval. Although very simple, this potential can

approximate a generic standard slow-roll potential during inflation for a few e-folds. By neglecting the presence of gauge fields, the inflaton velocity is constant in our scenario:

$$\dot{\phi}|_{\text{std SR}} = -\frac{V'}{3H} = \frac{\kappa}{3H} = \text{const}. \quad (6.3)$$

where *std SR* stands for standard Slow-Roll.

Accounting for the backreaction of gauge fields, a solution of constant inflaton velocity can be achieved through an analytical solution in de Sitter spacetime. In these approximations we can fix $\xi = \bar{\xi} = \text{const}$ and the mode equation

$$A_\lambda''(\tau, k) + \left[k^2 + \frac{2\lambda k \xi}{\tau} \right] A_\lambda(\tau, k) = 0 \quad (6.4)$$

takes the form of a Whittaker equation [62], the solution of which is the AS solution $A_\lambda(\tau, k) = e^{\lambda\pi\xi/2} W_{i\lambda\xi, \frac{1}{2}}(-2ik\tau)/\sqrt{2k}$, as previously stated in the beginning of the chapter. Using this analytical estimate we can compute the following correlator appearing in the axion equation:

$$\langle \mathbf{E} \cdot \mathbf{B} \rangle = -H^4 \bar{g}_0(\bar{\xi}) \quad (6.5)$$

with

$$\bar{g}_0(\bar{\xi}) := \frac{1}{4\pi^2} \int_0^{2|\bar{\xi}|} dx \sum_{\lambda=\pm} \lambda x^2 e^{\lambda\pi\bar{\xi}} \text{Re} \left[W_{i\lambda\bar{\xi}, \frac{1}{2}}(2ix) W_{1-i\lambda\bar{\xi}, \frac{1}{2}}(-2ix) \right], \quad (6.6)$$

where the integration variable is $x = k/(aH)$ and we have used the following property of the Whittaker functions: $dW_{\sigma,\mu}(y)/d\ln y = (y/2 - \sigma)W_{\sigma,\mu}(y) - W_{1+\sigma,\mu}(y)$, from eq. (13.4.33) in [62].

We notice that we are actively summing over both gauge mode polarizations, $\lambda = \pm$. This is because this analysis is done within the GEF and in this formalism physical gauge modes are the ones of momentum $k \leq 2aH\xi$, irrespectively on whether they are or they are not affected by the tachyonic instability, as highlighted in Sec. 4.2.1.

Assuming $V_0 \gg \dot{\phi}^2$, ρ_{EM} , the Friedmann equation just gives de Sitter spacetime, while we can express the axion equation as follows

$$\ddot{\phi} = 0 = -\frac{6H^2 M_p}{\beta} \bar{\xi} - V' - \frac{\beta}{M_p} H^4 \bar{g}_0(\bar{\xi}), \quad (6.7)$$

where we used that $\dot{\phi} = 2HM_p\bar{\xi}/\beta$. Eq. (6.7) can be expressed in a simpler form as

$$6\bar{\xi} - bv + b^2 \bar{g}_0(\bar{\xi}) = 0, \quad (6.8)$$

where we have defined the following dimensionless parameters in place of the coupling strength β and of the slope of the potential κ , that essentially are re-writings in Hubble units:

$$\begin{cases} b := \frac{\beta H}{M_p} \\ v := \frac{-V'}{H^3} = \frac{3\sqrt{2\epsilon_V}}{H/M_p} \stackrel{\text{SR}}{\approx} \frac{3}{2\pi\sqrt{\mathcal{P}_{\mathcal{R}}}} \end{cases} \quad (6.9)$$

where ϵ_V is the first slow-roll parameter and $\mathcal{P}_{\mathcal{R}} = H^2/(8\pi^2\epsilon_V M_p^2)$ is the power spectrum of scalar perturbations generated in slow-roll inflation.

6.1.2 The linearized gradient expansion formalism

In this subsection we focus on introducing small perturbations onto the AS background solution. Due to the fact that we are working in the homogeneous inflaton approximation, we consider perturbations that only depend on time, or, equivalently, on the number of e-folds N . For the instability variable, for example, we use the following decomposition:

$$\xi(N) = \bar{\xi} + \delta\xi(N). \quad (6.10)$$

Introducing perturbations for the electromagnetic expectation values, such as $\langle \mathbf{E} \cdot \mathbf{B} \rangle$, is more delicate: they are not just functionals of ξ and H computed at the time when we are evaluating them, but we should also account for a *memory effect* by which changes in ξ affect gauge field bilinears at a later time. This delay in the response is present due to the time delay between the moment in which a mode becomes tachyonically unstable and the moment in which its contribution becomes significant in the computation of backreaction. This type of behavior generates oscillations in the evolution of the instability variable and of the energy densities, as studied in [11].

This *memory effect* is accounted for in the first study of stability of the AS solution in the strong backreaction regime [15], where the growth of perturbations is studied by solving complicated transcendental equations. This approach is very expensive from a computational point of view. Thus, in [88] the *linearized gradient expansion formalism* (LGEF) was developed in order to study perturbations of the AS solution at the linear level in a fast and efficient manner. Now we outline the LGEF approach. We first define the following functions:

$$e_n(N) := \frac{1}{a^n H^{n+4}} \langle \mathbf{E} \cdot [(\nabla \times)^n \mathbf{E}] \rangle = \left(\frac{k_h}{aH} \right)^{n+4} \mathcal{F}_E^{(n)}, \quad (6.11)$$

$$g_n(N) := -\frac{1}{a^n H^{n+4}} \langle \mathbf{E} \cdot [(\nabla \times)^n \mathbf{B}] \rangle = \left(\frac{k_h}{aH} \right)^{n+4} \mathcal{F}_G^{(n)}, \quad (6.12)$$

$$b_n(N) := \frac{1}{a^n H^{n+4}} \langle \mathbf{B} \cdot [(\nabla \times)^n \mathbf{B}] \rangle = \left(\frac{k_h}{aH} \right)^{n+4} \mathcal{F}_B^{(n)}, \quad (6.13)$$

where n is a non-negative integer and $\mathcal{F}_X^{(n)}$ are the following electromagnetic bilinears of the full GEF:

$$\mathcal{F}_E^{(n)} := \frac{a^4}{k_h^{n+4}} \langle \mathbf{E} \cdot [(\nabla \times)^n \mathbf{E}] \rangle = \int_0^{k_h(t)} \frac{dk}{k} \frac{a^2 k^{n+3}}{2\pi^2 k_h^{n+4}} \sum_{\lambda} \lambda^n |\dot{A}_{\lambda}(t, k)|^2, \quad (6.14)$$

$$\mathcal{F}_G^{(n)} := -\frac{a^4}{k_h^{n+4}} \langle \mathbf{E} \cdot [(\nabla \times)^n \mathbf{B}] \rangle = \int_0^{k_h(t)} \frac{dk}{k} \frac{a k^{n+4}}{2\pi^2 k_h^{n+4}} \sum_{\lambda} \lambda^{n+1} \text{Re}[\dot{A}_{\lambda}(t, k) A_{\lambda}^*(t, k)], \quad (6.15)$$

$$\mathcal{F}_B^{(n)} := \frac{a^4}{k_h^{n+4}} \langle \mathbf{B} \cdot [(\nabla \times)^n \mathbf{B}] \rangle = \int_0^{k_h(t)} \frac{dk}{k} \frac{k^{n+5}}{2\pi^2 k_h^{n+4}} \sum_{\lambda} \lambda^n |A_{\lambda}(t, k)|^2, \quad (6.16)$$

where $k_h := \max_{t' \leq t} \{2a(t')H(t')|\xi(t')|\}$ is the momentum cutoff. For the background stationary solution, that has $\xi = \bar{\xi} = \text{const}$, these functions depend only on $\bar{\xi}$ and, after substituting the AS

analytical expression, (6.11)–(6.13) acquire the following form:

$$\bar{e}_n(\bar{\xi}) := \frac{1}{4\pi^2} \int_0^{2|\bar{\xi}|} dx \sum_{\lambda=\pm} \lambda^n x^{n+1} e^{\lambda\pi\bar{\xi}} \left| (x - \lambda\bar{\xi}) W_{-i\lambda\bar{\xi}, \frac{1}{2}}(-2ix) - i W_{1-i\lambda\bar{\xi}, \frac{1}{2}}(-2ix) \right|^2, \quad (6.17)$$

$$\bar{g}_n(\bar{\xi}) := \frac{1}{4\pi^2} \int_0^{2|\bar{\xi}|} dx \sum_{\lambda=\pm} \lambda^{n+1} x^{n+2} e^{\lambda\pi\bar{\xi}} \operatorname{Re} \left[W_{i\lambda\bar{\xi}, \frac{1}{2}}(2ix) W_{1-i\lambda\bar{\xi}, \frac{1}{2}}(-2ix) \right], \quad (6.18)$$

$$\bar{b}_n(\bar{\xi}) := \frac{1}{4\pi^2} \int_0^{2|\bar{\xi}|} dx \sum_{\lambda=\pm} \lambda^n x^{n+3} e^{\lambda\pi\bar{\xi}} \left| W_{-i\lambda\bar{\xi}, \frac{1}{2}}(-2ix) \right|^2. \quad (6.19)$$

By employing the expressions just defined, we can introduce the linear perturbations of the electro-magnetic bilinears:

$$\delta e_n(N) := e_n(N) - \bar{e}_n(\bar{\xi}), \quad (6.20)$$

$$\delta g_n(N) := g_n(N) - \bar{g}_n(\bar{\xi}), \quad (6.21)$$

$$\delta b_n(N) := b_n(N) - \bar{b}_n(\bar{\xi}). \quad (6.22)$$

The full system describing the evolution of the perturbations on the AS stationary background is obtained by considering the complete system of GEF equations plus eq. (6.8), substituting our Ansätze for ξ , e_n , g_n and b_n and expanding at the linear level. The full set reads

$$\begin{cases} \frac{d\delta\xi}{dN} + 3\delta\xi = -\frac{b^2}{2}\delta g_0 \\ \frac{d\delta e_n}{dN} + (n+4)\delta e_n - 4\bar{\xi}\delta g_n - 4\bar{g}_n(\bar{\xi})\delta\xi + 2\delta g_{n+1} = \delta S_n^{(e)} \\ \frac{d\delta g_n}{dN} + (n+4)\delta g_n - 2\bar{\xi}\delta b_n - 2\bar{b}_n(\bar{\xi})\delta\xi + \delta b_{n+1} - \delta e_{n+1} = \delta S_n^{(g)} \\ \frac{d\delta b_n}{dN} + (n+4)\delta b_n - 2\delta g_{n+1} = \delta S_n^{(b)} \end{cases} \quad (6.23)$$

where the R.H.S. of the last three equations have the form

$$\delta S_n^{(e)} := \frac{(2\bar{\xi})^{n+4}}{4\pi^2} \sum_{\lambda=\pm} \lambda^n \left\{ \frac{E_\lambda(\bar{\xi})}{\bar{\xi}} \left[(n+4)\delta\xi + \frac{d\delta\xi}{dN} \right] + \frac{dE_\lambda(\bar{\xi})}{d\bar{\xi}} \delta\xi \right\}, \quad (6.24)$$

$$\delta S_n^{(g)} := \frac{(2\bar{\xi})^{n+4}}{4\pi^2} \sum_{\lambda=\pm} \lambda^{n+1} \left\{ \frac{G_\lambda(\bar{\xi})}{\bar{\xi}} \left[(n+4)\delta\xi + \frac{d\delta\xi}{dN} \right] + \frac{dG_\lambda(\bar{\xi})}{d\bar{\xi}} \delta\xi \right\}, \quad (6.25)$$

$$\delta S_n^{(b)} := \frac{(2\bar{\xi})^{n+4}}{4\pi^2} \sum_{\lambda=\pm} \lambda^n \left\{ \frac{B_\lambda(\bar{\xi})}{\bar{\xi}} \left[(n+4)\delta\xi + \frac{d\delta\xi}{dN} \right] + \frac{dB_\lambda(\bar{\xi})}{d\bar{\xi}} \delta\xi \right\}, \quad (6.26)$$

and the functions E_λ , G_λ , and B_λ are defined as

$$E_\lambda(\xi) := \frac{e^{\lambda\pi\xi}}{4\xi^2} \left| (2i|\xi| - i\lambda\xi) W_{-i\lambda\xi, \frac{1}{2}}(-4i|\xi|) + W_{1-i\lambda\xi, \frac{1}{2}}(-4i|\xi|) \right|^2, \quad (6.27)$$

$$B_\lambda(\xi) := \frac{e^{\lambda\pi\xi}}{2|\xi|} \operatorname{Re} \left\{ W_{1-i\lambda\xi, \frac{1}{2}}(-4i|\xi|) W_{i\lambda\xi, \frac{1}{2}}(4i|\xi|) \right\}, \quad (6.28)$$

$$B_\lambda(\xi) := e^{\lambda\pi\xi} \left| W_{-i\lambda\xi, \frac{1}{2}}(-4i|\xi|) \right|^2. \quad (6.29)$$

The system (6.23) is composed of an infinite number of equations, therefore we must truncate it in order to close the system of equations. The way we choose is setting to zero all of the perturbations of order higher than some $n_{\max}$:

$$\delta e_n = \delta g_n = \delta b_n = 0 \quad \text{for} \quad n > n_{\text{cut}}. \quad (6.30)$$

Lyapunov exponents

The full system of equations (6.23) we just obtained with LGEF is made of first order differential equations, hence we can express it in matrix notation as follows

$$\frac{d}{dN} \mathbf{x}(N) = \underline{\mathbf{A}} \cdot \mathbf{x}(N), \quad (6.31)$$

where

$$\mathbf{x} := \left[\begin{array}{c} \delta\xi \\ \left\{ \begin{array}{c} \delta e_n \\ \delta g_n \\ \delta b_n \end{array} \right\} \end{array} \right] \quad (6.32)$$

is the vector of all of the perturbations. The curly brackets $\{\dots\}$ indicate a set of $3(n_{max} + 1)$ variables and the matrix $\underline{\mathbf{A}}$ is fixed by the set of LGEF equations (6.23).

In order to solve eq. (6.31) we impose the following Ansatz:

$$\mathbf{x}(N) = \mathbf{x}_0 e^{\zeta N}, \quad (6.33)$$

where $\mathbf{x}_0$ is a constant vector and ζ is, in general, a complex number. Hence, thanks to this Ansatz we are left with just a linear eigenvalue problem:

$$(\underline{\mathbf{A}} - \zeta \underline{\mathbf{1}}) \cdot \mathbf{x}_0 = \mathbf{0}. \quad (6.34)$$

The eigenvalues ζ are called Lyapunov exponents and they fully characterize the stability behavior of the AS solution at the linear level for a given $\bar{\xi}$. We highlight that solving this eigenvalue problem is just an algebraic task that does not involve solving differential equations.

By ordering the eigenvalues ζ by increasing real part, we call ζ_1 the Lyapunov exponent with the largest real part, namely

$$\text{Re}(\zeta_1) \geq \text{Re}(\zeta_2) \geq \text{Re}(\zeta_3) \geq \dots \quad (6.35)$$

Therefore, the value of $\text{Re}(\zeta_1)$ determines the stability or instability of the solution: if $\text{Re}(\zeta_1) < 0$, small perturbation around the stationary AS solution will decay, while if $\text{Re}(\zeta_1) > 0$ perturbations will grow making the solution unstable. Hence, the stability criterion is given by

$$\text{Re}(\zeta_1) \leq 0. \quad (6.36)$$

The imaginary part of ζ_1 , instead, will be responsible for the oscillations of the perturbations around the AS solution.

6.1.3 Stability in the strong backreaction regime

Colors in Fig. 6.1, taken from ref. [16], indicate the value of $\text{Re}(\zeta_1)$ as a function of the rescaled slope parameter v and the rescaled coupling strength b . The area shaded in blue corresponds to negative values of $\text{Re}(\zeta_1)$, thus marking the region of parameter space in which the AS solution is stable against small perturbations.¹ On the other hand, the area shaded in yellow indicates the instability region where small perturbations grow. The contour line in red draws the boundary between the stability and instability regions along $\text{Re}(\zeta_1) = 0$. The black lines in Fig. 6.1, superimposed to the colors, are $\bar{\xi}$ contour lines for $10^{-5} \leq \bar{\xi} \leq 7$. We observe that for smaller values of b they become straight lines of negative slope: in fact, for $b \ll 1$, eq. (6.8) reduces to

$$0 = 6\bar{\xi} - bv + b^2 \bar{g}_0(\bar{\xi}) \stackrel{b \ll 1}{\approx} 6\bar{\xi} - bv \implies \ln b \approx -\ln v + \ln(6\bar{\xi}), \quad (6.37)$$

¹More accurately, against perturbations that keep the axion homogeneous.

hence for a fixed value of $\bar{\xi}$ this is the equation for a straight line of slope -1 in a log-log plot. Note that the relation (6.37) obtained in the small coupling limit is the same we get by neglecting the contribution of the gauge fields, $\bar{g}_0(\bar{\xi}) \rightarrow 0$. We can define the value the instability parameter would have in absence of gauge fields as

$$\bar{\xi}_{SR} := \frac{bv}{6} \quad (6.38)$$

for some pair (v, b) . If the contribution of gauge fields is negligible, the equation in the first row of (6.23) has a simple solution

$$\frac{d\delta\xi}{dN} + 3\delta\xi = -\frac{b^2}{2}\delta g_0 \approx 0 \implies \delta\xi \propto e^{-3N}, \quad \zeta_1 = -3, \quad (6.39)$$

therefore perturbations get exponentially damped and we recover the stable standard slow-roll regime.

By taking the large coupling limit instead, i.e. $b \gg 1$, eq. (6.8) tends to the following:

$$0 = 6\bar{\xi} - bv + b^2\bar{g}_0(\bar{\xi}) \stackrel{b \gg 1}{\approx} b(-v + b\bar{g}_0(\bar{\xi})) \implies \ln b \approx \ln v - \ln \bar{g}_0(\bar{\xi}), \quad (6.40)$$

that, for a specific value of $\bar{\xi}$, implies that the contour line tends to a straight line of slope $+1$ in a log-log plot.

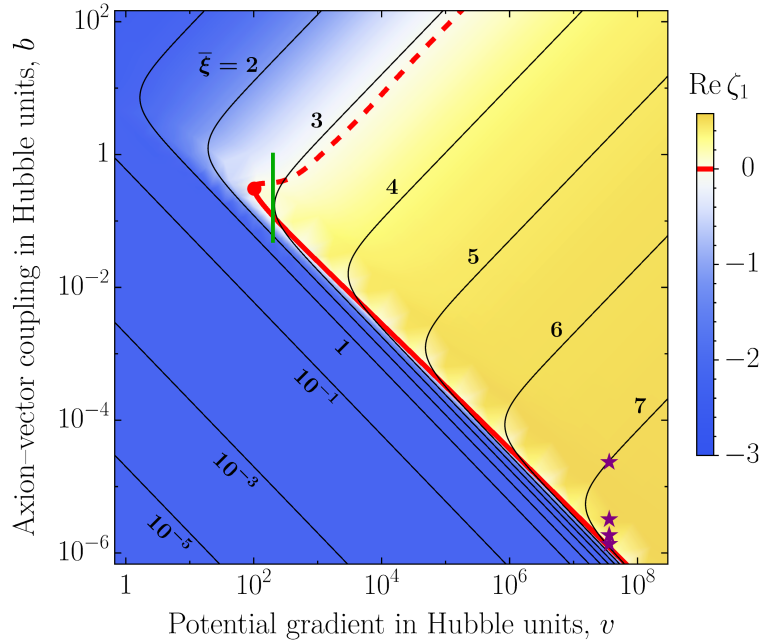


Figure 6.1: Stability diagram in LGEF approach with $n_{max}=70$. The color in the background indicates the value of $\text{Re}(\zeta_1)$ as a function of the b and v parameters (6.9). The solid and red line mark the $\text{Re}(\zeta_1) = 0$ contour line when crossed from the weak and strong backreaction regimes respectively. The black lines represent the $\bar{\xi}$ contour lines. The red dot separating the solid and dashed red lines represents a codimension-2 bifurcation point, the properties of which are not discussed in the present work, but it is examined in detail in Sec. 3.4 of [16]. In the same section a bifurcation diagram is built from the solid green segment. The purple stars denote benchmark points considered in Sec. 3.5 of [16], from which this figure was taken.

Notably, a peculiar feature of Fig. 6.1 is that, fixing a value of $v > 102$ and starting from the stable blue region corresponding to $\bar{\xi} \lesssim \mathcal{O}(1)$, by increasing the value of b we first enter the yellow instability region, but then for values of b sufficiently large we reach another region of stability. This blue region in the top left corner corresponds to a stable AS configuration in the strong backreaction regime.

In [16], the strong backreaction regime is defined as follows:

$$\delta_{\text{BR}} = \left| \frac{\beta \langle \mathbf{E} \cdot \mathbf{B} \rangle / M_p}{3H\dot{\phi}} \right| \geq 1, \quad (6.41)$$

where δ_{BR} is called the backreaction parameter. In Fig. 6.2 we can see the same region of parameter space shown in Fig. 6.1, but in less detail. In this figure, the strong-weak backreaction divide line is shown (solid blue) and three different regions can be clearly distinguished: the white area on the bottom left side corresponds to cases in which the backreaction is weak, namely $\delta_{\text{BR}} < 1$, instead the strong backreaction region above the blue line is divided into two regions a green one and a yellow one that respectively indicate stable and unstable AS solutions. The green region in particular is a new region of parameter space that has not been explored before and it will be analyzed in this chapter. We remark that, while very interesting for understanding the general stability structure of this system, the stable region corresponds to small values of ξ , that are unlikely to lead to observable phenomenology.

We also observe that between the red solid line and the blue line there is a thin region of parameter space in which the AS solution is unstable even though the backreaction is still weak according to (6.41). This means that there are configurations in which the presence of gauge fields, although subdominant, can ultimately destabilize the system. Hence, the stability criterion in (6.36), i.e. $\text{Re}(\zeta_1) \leq 0$, does not perfectly overlap with the weak backreaction regime. In particular, the effect of axion gradients might be suppressed, even though the AS solution is ultimately unstable. A similar regime was recently investigated in detail in Ref. [82]. In future work, it would be interesting to map the results of Ref. [82] onto the parameter space in Fig. 6.2 and check this expectation.

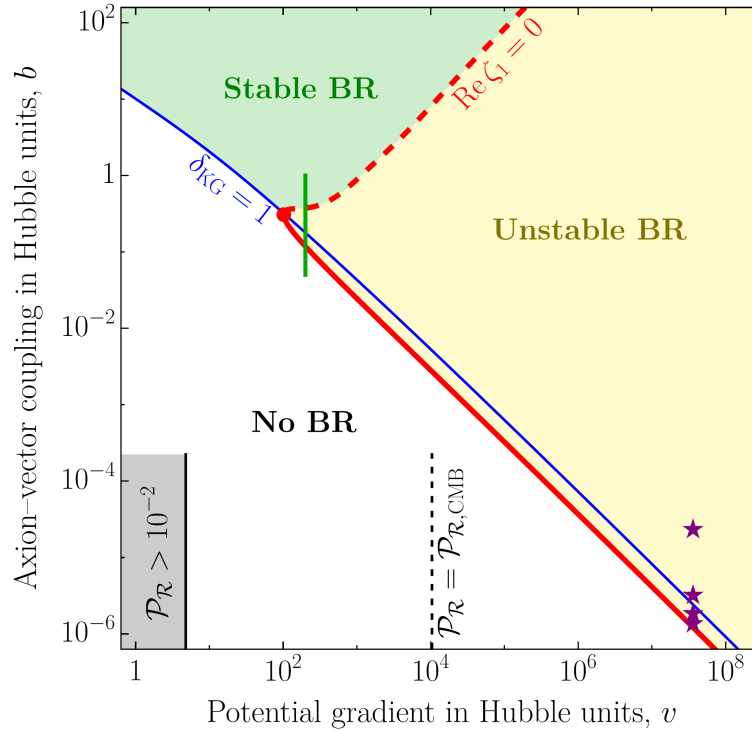


Figure 6.2: In this diagram we can see the region of parameter space corresponding to weak backreaction (white), unstable strong backreaction (yellow) and stable strong backreaction (green). The blue line indicates the $\delta_{\text{BR}} = 1$ contour line. The vertical dashed line marks the value of v fixed by the CMB polarization. The gray area covers the region of excessively large vacuum scalar perturbations, $\mathcal{P}_{\mathcal{R}} > 10^{-2}$, where the homogeneous axion approximation breaks down even in absence of gauge fields. The rest of the notation is the same as in Fig. 6.1. This figure is borrowed from [16].

6.2 Numerical exploration of the stability region

In this section, we extend the work of [16] by exploring the recently found region of parameters space in which the Anber-Sorbo solution is stable. In the analysis of [16], this region was found considering perturbations around an AS background in the assumptions of constant H and constant $\dot{\phi}$. Here, we relax these assumptions by evolving the full system of equations, making H and ϕ dynamical variables.

Firstly, we prepare the system with slow-roll initial conditions on a branch of the potential flat enough to accommodate a standard slow-roll regime, then we steepen the slope of the potential so that gauge field production becomes highly efficient. To achieve this, we build a toy model characterized by a potential constructed by joining two linear slopes via a sigmoid function:

$$V(\phi) := V_0 + \left[\frac{\kappa_{AS} + \kappa_{SR}}{2} + \frac{\kappa_{AS} - \kappa_{SR}}{2} \sigma(\phi) \right] (\phi - \bar{\phi}) + \left[\frac{\kappa_{AS} + \kappa_{SR}}{2} + \frac{\kappa_{AS} - \kappa_{SR}}{2} \sigma(0) \right] \bar{\phi}, \quad (6.42)$$

where

$$\sigma(\phi) := \frac{(\phi - \bar{\phi})/\Delta\phi}{1 + |(\phi - \bar{\phi})/\Delta\phi|} \quad (6.43)$$

is a sigmoid function, V_0 is the value of the potential at $\phi = 0$, κ_{SR} and κ_{AS} are the asymptotic absolute values of the slope, $\bar{\phi}$ is the field value at which the two slopes are evenly mixed ($\sigma(\bar{\phi}) = 0$) and $\Delta\phi$ is the *smoothness* parameter regulating the sharpness of the transition. Notice that for $\phi - \bar{\phi} \gg \Delta\phi$, $\sigma(\phi)$ approaches the value of 1 and the slope of the potential tends to $-\kappa_{AS}$, while for $\phi - \bar{\phi} \ll -\Delta\phi$ the sigmoid tends to -1 and the slope of the potential to $-\kappa_{SR}$.

In the following figure the silhouette of the potential is shown.

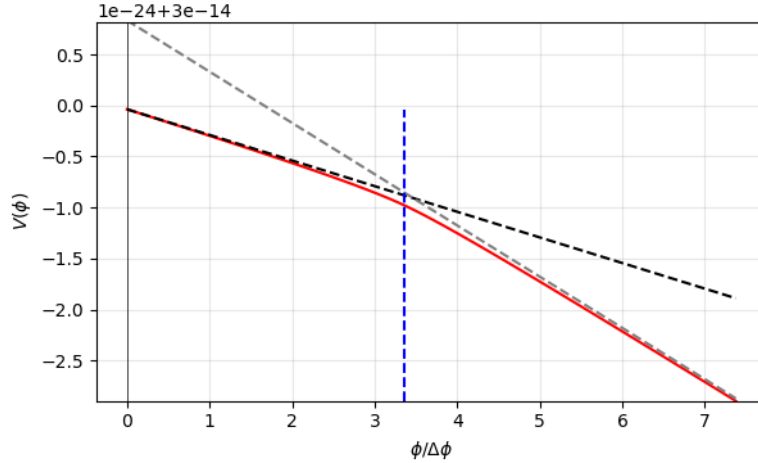


Figure 6.3: The potential is in red, the black and the gray dashed lines are the asymptotic values of the slope, $-\kappa_{SR}$ and $-\kappa_{AS}$ respectively. The blue dashed vertical line marks $\phi = \bar{\phi}$. The x-axis is rescaled by the smoothness parameter $\Delta\phi$.

Let us outline how the various parameters are fixed in the current analysis. We fix the initial value of the Hubble rate to $\bar{H} = 10^{-7} M_p$, so that for a given point (v, b) in the parameter space shown in Fig. 6.1, we choose the values of the coupling strength and of the slope at asymptotically large values of ϕ by inverting the definitions in (6.9):

$$\beta = b \frac{M_p}{\bar{H}}, \quad \kappa_{AS} = v \bar{H}^3. \quad (6.44)$$

On the other hand, evolving the system from $\phi = 0$, the initial value of the potential is found by inverting the Friedmann equation:

$$V_0 = 3M_p^2 \bar{H}^2 - \frac{1}{2} \bar{H}^4 (\bar{e}_0(\bar{\xi}) + \bar{b}_0(\bar{\xi})) - \frac{1}{2} \dot{\phi}^2, \quad (6.45)$$

where $\dot{\phi} := 2\bar{H}M_p\bar{\xi}/\beta$ is obtained by inverting the definition of ξ , $\bar{e}_0(\bar{\xi})$ and $\bar{b}_0(\bar{\xi})$ are the functions defined in (6.17) and (6.19), $\bar{\xi}$ is the value of the instability variable found by solving eq. (6.8) for a given pair (v, b) .

In this way, when the inflaton reaches the final branch of the potential, the system will enter the AS configuration defined by the targeted pair (v, b) in Fig. 6.1. Even though the Hubble rate is dynamical and its value will not stay exactly constant during the simulation, it remains practically constant, as we will show.

In order to approximately keep $\dot{\phi}$ of the same order of $\dot{\phi}$ throughout the evolution of the system, we fix the slope of the first branch of the potential from the slow-roll equation $\dot{\phi} = -V'(\phi)/(3H)$ as follows

$$\kappa_{SR} = 3\bar{H}\dot{\phi}. \quad (6.46)$$

Then, to initialize the inflaton velocity, via eq. (6.8) we find the value ξ^* of the instability variable corresponding to the pair $(v_{SR} = \kappa_{SR}/\bar{H}^3, b)$:

$$\dot{\phi}(N=0) = 2\xi^* \bar{H} M_p / \beta. \quad (6.47)$$

The last free parameter to fix is the mean value of the sigmoid function and we rewrite it as follows

$$\bar{\phi} = \bar{N} \frac{\kappa_{SR}}{V_0} M_p^2, \quad (6.48)$$

where $\bar{N}$ is a free parameter determining the number of e-folds the inflaton would approximately roll for from $\phi = 0$ to $\phi = \bar{\phi}$ on the potential $V_{SR}(\phi) := V_0 - \kappa_{SR}\phi$ neglecting the gauge mode backreaction. The parameter $\bar{N}$ is chosen depending on how long it takes for the system to stabilize to the attractor corresponding to the first branch of the potential.

6.2.1 Results

The stable AS solution we target in this subsection is characterized by the coordinates $(\bar{v} = 114.755561, \bar{b} = 1)$ in the parameter space shown in Fig. 6.1. This specific choice for the slope parameter v is made so that $(10\bar{v}, \bar{b})$ lies on the red contour line,² marking the stability boundary. We will compare the stable solution with this other benchmark point in the following, as well as with $(100\bar{v}, \bar{b})$.

Recovering the Anber-Sorbo regime

In Fig. 6.4, we show the evolution of the kinetic and electromagnetic energies normalized with respect to the total energy (left panel) and the evolution of the terms of the axion equation normalized by the opposite of the slope of the potential (right panel). The potential energy is not shown in the left panel, but it is dominating the energy budget of the system throughout the whole dynamical evolution in such a way that $H^2 \approx V(\phi)/(3M_p^2)$ with a relative deviation $\lesssim \mathcal{O}(10^{-13})$, as can be inferred from the value of the other energy components shown in the figure. We examine this benchmark point with both *BackHom* (solid) and GEFF (dot-dashed) in order to give greater validity to this finding.

We fix the shape of the potential by choosing $\Delta\phi = 10^{-6}$, so that the transition from the first branch of the potential to the second one happens smoothly, and by setting $\bar{N} = 60$, in such a way that the transition happens late enough for the system to stabilize. In the beginning, the system settles in a regime in which the backreaction is weak, but not negligible: in the right panel of Fig. 6.4, we see that, initially, the gauge friction contribution is roughly half that of the Hubble friction; therefore, the

²We thank Oleksandr Sobol for giving us the precise value of $\bar{v}$, so that we can directly compare with the result of [16].

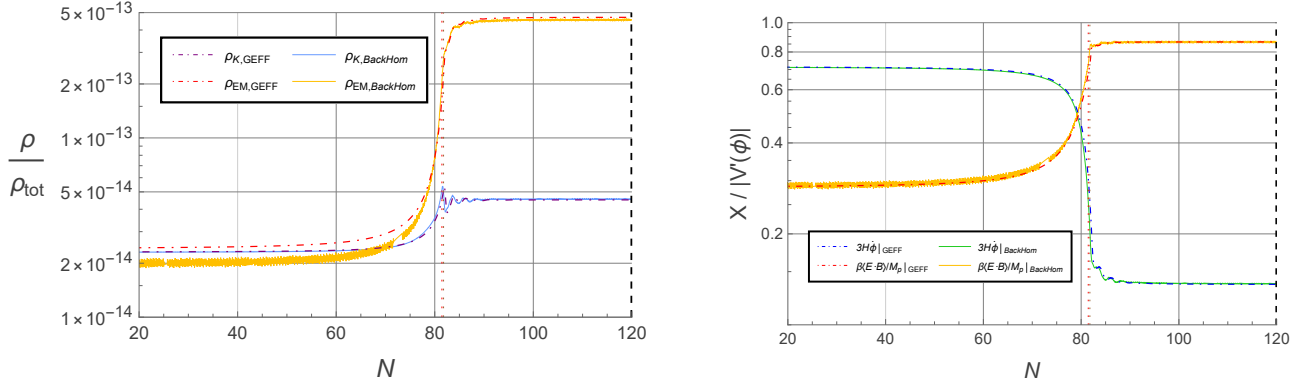


Figure 6.4: The evolution of the kinetic (solid light blue/dot-dashed purple) and electromagnetic (solid yellow/dot-dashed red) energies normalized by $\rho_{\text{tot}} = 3H^2 M_p^2$ is shown in the left panel. The evolution of the terms of the axion equation normalized by $-V'(\phi)$ can be seen in the right panel: the Hubble friction (solid green/dot-dashed blue) and the gauge friction (solid yellow/dot-dashed red). The *BackHom* run is marked with the solid lines, while the GEFF one with the dot-dashed lines. The vertical dashed line indicates the end of the simulation, while the red and brown vertical dotted lines mark the crossing of $\phi = \bar{\phi}$ for the *BackHom* and GEFF runs respectively. These two lines almost overlap. The system evolved in these runs is characterized by $(\bar{v} = 114.755561, \bar{b} = 1)$, $\Delta\phi = 10^{-6}$ and $\bar{N} = 60$. The *BackHom* run was taken with $i_{\text{max}} = 1000$ gauge modes and the GEFF one with $n_{\text{max}} = 200$.

fact that at $t = 0$ the gauge field contribution is vanishing in both codes forces us to select a big value for $\bar{N}$, in such a way that the system can correctly stabilize to the attractor before the slope changes significantly. This is just a limitation coming from the fact that the two codes neglect the presence of gauge fields initially.

The $\phi = \bar{\phi}$ value is crossed at ≈ 81 e-folds (vertical dotted lines), later than the 60 e-folds we would have expected in the absence of gauge fields, as confirmation of the impact of gauge friction on the inflaton velocity. We observe that the transition from one regime to the other happens in about 40 e-folds, starting from $N = 60$. Some oscillations, which are more visible in the kinetic energy and in the Hubble friction, take place, but they get damped away, as we expected from the linear study of perturbations of [16], since in this region of parameter space the real part of the Lyapunov exponent ζ_1 is negative. During the last 30 e-folds of the simulation, we note that the system is stabilizing into a regime of strong backreaction since the gauge friction term is coupled to the slope.

We note that the *BackHom* run is affected by “numerical noise”, especially in the first stage. We chose a remarkably high number of gauge modes, $i_{\text{max}} = 1000$, to limit it, but, in any case, the “noisy behavior” is located in a range in which backreaction is weak and, then, it converges to good agreement with GEFF.

While in the right panel the GEFF evolution of the gauge friction overlaps with the one computed by *BackHom*, the electromagnetic energy takes clearly different values before the transition happens. As we commented for the first benchmark scenario in the previous chapter, we attribute this difference to the fact that GEFF accounts for both polarizations in the computation of backreaction, to which a mode contributes if its momentum is below the UV threshold, while *BackHom* features only the plus polarization, which is the one that undergoes the tachyonic instability. This difference effectively vanishes when entering the strong backreaction regime.

We display the evolution of the inflaton and of the instability variable in Fig. 6.5 while in Fig. 6.6 we show the evolution of the equation of state and of the Hubble rate after shifting and rescaling them for better visualization. It is interesting to note, from the right panel of Fig. 6.5 that ξ evolves from a regime in which it is constant, ≈ 1.86 , to another one characterized by a value of ≈ 2.60 , through a transitioning phase in which oscillations are featured. These oscillations arise because the potential is modulating too quickly, hence when the slope changes, the system has not yet adapted to the attractor solution corresponding to that specific combination of v and b , meaning that the dynamical variables are in a perturbed version of the proper attractor and we see damped oscillations.

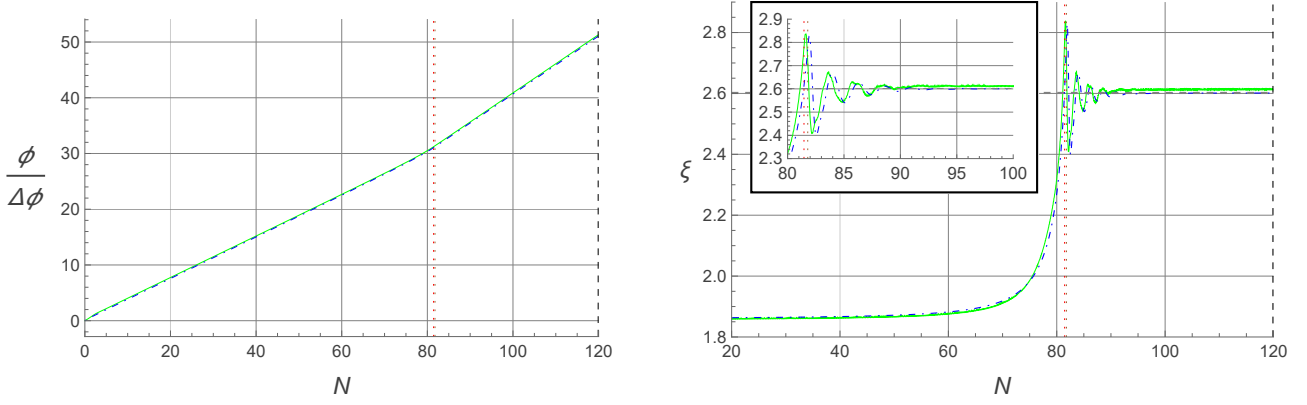


Figure 6.5: In this figure we show: the evolution of the inflaton field rescaled by the smoothness parameter $\Delta\phi = 10^{-6}$ (left panel) and of the instability variable (right panel) with a close-up on the oscillations. The solid green line represents the *BackHom* run and the dot-dashed one the GEFF run. The horizontal dashed gray line marks the value of $\bar{\xi}$. The rest of the notation follows from the previous figure.

We notice that, in the left panel of Fig. 6.5, the evolutions of the ξ variable in the two codes do not overlap and look as if the output of one code was delayed with respect to the other; in any case, from a qualitative point of view, the pattern they follow features the same number of oscillations with very similar amplitude to then stabilize on slightly different constant values. To quantify this delay, the inset of the left panel shows the minimal difference between the two values at which the inflaton crosses $\bar{\phi}$ in the two codes: $N_{\text{BackHom}} = 81.45$ and $N_{\text{GEFF}} = 81.79$. This delay between the two simulations could be related to two factors. As stated before, the system needs to stabilize after it is initialized, since backreaction is weak but not negligible in the beginning. Hence, due to the fact that it takes a different amount of e-folds for the two simulations to start properly accounting for gauge fields, the delay could be caused by *BackHom* taking longer to completely stabilize to the attractor with respect to GEFF. Moreover, another possible origin of the delay could lie in the fact that GEFF includes both polarization modes in the backreaction, while *BackHom* accounts only for one: in the first benchmark scenario of the previous chapter, we saw that ρ_{EM} is greater for GEFF than in *BackHom* during the weak backreaction stage, but then, this difference vanishes when backreaction becomes strong and ρ_{EM} grows by several orders of magnitude. In the case analyzed in this section, ρ_{EM} grows by roughly one order of magnitude, hence the discrepancy present in the weak backreaction regime might not have been erased fully. We remark how this difference in the electromagnetic energy seems to be tied to the strength of backreaction, rather than the overall magnitude of ρ_{EM} since in the chaotic benchmark scenario of the previous chapter the discrepancy appeared for $\rho_{\text{EM}}/\rho_{\text{tot}} \approx \mathcal{O}(10^{-10})$.

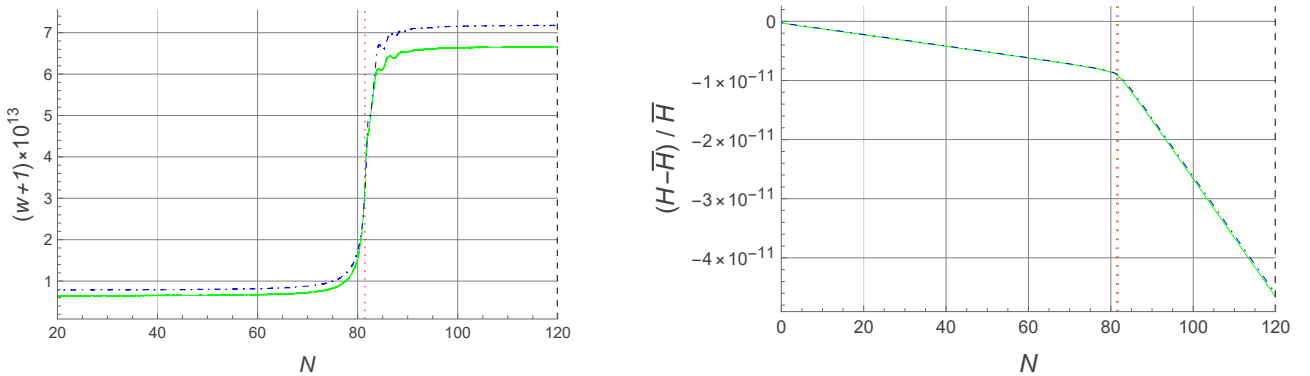


Figure 6.6: In this figure we show: the evolution of the equation of state w shifted by 1 and rescaled by a factor 10^{13} (left panel) and the evolution of the relative deviation of the Hubble rate with respect to its initial value. We observe that both these variables remain essentially constant. The notation follows from Fig. 6.4.

In Fig. 6.6, we observe that the Hubble rate remains always essentially constant, with a total relative excursion of $< \mathcal{O}(10^{-10})$. The equation of state lies very close to -1 throughout the whole evolution,

therefore accelerated expansion is always sustained throughout the simulation. The discrepancy we see between the two codes in the left panel is $\mathcal{O}(10^{-13})$, therefore it is negligible.

Connecting all of the information gained in the last three figures, we notice that after the transition took place, we have successfully entered an inflationary regime of strong backreaction in which gauge friction couples to the slope in the inflaton equation and the instability variable ξ is constant as well as the Hubble rate, exactly as described by Anber and Sorbo in [8].

Changing the value of v

This sub-subsection focuses on the analysis of two other benchmark points with the same value of $\bar{b}$ as the one just examined, but with greater slope: $(10\bar{v}, \bar{b})$ and $(100\bar{v}, \bar{b})$. The first benchmark point lies on the red contour line of Fig. 6.7 corresponding to $\text{Re}(\zeta_1) = 0$, while the second one is inside the yellow instability region $\text{Re}(\zeta_1) > 0$. Therefore, from the linear perturbative analysis we expect perturbations about the AS solution to get amplified and oscillate when we reach the second branch of the potential.

In Figs. 6.7 and 6.8, we display the evolution of ρ_K and ρ_{EM} (left column), of the terms of the inflaton equation (central column) and of the instability parameter (right column) for the two benchmark points, characterized by $(10\bar{v}, \bar{b})$ and $(100\bar{v}, \bar{b})$. For clearness reasons, firstly we show only the *BackHom* run, and then the runs obtained with the two codes.

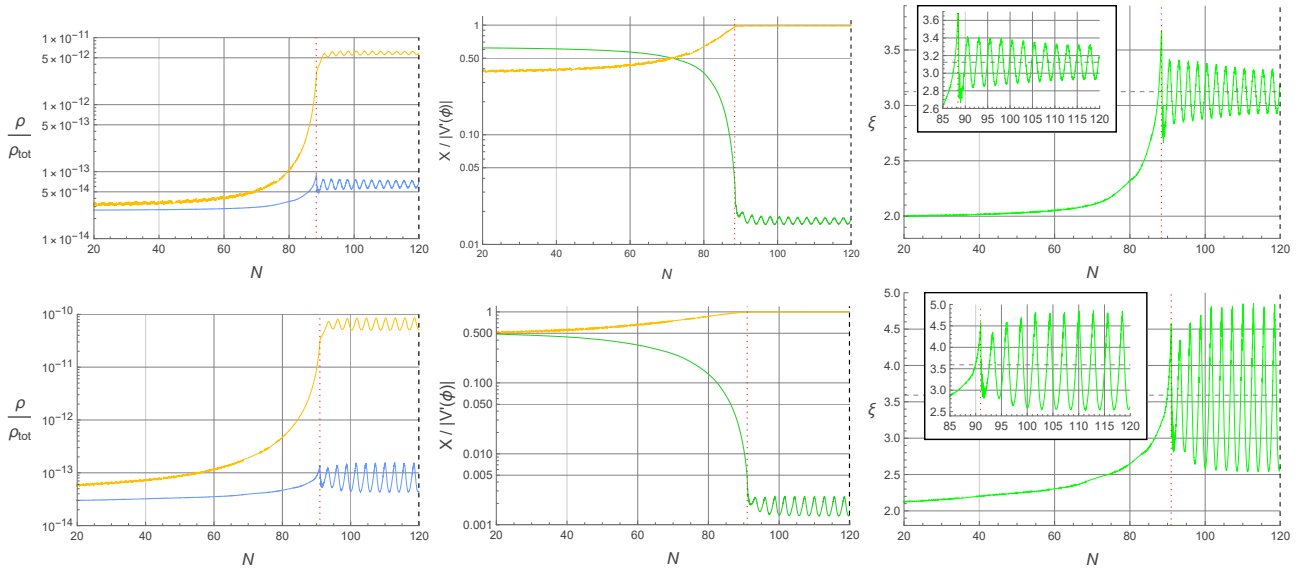


Figure 6.7: This figure shows runs taken with *BackHom* corresponding to $(10\bar{v}, \bar{b})$ (top row) and to $(100\bar{v}, \bar{b})$ (bottom row): in the left column the kinetic (light blue) and electromagnetic (yellow) energy densities normalized by the total energy; the middle column contains the evolution of the terms of the axion equation $X = \{3H\dot{\phi}(\text{green}), \beta\langle\mathbf{E} \cdot \mathbf{B}\rangle/M_p(\text{yellow})\}$ normalized by the absolute value of the slope; the right column shows the evolution of the instability variable ξ , where the horizontal dashed gray line stands at $\xi = \bar{\xi}$. The dotted vertical red line marks the e-folds at which $\phi = \bar{\phi}$, while the vertical dashed black line the end of the simulation.

In all panels of Fig. 6.7, we observe oscillations settling in, starting from the vertical dotted line that marks the moment when $\phi = \bar{\phi}$. We notice that in both panel rows these oscillations seem to stabilize around some fixed amplitude, which is clearly greater for the benchmark point inside the instability region. But, the amplitudes of these oscillations follow a different pattern in the two cases: for example, in the top right panel, the oscillations in ξ get smaller in the beginning and then, during the last ≈ 10 e-folds, their amplitude stops decreasing; on the other hand, the bottom right panel shows us that the oscillations at $\approx 90 - 95$ e-folds get damped, but then the amplitude increases until we see it stabilizing. While the maxima of each oscillation is slightly oscillating, the minima remain constant.

From the moment we observe the system oscillating, the reduction phase in which the amplitude of the perturbations diminishes is longer in the first case and shorter in the second. We can attribute this initial decrease in amplitude to the slope of the potential being $< 10\bar{v}$, because for these values the system is still lying within the stability region, where perturbations get damped. The fact that perturbations start oscillating with the same amplitude can be related only in the first case to the Lyapunov exponents, since for $\text{Re}(\zeta_1) = 0$ there is no real exponential convoluting the oscillations and the amplitude remains constant; instead, in the second case, perturbations should always grow according to the linear analysis of [16], therefore we relate the stabilization of the amplitude to the breaking of the linear approximation, when non-linear terms become relevant during the last 20 e-folds of the simulation. The LGEF can only make a statement about linear departures from the AS solution. But then, at some point, non-linear effects kick in, and we reach the limit cycle.

Figure 6.8 shows both the output of *BackHom* and GEFF. We clearly notice in all panels that the GEFF run recovers the same behavior of the variables computed with *BackHom*, but delayed. We observe that for the benchmark point in the first panel row the inflaton value $\bar{\phi}$ is crossed at $N_{\text{GEFF}}^{\text{top}} = 89.40$ and $N_{\text{BackHom}}^{\text{top}} = 88.36$ in the two codes ($\Delta N \approx 1$), while for the case shown in the bottom row we have that $N_{\text{GEFF}}^{\text{bottom}} = 92.14$ and $N_{\text{BackHom}}^{\text{bottom}} = 90.96$ ($\Delta N \approx 1.2$). Hence, recalling that for $(\bar{v}, \bar{b})$ $\Delta N \approx 0.3$, there is an increasing trend monotonic with the steepness of the targeted slope. Also, we note that in the bottom right panel at $\approx 98 - 100$ e-folds, GEFF clearly deviates from the prediction of *BackHom* since it shows very sudden turns that are likely due to numerical instabilities.

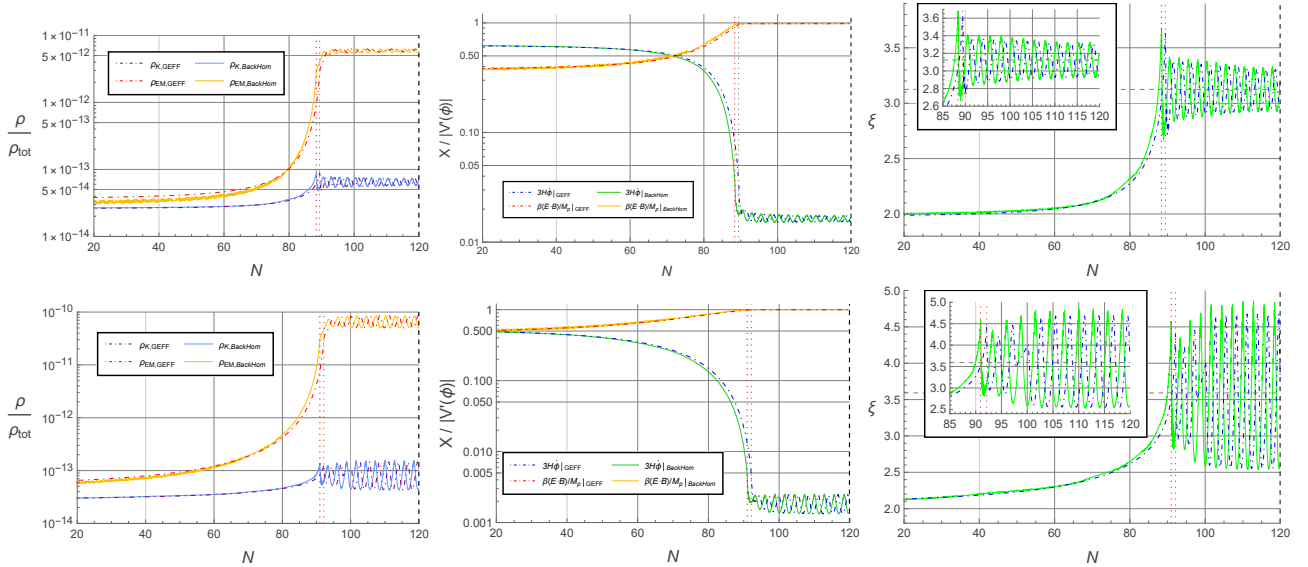


Figure 6.8: This figure shows the *BackHom* runs (solid) displayed in the previous figure overlapped with the GEFF runs (dot-dashed) corresponding to the same two benchmark points. The notation follows from the figures previously used in this subsection.

This increasing trend of the delay can be related to the slope of the potential where the inflaton is initialized: we are fixing directly the parameter $\bar{N}$ controlling the approximate number of e-folds that it takes to reach $\phi = \bar{\phi}$, but not the initial value of the slope. Hence, by choosing a larger value for v , we are effectively making the potential steeper also at $\phi = 0$, if we do not increase $\bar{N}$. This idea is further investigated in the next sub-subsection.

Adiabaticity of the transitioning phase

In this subsection, we vary the smoothness parameter $\Delta\phi$ in order to show how it affects the transition from the slow-roll regime to the stable AS solution identified by $(\bar{v}, \bar{b})$. In Fig. 6.9, we show the evolution of the kinetic and electromagnetic energy densities, of the terms of the axion equation and of the instability variable ξ for values of $\Delta\phi$ equal to 10^{-5} and 10^{-7} . We chose these two values, because they are one factor of 10 larger and smaller than the case analyzed in the beginning of the

subsection. The parameter $\bar{N}$ is kept equal to 60.

The panels in the top row of Fig. 6.9, corresponding to $\Delta\phi = 10^{-5}$, display a smooth transition in which no oscillations appear at any point in the simulation. Only in the top right panel, in correspondence with the crossing of $\bar{\phi}$, we notice a bump in the *BackHom* run that is not present for GEFF. In this case we had to prolong the evolution of the system to 150 e-folds, unlike the 120 e-folds all the other simulations are lasting, in order to get closer to the targeted value of $\bar{\xi}$. A transition to the stable AS solution without featuring oscillations can be achieved, but it takes a large amount of e-folds to complete it.

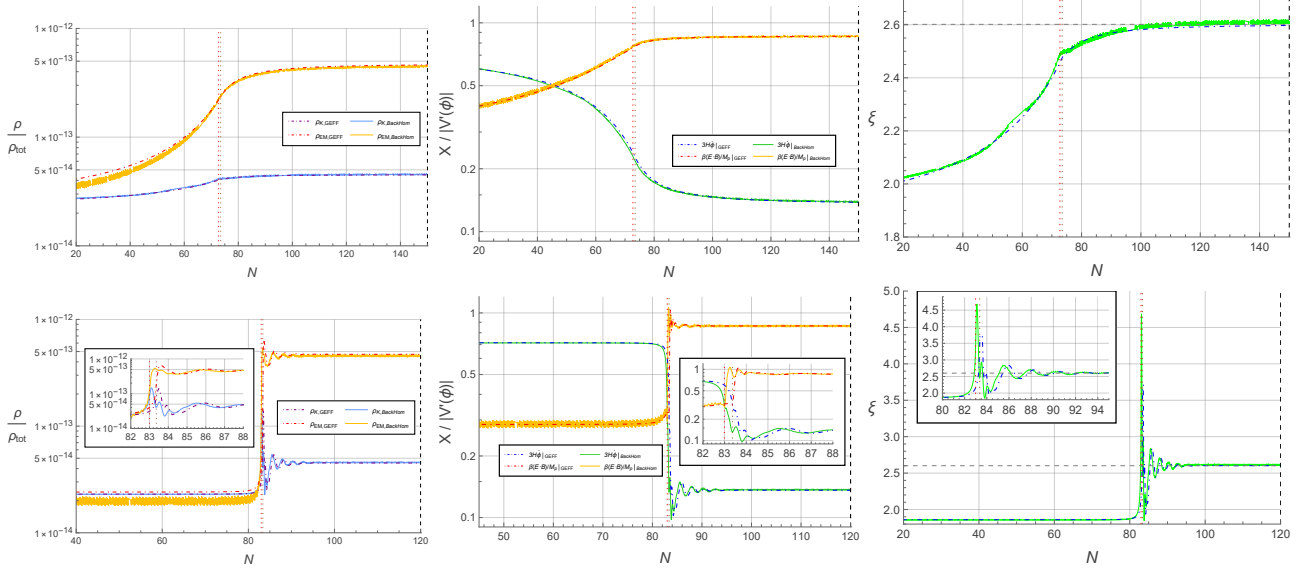


Figure 6.9: This figure shows runs taken with *BackHom* (solid) and GEFF (dot-dashed) corresponding to a potential defined by $\bar{N} = 60$ and respectively $\Delta\phi = 10^{-5}$ (top row) and $\Delta\phi = 10^{-7}$ (bottom row). The left column shows the energy densities, the central one the terms of the axion equation and the right one the instability variable; the notation follows from the previous figures of this subsection. The simulation in the top row lasts 150 e-folds, while all the others last 120.

The panels in the bottom row, instead, show the case of a very sharp transition, $\Delta\phi = 10^{-7}$. We notice that large oscillations take place, e.g. for *BackHom* the instability variable reaches the value of ≈ 4.6 and for GEFF it goes up to ≈ 3.8 , but in any case the system is able to stabilize at $N \approx 100$, condensing the whole transitioning phase in less than 20 e-folds.

While, in the case of a smoother transition, we see that the *BackHom* and GEFF runs overlap almost always, in the case of a smaller $\Delta\phi$ the two codes give clearly different predictions during the transitioning phase, in particular we note the different height in the maxima of ξ (bottom right panel). The silhouette of the observables we are considering is still similar, although the evolution with GEFF is delayed with respect to *BackHom*. Nevertheless, both simulations collapse to the same values of the observables starting from $N \approx 100$. The runs considered in this sub-subsection were taken with $i_{\max} = 1000$ (number of gauge modes) and $n_{\max} = 200$ (truncation order). Hence, since in Chapter 5 we saw that the relative accuracy of the two codes does not improve for $i_{\max} \geq 200$ and $n_{\max} \geq 100$, we do not think that this discrepancy would disappear by enhancing the degree of the approximation. The scope of this chapter is the realization of the stable AS solution in a more realistic context, in which H and $\dot{\phi}$ evolve in a dynamically consistent way, and we succeeded in it. Instead, we were not able to find agreement between the two codes for sharp transitions, but we quantified how sensitive the technical realization is by comparing the two methods. We leave this study to future work, since it is obviously interesting to study how to cope with sharp features in the potential, but this was not our goal.

In the simulations with $\Delta\phi = 10^{-5}$, the axion crosses the value $\bar{\phi}$ at $N_{\text{GEFF}}^{\text{top}} = 73.47$ and $N_{\text{BackHom}}^{\text{top}} = 72.70$ ($\Delta N \approx 0.8$), respectively for GEFF and *BackHom*, while for $\Delta\phi = 10^{-7}$, $\phi = \bar{\phi}$ is reached at

$N_{\text{GEFF}}^{\text{bottom}} = 83.36$ and $N_{\text{BackHom}}^{\text{bottom}} = 82.99$ ($\Delta N \approx 0.4$). We also notice that the smoother a transition is the earlier the strong backreaction regime settles in ($\bar{\phi}$ is crossed at ≈ 81 e-folds for $\Delta\phi = 10^{-6}$). Let us argue why this is related to the delay between the two codes. For shorter transitions, the inflaton is practically initialized on a slope $-\kappa_{SR}$ and then it encounters the steeper slope $-\kappa_{AS}$, but increasing $\Delta\phi$ enlarges the region of the potential in which the slopes $-\kappa_{SR}$ and $-\kappa_{AS}$ are mixed to a significant degree. Hence, for $\Delta\phi = 10^{-5}$, the inflaton is initialized in a point of the potential where the slope is already interpolating between the two asymptotic slopes. Due to the fact that it takes more time for *BackHom* to start accounting for the gauge field contribution, as we discussed in the previous chapter, the axion does not feel gauge friction for longer and it rolls faster advancing more on the potential than it happens for GEFF. The steeper the initial slope, the further the axion rolls away in *BackHom* with respect to GEFF. Therefore, in *BackHom* the $\phi = \bar{\phi}$ value is encountered earlier. We can connect this issue of initializing on a slope substantially steeper than $-\kappa_{SR}$ to our observations at the end of the previous sub-subsection for the exact same reasons. We suggest that a solution to this issue can be found by initializing the gauge fields directly in the AS regime already from $t = 0$. We leave this possibility open for future works, since this is beyond the scope of this work.

Conclusions

The analysis brought forward in this subsection shows that the linear study of perturbations of [16] predicted correctly the existence of a region of parameter space in which the stable AS solution can be achieved. We also showed that, by crossing the stability contour line corresponding to $\text{Re}(\zeta_1)$, the AS solution becomes indeed unstable and oscillations around it settle in, further confirming the results of [16]. Moreover, through our cross analysis with *BackHom* and GEFF, we were able to find that the AS solution can be reached without the onset of oscillations and that, even though the two codes do not always agree perfectly, both of them undoubtedly confirm the realization of the stable AS regime.

Chapter 7

Conclusions

In this thesis, we focused on an inflationary model in which an axion-inflaton scalar field is coupled to $U(1)$ gauge fields via a Chern–Simons-type interaction. The phenomenology produced by this kind of theory is studied in the literature numerically via different techniques. Two widely used schemes are the mode-by-mode approach and the gradient expansion formalism. The core of this work consisted in comparing the code implementations of these two methods. The comparison was brought forward in two benchmark scenarios and also in a region of parameter space in which the Anber-Sorbo (AS) solution can be achieved.

In Chapter 2, we introduced first the basics of FLRW cosmology, then the issue of initial conditions of Standard Cosmology, namely the horizon and flatness problems, explaining how they can be solved by a primordial epoch of accelerated expansion that can be realized by the slow-roll mechanism. We also gave a brief treatment of cosmological perturbations and how they can be generated in the inflationary paradigm.

Chapter 3 is dedicated to axion inflation in which the inflaton is coupled to $U(1)$ gauge fields. Firstly, we introduced the reasons why this model was proposed and then we derived the equations of motion in the homogeneous inflaton approximation, specifying how a cutoff in momentum space must be imposed in order to avoid divergences and to solve the equations numerically. Moreover, we reviewed how an inflationary era can be sustained thanks to gauge friction as presented in [8] and in the last section we went through the mechanism by which gravitational waves are produced in this model.

In Chapter 4 we presented the two codes, *BackHom* and GEF, in which the MbM approach and the GEF are realized. We described first *BackHom*, in which a one dimensional grid in momentum space is considered, and then GEF, in which a tower of equations for the electromagnetic bilinears with increasing insertions of spatial derivatives is built. While in the first case the approximation relies on the evolution of a finite amount of gauge modes, in the GEF the approximation lies in the truncation order at which we cut the infinite chain of coupled differential equations. In this chapter, we specified the way the variables are rescaled, what equations we consider in the two methods and how the initial conditions are set.

The focus of Chapter 5 was the comparison of the two codes, *BackHom* and GEF. After choosing the three energy densities, ρ_K , ρ_V and ρ_{EM} , normalized by the total energy density $\rho_{tot} = 3H^2 M_p^2$ as the reference quantities to compare, we took various runs with both codes, increasing the level of approximation. For *BackHom* we modified the number of gauge modes i_{max} that are evolved, while for GEF the truncation order n_{max} of the chain of differential equations. Firstly, we computed the relative errors within a method in order to establish the level after which the precision is not improving anymore. Then, we compared runs of various degrees of approximation obtained with one method with a simulation done with the other at a very high number of either gauge modes (MbM) or differential equations in electromagnetic bilinears (GEF). This last step is crucial since it assesses the level of accuracy of the codes: numerical instabilities in an algorithm might make one code converge to a spurious solution, but confronting two completely independent numerical schemes guarantees

accuracy more strongly as it is extremely unlikely that they would converge to the same incorrect solution. We concluded that the two codes agree with a precision of $\lesssim \mathcal{O}(10^{-3})$ for $i_{\max} = 200$ and $n_{\max} = 80$.

In Chapter 6, the first section is dedicated to reviewing the linear stability analysis of [16] in which the linear gradient expansion formalism (LGEF) is adopted to study perturbations around the AS solution and to determine the Lyapunov exponents. Thanks to this analysis, we were able to determine the existence of a region of parameter space where, for a strong enough coupling, the AS solution is stable in presence of strong backreaction. In the second section, we defined a potential by joining two linear slopes with a sigmoid function: in this way, we initialized the system in the weak backreaction regime and then we entered the strong one by steepening the slope. We took advantage of both codes studied in the previous chapter to explore the parameter region in which the AS solution is stable. We successfully reached the stable AS solution and noticed that by modifying the sharpness of the transition we can shorten the amount of e-folds that it takes to attain it, but we could also make the transition longer eliminating the amount of oscillations.

This thesis showed for the first time that it is actually possible to dynamically roll into a stable AS solution, in the simplified case of a perfectly homogeneous axion field. Earlier attempts to dynamically roll into a stable AS solution have always failed and led to nonlinear oscillations in ξ . We could now show for the first time that it is possible to reach a constant- ξ solution. But to further validate the existence of the stable AS solution, one has to check the effect of axion gradients nonetheless. In future work, one therefore needs to study the same regime in a lattice simulation and check whether the stable AS solution survives or not.

Bibliography

- [1] GUTH, Alan H.: The Inflationary Universe: A Possible Solution to the Horizon and Flatness Problems. In: *Phys. Rev. D* 23 (1981), S. 347–356. <http://dx.doi.org/10.1103/PhysRevD.23.347>. – DOI 10.1103/PhysRevD.23.347
- [2] AKRAMI, Y. u. a.: Planck 2018 results. X. Constraints on inflation. In: *Astron. Astrophys.* 641 (2020), S. A10. <http://dx.doi.org/10.1051/0004-6361/201833887>. – DOI 10.1051/0004-6361/201833887
- [3] ADE, P. A. R. u. a.: Improved Constraints on Primordial Gravitational Waves using Planck, WMAP, and BICEP/Keck Observations through the 2018 Observing Season. In: *Phys. Rev. Lett.* 127 (2021), Nr. 15, S. 151301. <http://dx.doi.org/10.1103/PhysRevLett.127.151301>. – DOI 10.1103/PhysRevLett.127.151301
- [4] SMOOT, G. F. u. a.: Structure in the COBE Differential Microwave Radiometer First-Year Maps. In: *Astrophysical Journal Letters* 396 (1992), Nr. 1, S. L1–L5. <http://dx.doi.org/10.1086/186504>. – DOI 10.1086/186504. – S2CID 120701913
- [5] BENNETT, C. L. u. a.: Four-Year COBE DMR Cosmic Microwave Background Observations: Maps and Basic Results. In: *Astrophysical Journal Letters* 464 (1996), S. L1–L4. <http://dx.doi.org/10.1086/310075>. – DOI 10.1086/310075. – S2CID 18144842
- [6] FREESE, Katherine ; FRIEMAN, Joshua A. ; OLINTO, Angela V.: Natural inflation with pseudo Nambu-Goldstone bosons. In: *Phys. Rev. Lett.* 65 (1990), Dec, 3233–3236. <http://dx.doi.org/10.1103/PhysRevLett.65.3233>. – DOI 10.1103/PhysRevLett.65.3233
- [7] BANKS, Tom ; DINE, Michael ; FOX, P. J. ; GORBATOV, E: On the possibility of large axion decay constants. In: *Journal of Cosmology and Astroparticle Physics* 2003 (2003), Nr. 06, 001–001. <http://dx.doi.org/10.1088/1475-7516/2003/06/001>. – DOI 10.1088/1475-7516/2003/06/001. – ISSN 1475–7516
- [8] ANBER, Mohamed M. ; SORBO, Lorenzo: Naturally inflating on steep potentials through electromagnetic dissipation. In: *Phys. Rev. D* 81 (2010), S. 043534. <http://dx.doi.org/10.1103/PhysRevD.81.043534>. – DOI 10.1103/PhysRevD.81.043534
- [9] GORBAR, E. V. ; SCHMITZ, K. ; SOBOLOV, O. O. ; VILCHINSKII, S. I.: Gauge-field production during axion inflation in the gradient expansion formalism. In: *Phys. Rev. D* 104 (2021), Nr. 12, S. 123504. <http://dx.doi.org/10.1103/PhysRevD.104.123504>. – DOI 10.1103/PhysRevD.104.123504
- [10] GARCIA-BELLIDO, Juan ; PAPAGEORGIOU, Alexandros ; PELOSO, Marco ; SORBO, Lorenzo: A flashing beacon in axion inflation: recurring bursts of gravitational waves in the strong backreaction regime. In: *JCAP* 01 (2024), S. 034. <http://dx.doi.org/10.1088/1475-7516/2024/01/034>. – DOI 10.1088/1475-7516/2024/01/034
- [11] DOMCKE, Valerie ; GUIDETTI, Veronica ; WELLING, Yvette ; WESTPHAL, Alexander: Resonant backreaction in axion inflation. In: *JCAP* 09 (2020), S. 009. <http://dx.doi.org/10.1088/1475-7516/2020/09/009>. – DOI 10.1088/1475-7516/2020/09/009

- [12] CHENG, Shu-Lin ; LEE, Wolung ; NG, Kin-Wang: Numerical study of pseudoscalar inflation with an axion-gauge field coupling. In: *Phys. Rev. D* 93 (2016), Nr. 6, S. 063510. <http://dx.doi.org/10.1103/PhysRevD.93.063510>. – DOI 10.1103/PhysRevD.93.063510
- [13] NOTARI, Alessio ; TYWONIUK, Konrad: Dissipative Axial Inflation. In: *JCAP* 12 (2016), S. 038. <http://dx.doi.org/10.1088/1475-7516/2016/12/038>. – DOI 10.1088/1475-7516/2016/12/038
- [14] DALL’AGATA, Gianguido ; GONZÁLEZ-MARTÍN, Sergio ; PAPAGEORGIOU, Alexandros ; PELOSO, Marco: Warm dark energy. In: *JCAP* 08 (2020), S. 032. <http://dx.doi.org/10.1088/1475-7516/2020/08/032>. – DOI 10.1088/1475-7516/2020/08/032
- [15] PELOSO, Marco ; SORBO, Lorenzo: Instability in axion inflation with strong backreaction from gauge modes. In: *JCAP* 01 (2023), S. 038. <http://dx.doi.org/10.1088/1475-7516/2023/01/038>. – DOI 10.1088/1475-7516/2023/01/038
- [16] SOBOL, Oleksandr O. ; ECKARDSTEIN, Richard von ; KOCH, Elias R. ; GUREVICH, Svetlana V. ; THIELE, Uwe ; SCHMITZ, Kai: Nonlinear physics of axion inflation. In: *JHEP* 06 (2026), S. 166. [http://dx.doi.org/10.1007/JHEP06\(2026\)166](http://dx.doi.org/10.1007/JHEP06(2026)166). – DOI 10.1007/JHEP06(2026)166
- [17] SLIPHER, V. M.: The Radial Velocity of the Andromeda Nebula. In: *Lowell Observatory Bulletin* 1 (1913), Nr. 8, S. 56–57
- [18] FRIEDMAN, A.: Über die Krümmung des Raumes. In: *Zeitschrift für Physik* 10 (1922), Nr. 1, S. 377–386. <http://dx.doi.org/10.1007/BF01332580>. – DOI 10.1007/BF01332580. – S2CID 125190902
- [19] FRIEDMANN, A.: On the Curvature of Space. In: *General Relativity and Gravitation* 31 (1999), Nr. 12, S. 1991–2000. <http://dx.doi.org/10.1023/A:1026751225741>. – DOI 10.1023/A:1026751225741. – S2CID 122950995
- [20] HUBBLE, E. P.: A Spiral Nebula as a Stellar System, Messier 31. In: *The Astrophysical Journal* 69 (1929), S. 103. <http://dx.doi.org/10.1086/143167>. – DOI 10.1086/143167
- [21] LEMAÎTRE, Georges: Un Univers homogène de masse constante et de rayon croissant rendant compte de la vitesse radiale des nébuleuses extra-galactiques. In: *Annales de la Société Scientifique de Bruxelles A47* (1927), 49–59. <http://adsabs.harvard.edu/abs/1927ASSB...47...49L>. – English translation title: A homogeneous universe of constant mass and increasing radius accounting for the radial speed of extra-galactic nebulae
- [22] ROBERTSON, H. P.: Kinematics and World Structure. In: *Astrophysical Journal* 82 (1935), S. 284–301. <http://dx.doi.org/10.1086/143681>. – DOI 10.1086/143681
- [23] ROBERTSON, H. P.: Kinematics and World Structure II. In: *Astrophysical Journal* 83 (1936), S. 187–201. <http://dx.doi.org/10.1086/143716>. – DOI 10.1086/143716
- [24] ROBERTSON, H. P.: Kinematics and World Structure III. In: *Astrophysical Journal* 83 (1936), S. 257–271. <http://dx.doi.org/10.1086/143726>. – DOI 10.1086/143726
- [25] WALKER, A. G.: On Milne’s Theory of World-Structure. In: *Proceedings of the London Mathematical Society* 42 (1937), Nr. 1, S. 90–127. <http://dx.doi.org/10.1112/plms/s2-42.1.90>. – DOI 10.1112/plms/s2-42.1.90
- [26] PENZIAS, A. A. ; WILSON, R. W.: A Measurement of Excess Antenna Temperature at 4080 Mc/s. In: *The Astrophysical Journal* 142 (1965), Nr. 1, S. 419–421. <http://dx.doi.org/10.1086/148307>. – DOI 10.1086/148307
- [27] AGHANIM, N. u. a.: Planck 2018 results. I. Overview and the cosmological legacy of Planck. In: *Astron. Astrophys.* 641 (2020), S. A1. <http://dx.doi.org/10.1051/0004-6361/201833880>. – DOI 10.1051/0004-6361/201833880

- [28] PEEBLES, P. J. E.: *Principles of Physical Cosmology*. Princeton University Press, 2020. – ISBN 978-0-691-20981-4
- [29] CARROLL, Sean M.: *Spacetime and Geometry: An Introduction to General Relativity*. Cambridge University Press, 2019. <http://dx.doi.org/10.1017/9781108770385>, <http://dx.doi.org/10.1017/9781108770385>. – ISBN 978-0-8053-8732-2, 978-1-108-48839-6, 978-1-108-77555-7
- [30] RIESS, Adam G. ; FILIPPENKO, A. V. ; CHALLIS, P. ; CLOCCHIATTI, A. ; DIERCKS, A. ; GARNAVICH, P. M. ; GILLILAND, R. L. ; HOGAN, C. J. ; JHA, S. ; KIRSHNER, R. P. ; LEIBUNDGUT, B. ; PHILLIPS, M. M. ; REISS, D. ; SCHMIDT, B. P. ; SCHOMMER, R. A. ; SMITH, R. C. ; SPYROMILIO, J. ; STUBBS, C. ; SUNTZEFF, N. B. ; TONRY, J.: Observational Evidence from Supernovae for an Accelerating Universe and a Cosmological Constant. In: *Astronomical Journal* 116 (1998), Nr. 3, S. 1009–1038. <http://dx.doi.org/10.1086/300499>. – DOI 10.1086/300499. – S2CID 15640044
- [31] PERLMUTTER, S. ; ALDERING, G. ; GOLDBERGER, G. ; KNOP, R. A. ; NUGENT, P. ; CASTRO, P. G. ; DEUSTUA, S. ; FABBRO, S. ; GOOBAR, A. ; GROOM, D. E. ; HOOK, I. M. ; KIM, A. G. ; KIM, M. Y. ; LEE, J. C. ; NUNES, N. J. ; PAIN, R. ; PENNYPACKER, C. R. ; QUIMBY, R. ; LIDMAN, C. ; ELLIS, R. S. ; IRWIN, M. ; MCMAHON, R. G. ; RUIZ-LAPUENTE, P. ; WALTON, N. ; SCHAEFER, B. ; BOYLE, B. J. ; FILIPPENKO, A. V. ; MATHESON, T. ; FRUCHTER, A. S. u. a.: Measurements of Omega and Lambda from 42 High Redshift Supernovae. In: *Astrophysical Journal* 517 (1999), Nr. 2, S. 565–586. <http://dx.doi.org/10.1086/307221>. – DOI 10.1086/307221. – S2CID 118910636
- [32] RINDLER, W.: Visual Horizons in World Models. In: *Monthly Notices of the Royal Astronomical Society* 116 (1956), December, Nr. 6, S. 662–677. <http://dx.doi.org/10.1093/mnras/116.6.662>. – DOI 10.1093/mnras/116.6.662
- [33] DICKE, Robert H.: *Gravitation and the Universe: Jayne Lectures for 1969*. American Philosophical Society, 1970. – ISBN 978-0-87169-078-4
- [34] BAUMANN, Daniel: Inflation. In: *Theoretical Advanced Study Institute in Elementary Particle Physics: Physics of the Large and the Small*, 2011, S. 523–686
- [35] RIOTTO, A.: Inflation and the Theory of Cosmological Perturbations, 2018
- [36] AGHANIM, N. u. a.: Planck 2018 results. VI. Cosmological parameters. In: *Astron. Astrophys.* 641 (2020), S. A6. <http://dx.doi.org/10.1051/0004-6361/201833910>. – DOI 10.1051/0004-6361/201833910. – [Erratum: *Astron. Astrophys.* 652, C4 (2021)]
- [37] ADAME, A. G. u. a.: DESI 2024 VI: cosmological constraints from the measurements of baryon acoustic oscillations. In: *JCAP* 02 (2025), S. 021. <http://dx.doi.org/10.1088/1475-7516/2025/02/021>. – DOI 10.1088/1475-7516/2025/02/021
- [38] GUTH, Alan H. ; PI, S. Y.: Fluctuations in the New Inflationary Universe. In: *Phys. Rev. Lett.* 49 (1982), S. 1110–1113. <http://dx.doi.org/10.1103/PhysRevLett.49.1110>. – DOI 10.1103/PhysRevLett.49.1110
- [39] LINDE, Andrei D.: A New Inflationary Universe Scenario: A Possible Solution of the Horizon, Flatness, Homogeneity, Isotropy and Primordial Monopole Problems. In: *Phys. Lett. B* 108 (1982), S. 389–393. [http://dx.doi.org/10.1016/0370-2693\(82\)91219-9](http://dx.doi.org/10.1016/0370-2693(82)91219-9). – DOI 10.1016/0370-2693(82)91219-9
- [40] ALBRECHT, Andreas ; STEINHARDT, Paul J.: Cosmology for Grand Unified Theories with Radiatively Induced Symmetry Breaking. In: *Phys. Rev. Lett.* 48 (1982), S. 1220–1223. <http://dx.doi.org/10.1103/PhysRevLett.48.1220>. – DOI 10.1103/PhysRevLett.48.1220
- [41] LINDE, Andrei D.: Chaotic Inflation. In: *Phys. Lett. B* 129 (1983), S. 177–181. [http://dx.doi.org/10.1016/0370-2693\(83\)90837-7](http://dx.doi.org/10.1016/0370-2693(83)90837-7). – DOI 10.1016/0370-2693(83)90837-7

- [42] SPOKOINY, B. L.: Inflation and Generation of Perturbations in Broken Symmetric Theory of Gravity. In: *Phys. Lett. B* 147 (1984), S. 39–43
- [43] FUTAMASE, T. ; MAEDA, K.: Chaotic Inflationary Scenario of the Universe with a Nonminimally Coupled 'Inflaton' Field. In: *Phys. Rev. D* 39 (1989), S. 399–404. <http://dx.doi.org/10.1103/PhysRevD.39.399>. – DOI 10.1103/PhysRevD.39.399
- [44] STAROBINSKY, A. A.: A New Type of Isotropic Cosmological Models Without Singularity. In: *Phys. Lett. B* 91 (1980), S. 99–102. [http://dx.doi.org/10.1016/0370-2693\(80\)90670-X](http://dx.doi.org/10.1016/0370-2693(80)90670-X). – DOI 10.1016/0370-2693(80)90670-X
- [45] STAROBINSKY, A. A.: The Perturbation Spectrum Evolving from a Nonsingular Initially De-Sitter Cosmology and the Microwave Background Anisotropy. In: *Sov. Astron. Lett.* 9 (1983), S. 302–304
- [46] BARROW, John D. ; OTTEWILL, A. C.: The Stability of General Relativistic Cosmological Theory. In: *J. Phys. A* 16 (1983), S. 2757–2776. <http://dx.doi.org/10.1088/0305-4470/16/12/022>. – DOI 10.1088/0305-4470/16/12/022
- [47] WHITT, B.: Fourth Order Gravity as General Relativity Plus Matter. In: *Phys. Lett. B* 145 (1984), S. 176–178. [http://dx.doi.org/10.1016/0370-2693\(84\)90332-0](http://dx.doi.org/10.1016/0370-2693(84)90332-0). – DOI 10.1016/0370-2693(84)90332-0
- [48] KOFMAN, L. A. ; LINDE, A. D. ; STAROBINSKY, A. A.: Inflationary Universe Generated by the Combined Action of a Scalar Field and Gravitational Vacuum Polarization. In: *Phys. Lett. B* 157 (1985), S. 361–367. [http://dx.doi.org/10.1016/0370-2693\(85\)90381-4](http://dx.doi.org/10.1016/0370-2693(85)90381-4). – DOI 10.1016/0370-2693(85)90381-4
- [49] ARMENDARIZ-PICON, C. ; DAMOUR, T. ; MUKHANOV, V. F.: k-Inflation. In: *Phys. Lett. B* 458 (1999), S. 209–218. [http://dx.doi.org/10.1016/S0370-2693\(99\)00603-6](http://dx.doi.org/10.1016/S0370-2693(99)00603-6). – DOI 10.1016/S0370-2693(99)00603-6
- [50] GARRIGA, J. ; MUKHANOV, V. F.: Perturbations in k-inflation. In: *Phys. Lett. B* 458 (1999), S. 219–225. [http://dx.doi.org/10.1016/S0370-2693\(99\)00602-4](http://dx.doi.org/10.1016/S0370-2693(99)00602-4). – DOI 10.1016/S0370-2693(99)00602-4
- [51] KOFMAN, L. A. ; LINDE, A. D.: Generation of Density Perturbations in the Inflationary Cosmology. In: *Nucl. Phys. B* 282 (1987), S. 555–588. [http://dx.doi.org/10.1016/0550-3213\(87\)90698-5](http://dx.doi.org/10.1016/0550-3213(87)90698-5). – DOI 10.1016/0550-3213(87)90698-5
- [52] SASAKI, Misao ; STEWART, Ewan D.: A General analytic formula for the spectral index of the density perturbations produced during inflation. In: *Prog. Theor. Phys.* 95 (1996), S. 71–78. <http://dx.doi.org/10.1143/PTP.95.71>. – DOI 10.1143/PTP.95.71
- [53] COPELAND, Edmund J. ; LIDDLE, Andrew R. ; LYTH, David H. ; STEWART, Ewan D. ; WANDS, David: False vacuum inflation with Einstein gravity. In: *Phys. Rev. D* 49 (1994), S. 6410–6433. <http://dx.doi.org/10.1103/PhysRevD.49.6410>. – DOI 10.1103/PhysRevD.49.6410
- [54] ARKANI-HAMED, Nima ; CHENG, Hsin-Chia ; CREMINELLI, Paolo ; RANDALL, Lisa: Pseudonatural inflation. In: *Journal of Cosmology and Astroparticle Physics* 2003 (2003), Nr. 07, 003–003. <http://dx.doi.org/10.1088/1475-7516/2003/07/003>. – DOI 10.1088/1475-7516/2003/07/003. – ISSN 1475–7516
- [55] PECCEI, R. D. ; QUINN, Helen R.: CP Conservation in the Presence of Pseudoparticles. In: *Phys. Rev. Lett.* 38 (1977), Jun, 1440–1443. <http://dx.doi.org/10.1103/PhysRevLett.38.1440>. – DOI 10.1103/PhysRevLett.38.1440
- [56] WEINBERG, Steven: A New Light Boson? In: *Phys. Rev. Lett.* 40 (1978), S. 223–226. <http://dx.doi.org/10.1103/PhysRevLett.40.223>. – DOI 10.1103/PhysRevLett.40.223

- [57] WILCZEK, F.: Problem of Strong P and T Invariance in the Presence of Instantons. In: *Phys. Rev. Lett.* 40 (1978), Jan, 279–282. <http://dx.doi.org/10.1103/PhysRevLett.40.279>. – DOI 10.1103/PhysRevLett.40.279
- [58] PAJER, Enrico ; PELOSO, Marco: A review of Axion Inflation in the era of Planck. In: *Class. Quant. Grav.* 30 (2013), S. 214002. <http://dx.doi.org/10.1088/0264-9381/30/21/214002>. – DOI 10.1088/0264-9381/30/21/214002
- [59] ANBER, Mohamed M. ; SORBO, Lorenzo: N-flationary magnetic fields. In: *JCAP* 10 (2006), S. 018. <http://dx.doi.org/10.1088/1475-7516/2006/10/018>. – DOI 10.1088/1475-7516/2006/10/018
- [60] FIGUEROA, Daniel G. ; FLORIO, Adrien ; TORRENTI, Francisco ; VALKENBURG, Wessel: The art of simulating the early Universe – Part I: Integration techniques and canonical cases. In: *JCAP* 04 (2021), S. 035. <http://dx.doi.org/10.1088/1475-7516/2021/04/035>. – DOI 10.1088/1475-7516/2021/04/035
- [61] BAEZA-BALLESTEROS, Jorge ; FIGUEROA, Daniel G. ; FLORIO, Adrien ; LIZARRAGA, Joanes ; LOAYZA, Nicolás ; MARSCHALL, Kenneth ; OPFERKUCH, Toby ; STEFANEK, Ben A. ; TORRENTÍ, Francisco ; URIO, Ander: The art of simulating the early Universe. Part II. Non-canonical cases & gravitational waves. In: *JCAP* 06 (2026), S. 087. <http://dx.doi.org/10.1088/1475-7516/2026/06/087>. – DOI 10.1088/1475-7516/2026/06/087
- [62] ABRAMOWITZ, Milton ; STEGUN, Irene A.: *Applied Mathematics Series*. Bd. 55: *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*. Washington, D.C. : National Bureau of Standards, 1964. – ISBN 0-486-61272-4. – Reprinted by Dover Publications, New York
- [63] BALLARDINI, M. ; BRAGLIA, M. ; FINELLI, F. ; MAROZZI, G. ; STAROBINSKY, A. A.: Energy-momentum tensor and helicity for gauge fields coupled to a pseudoscalar inflaton. In: *Phys. Rev. D* 100 (2019), Dec, 123542. <http://dx.doi.org/10.1103/PhysRevD.100.123542>. – DOI 10.1103/PhysRevD.100.123542
- [64] ECKARDSTEIN, Richard von ; SCHMITZ, Kai ; SOBOL, Oleksandr: Gravitational waves from axion inflation in the gradient expansion formalism. Part I. Pure axion inflation. In: *JHEP* 01 (2026), S. 018. [http://dx.doi.org/10.1007/JHEP01\(2026\)018](http://dx.doi.org/10.1007/JHEP01(2026)018). – DOI 10.1007/JHEP01(2026)018
- [65] BARBON, Matteo: *Tensor and scalar modes in axion inflation in the regime of strong homogeneous back-reaction*. Padua, Italy, University of Padua, Master’s thesis in Physics, 2024. <https://hdl.handle.net/20.500.12608/91183>. – Supervisor: Marco Peloso
- [66] BOYLE, Latham A. ; STEINHARDT, Paul J.: Probing the early universe with inflationary gravitational waves. In: *Phys. Rev. D* 77 (2008), Mar, 063504. <http://dx.doi.org/10.1103/PhysRevD.77.063504>. – DOI 10.1103/PhysRevD.77.063504
- [67] CAPRINI, Chiara ; FIGUEROA, Daniel G.: Cosmological backgrounds of gravitational waves. In: *Class. Quant. Grav.* 35 (2018), Nr. 16, S. 163001. <http://dx.doi.org/10.1088/1361-6382/aac608>. – DOI 10.1088/1361-6382/aac608
- [68] SAIKAWA, Ken’ichi ; SHIRAI, Satoshi: Primordial gravitational waves, precisely: The role of thermodynamics in the Standard Model. In: *JCAP* 05 (2018), S. 035. <http://dx.doi.org/10.1088/1475-7516/2018/05/035>. – DOI 10.1088/1475-7516/2018/05/035
- [69] ECKARDSTEIN, Richard von: GEFF: The Gradient Expansion Formalism Factory - A tool for inflationary gauge-field production. (2025), 10
- [70] BARBON, Matteo ; IJAZ, Nadir ; PELOSO, Marco: Axion inflation in the regime of homogeneous backreaction. In: *JCAP* 06 (2026), S. 013. <http://dx.doi.org/10.1088/1475-7516/2026/06/013>. – DOI 10.1088/1475-7516/2026/06/013

- [71] ABRAMOWITZ, M. ; RABINOWITZ, P.: Evaluation of Coulomb Wave Functions Along the Transition Line. In: *Phys. Rev.* 96 (1954), S. 77. <http://dx.doi.org/10.1103/PhysRev.96.77>. – DOI 10.1103/PhysRev.96.77
- [72] FRÖBERG, C.-E.: Numerical Treatment of Coulomb Wave Functions. In: *Rev. Mod. Phys.* 27 (1955), S. 399. <http://dx.doi.org/10.1103/RevModPhys.27.399>. – DOI 10.1103/RevModPhys.27.399
- [73] ISACSON, T.: Asymptotic Expansion of Coulomb Wave Functions on the Transition Line. In: *BIT* 8 (1968), S. 243. <http://dx.doi.org/10.1007/BF01933424>. – DOI 10.1007/BF01933424
- [74] BATEMAN, H. ; ERDÉLYI, A.: *Higher Transcendental Functions*. Bd. 1. New York : McGraw-Hill, 1953
- [75] ERDÉLYI, A. ; SWANSON, C. A.: Asymptotic Forms of Whittaker's Confluent Hypergeometric Functions. In: *Mem. Amer. Math. Soc.* 25 (1957), S. 1. <http://dx.doi.org/10.1090/memo/0025>. – DOI 10.1090/memo/0025
- [76] ECKARDSTEIN, Richard von ; SCHMITZ, Kai ; SOBOL, Oleksandr: Gravitational waves from axion inflation in the gradient expansion formalism. Part II. Fermionic axion inflation. In: *JHEP* 03 (2026), S. 072. [http://dx.doi.org/10.1007/JHEP03\(2026\)072](http://dx.doi.org/10.1007/JHEP03(2026)072). – DOI 10.1007/JHEP03(2026)072
- [77] ECKARDSTEIN, Richard von ; SCHMITZ, Kai ; SOBOL, Oleksandr: On the Schwinger effect during axion inflation. In: *JHEP* 02 (2025), S. 096. [http://dx.doi.org/10.1007/JHEP02\(2025\)096](http://dx.doi.org/10.1007/JHEP02(2025)096). – DOI 10.1007/JHEP02(2025)096
- [78] FRANCIOLINI, Gabriele ; IJAZ, Nadir ; PELOSO, Marco: Primordial black hole dark matter from axion inflation. (2026), 4
- [79] TRISTRAM, M. u. a.: Improved limits on the tensor-to-scalar ratio using BICEP and Planck data. In: *Phys. Rev. D* 105 (2022), Nr. 8, S. 083524. <http://dx.doi.org/10.1103/PhysRevD.105.083524>. – DOI 10.1103/PhysRevD.105.083524
- [80] ADSHEAD, Peter ; GIBLIN, John T. ; PIERONI, Mauro ; WEINER, Zachary J.: Constraining axion inflation with gravitational waves from preheating. In: *Phys. Rev. D* 101 (2020), Nr. 8, S. 8. <http://dx.doi.org/10.1103/PhysRevD.101.083534>. – DOI 10.1103/PhysRevD.101.083534
- [81] ADSHEAD, Peter ; GIBLIN, John T. ; PIERONI, Mauro ; WEINER, Zachary J.: Constraining Axion Inflation with Gravitational Waves across 29 Decades in Frequency. In: *Phys. Rev. Lett.* 124 (2020), Nr. 17, S. 17. <http://dx.doi.org/10.1103/PhysRevLett.124.171301>. – DOI 10.1103/PhysRevLett.124.171301
- [82] DOMCKE, Valerie ; PAPAGEORGIOU, Alexandros ; PELOSO, Marco ; SANDNER, Stefan: Axion Inflation: Perturbative control in the strong backreaction regime. (2026), 8
- [83] CARAVANO, Angelo ; KOMATSU, Eiichiro ; LOZANOV, Kaloian D. ; WELLER, Jochen: Lattice simulations of axion-U(1) inflation. In: *Phys. Rev. D* 108 (2023), Nr. 4, S. 043504. <http://dx.doi.org/10.1103/PhysRevD.108.043504>. – DOI 10.1103/PhysRevD.108.043504
- [84] FIGUEROA, Daniel G. ; LIZARRAGA, Joanes ; URIO, Ander ; URRESTILLA, Jon: Strong Backreaction Regime in Axion Inflation. In: *Phys. Rev. Lett.* 131 (2023), Nr. 15, S. 151003. <http://dx.doi.org/10.1103/PhysRevLett.131.151003>. – DOI 10.1103/PhysRevLett.131.151003
- [85] FIGUEROA, Daniel G. ; LIZARRAGA, Joanes ; LOAYZA, Nicolás ; URIO, Ander ; URRESTILLA, Jon: Nonlinear dynamics of axion inflation: A detailed lattice study. In: *Phys. Rev. D* 111 (2025), Nr. 6, S. 063545. <http://dx.doi.org/10.1103/PhysRevD.111.063545>. – DOI 10.1103/PhysRevD.111.063545
- [86] FIGUEROA, Daniel G. ; FLORIO, Adrien ; TORRENTI, Francisco ; VALKENBURG, Wessel: Cosmo-Lattice: A modern code for lattice simulations of scalar and gauge field dynamics in an expanding

- universe. In: *Comput. Phys. Commun.* 283 (2023), S. 108586. <http://dx.doi.org/10.1016/j.cpc.2022.108586>. – DOI 10.1016/j.cpc.2022.108586
- [87] DOMCKE, Valerie ; EMA, Yohei ; SANDNER, Stefan: Perturbatively including inhomogeneities in axion inflation. In: *JCAP* 03 (2024), S. 019. <http://dx.doi.org/10.1088/1475-7516/2024/03/019>. – DOI 10.1088/1475-7516/2024/03/019
- [88] ECKARDSTEIN, Richard von ; PELOSO, Marco ; SCHMITZ, Kai ; SOBOL, Oleksandr ; SORBO, Lorenzo: Axion inflation in the strong-backreaction regime: decay of the Anber-Sorbo solution. In: *JHEP* 11 (2023), S. 183. [http://dx.doi.org/10.1007/JHEP11\(2023\)183](http://dx.doi.org/10.1007/JHEP11(2023)183). – DOI 10.1007/JHEP11(2023)183